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A NEW HALPERN-TYPE AVERAGING ALGORITHM WITH INERTIAL AND ERROR TERMS FOR FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPS

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Abstract. We introduce and study a Halpern-type averaging algorithm with both inertial and error terms for the approximation of fixed points of asymptotically nonexpansive maps in real Hilbert spaces. Implementation of our algorithm is illustrated using numerical examples in both finite and infinite dimensional real Hilbert spaces. Our results extend recent results of Yekini, Iyiola and Ogbuisi, Numer Algor (2019), https://doi.org/10.1007/s11075-019-00727-5 from the important class of nonexpansive maps to the much more general class of asymptotically nonexpansive maps. Furthermore, our preliminary lemma is of independent interest.

Keywords: asymptotically nonexpansive mappings; nonexpansive mappings; Halpern-type averaging algorithm; inertial terms; Hilbert spaces; strong convergence.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $||.||$. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *L-Lipschitzian* if there

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exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \; \forall x, y \in C. \tag{1.1}$$

T is said to be a *contraction* if $L \in [0,1)$ and T is said to be *nonexpansive* if $L = 1$ (see for example [3, 5, 9, 25]). T is said to be *asymptotically nonexpansive* (see for example [3, 9, 12, 15, 17, 33, 34]) if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||, \; \forall x, y \in C. \tag{1.2}$$

It is well known (see for example [12, 15]) that the class of nonexpansive mappings is a proper subclass of the class of asymptotically nonexpansive mappings. The following example is also a simple example of an asymptotically nonexpansive mapping in a finite dimensional real Hilbert space which is not nonexpansive.

Example 1.1 Let $\Re$ denote the reals with the usual norm and define $T : \Re \to \Re$ by

$$Tx = \begin{cases} -3x, \; x \in (-\infty, 0] \\ 0, \; x \in (0, \infty) \end{cases}.$$

Then $\forall x, y \in (-\infty, 0]$, we obtain $|Tx - Ty|^2 = 9|x - y|^2$, $|x - Tx - (y - Ty)|^2 = 16|x - y|^2$, and hence

$$|Tx - Ty|^2 = 9|x - y|^2 = |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2.$$

Observe also that $\forall x, y \in (0, \infty)$ we have

$$|Tx - Ty|^2 = 0 \leq |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2.$$

Furthermore, for all $x \in (-\infty, 0]$ and $y \in (0, \infty)$ we have $|Tx - Ty|^2 = 9x^2$ and

$$\begin{aligned} |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2 &= |x - y|^2 + \frac{1}{2}|4x - y|^2 \\ &= x^2 - 2xy + y^2 + 8x^2 + \frac{y^2}{2} - 4xy \\ &= 9x^2 + \frac{3y^2}{2} - 6xy \geq 9x^2 = |Tx - Ty|^2. \end{aligned}$$

Thus

$$|Tx - Ty|^2 \leq |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2, \; \forall x, y \in \Re,$$

and

$$|Tx-Ty| \leq \frac{1+\sqrt{2}}{\sqrt{2}-1}|x-y|, \ \forall x,y \in \Re.$$

Observe that for all integer $n \geq 2$ we have $T^n x = 0, \ \forall x \in \Re$. Thus for all $x,y \in \Re$, $n \geq 2$ we have

$$|T^n x - T^n y|^2 \leq |x-y|^2.$$

It follows that T is asymptotically nonexpansive with

$$k_n = \begin{cases} \frac{1+\sqrt{2}}{\sqrt{2}-1}, \ n=1, \\ 1, \ n \geq 2. \end{cases}$$

T is not nonexpansive.

T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x-y||, \ \forall x,y \in C. \tag{1.3}$$

T is said to be demiclosed at p if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which converges weakly to $x^* \in C$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx^* = p$. It is well-known (see for example [3, 9, 27]) that if C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ is an asymptotically nonexpansive mapping with a nonempty fixed point-set, $F(T)$, then $(I-T)$ is demiclosed at zero.

Let $P_C : H \to C$ denote the metric projection (the proximity map) which assigns to each point $x \in H$ the unique nearest point in C, denoted by $P_C(x)$. It is well known that $z = P_C(x)$ if and only if $\langle x-z, z-y \rangle \geq 0, \ \forall y \in C$, and that P_C is nonexpansive.

In the iterative approximation of fixed points of asymptotically nonexpansive maps, the modified averaging iterative scheme of Mann:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1; \tag{1.4}$$

and Ishikawa:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n[(1-\beta_n)x_n + \beta_n T^n x_n], \ n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are suitable sequences in $[0,1]$ have played pivotal role. These schemes were first studied by Schu ([33, 34]) in 1991 and the schemes have played pivotal roles in approximation of fixed points of maps with asymptotic type behaviours (see for example [3, 6, 7, 17, 26, 27, 28, 31, 33, 34]). However, these two iteration schemes yield only weak convergence usually obtained mostly from $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$; and require "compactness" assumption either on the operator or the domain of the operator or even both to yield strong convergence. Even for nonexpansive maps, k-strictly pseudocontractive maps and other generalizations that do not exhibit asymptotic behaviours, sometimes very strong conditions are imposed on the fixed-point set, $F(T)$ to obtain strong convergence using the usual Mann or the Ishikawa iteration process (see for example [3, 5, 9, 29, 32, 40]). For instance in [32], the author required that $F(T)$ is finite where T is a continuous pseudocontractive-type self-mapping of a nonempty convex compact subset of a Hilbert space, and in [40] the authors required that the interior of $F(T)$ is nonempty where T is a Lipschitz pseudocontractive self-mapping of a nonempty closed convex subset of a Hilbert space. Thus many other schemes have been recently studied by several authors to achieve relatively fast strong convergence with mild assumptions on the operator, its domain, its set of fixed points and other necessary components (see for example [1, 2, 8, 10, 11, 13, 14, 16, 18, 19, 20, 21, 22, 23, 24, 30, 36, 38, 39, 41]). In [35] the authors introduced a Halpern-type algorithm with both inertial and error terms for approximating fixed points of nonexpansive mappings in real Hilbert spaces. They proved the following main convergence theorem:

Theorem 1.1([35, Theorem 4.2]) Let H be a real Hilbert space and let $T : H \to H$ be a nonexpansive mapping with a nonempty fixed point set $F(T)$. Let $\{x_n\}$ be the sequence generated from arbitrary $x_0, x_1 \in H$ by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}),\ n \geq 1 \\ x_{n+1} = \alpha_n x_0 + \beta_n y_n + \gamma_n T y_n + e_n,\ n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0,1)$; $\{\varepsilon_n\}$ is a positive sequence and $\{e_n\} \subseteq H$ is a sequence of errors satisfying the conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\varepsilon_n = o(\alpha_n)$, where $\varepsilon_n = o(\alpha_n)$ means $\lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_n} = 0$.

(ii) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$ and $\liminf_{n\to\infty}\beta_n\gamma_n > 0$.

(iii) Either $\sum_{n=1}^{\infty}\|e_n\| < \infty$ or $\lim_{n\to\infty}\frac{\|e_n\|}{\alpha_n} = 0$.

(iv) $\theta \in (0,1)$, $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, x_n \neq x_{n-1} \\ \theta,\ \text{otherwise.} \end{cases}$$

Then the algorithm (1.6) converges strongly to $z = P_{F(T)}x_0$.

It is our purpose in this paper to consider a modified averaging Halpern-type algorithm with both inertial and error terms suitable for a class of asymptotically nonexpansive maps. Our strong convergence theorems extend the corresponding convergence theorems of [35] for nonexpansive maps to the much more general class of asymptotically nonexpansive maps.

2. PRELIMINARIES

We shall need the following results:

Lemma 2.1([3, 9, 35, 37]) Let H be a real Hilbert space. Then, the following well-known results hold:

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2,\ \forall x,y \in H,$

(ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle,\ \forall x,y \in H,$

(iii) $\|\alpha x + \beta y\|^2 = \alpha(\alpha+\beta)\|x\|^2 + \beta(\alpha+\beta)\|y\|^2 - \alpha\beta\|x-y\|^2,\ \forall x,y \in H.$

Lemma 2.2([3, 9, 27]) Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \longrightarrow C$ be a an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0. i.e, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.

Lemma 2.3([18]) Let $\{\Gamma_n\}$ be a sequence of real numbers. Assume $\{\Gamma_n\}$ does not decrease at infinity, that is , there exists at least a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_k} \le \Gamma_{n_k+1}$ for all $k \ge 0$. For every $n \ge n_0$, define an integer sequence $\{\tau(n)\}$ as

$$\tau(n) = \max\{k \le n : \Gamma_k \le \Gamma_{k+1}\}.$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$, and for all $n \ge n_0$,

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \le \Gamma_{\tau(n)+1}.$$

3. MAIN RESULTS

We begin with the following important lemma which will play crucial role in the proof of our convergence results.

Lemma 3.1 Let $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{e_n\}_{n=1}^{\infty} \subset \Re^+ = [0,\infty)$, $\{b_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{d_n\}_{n=1}^{\infty} \subset \Re$ be sequences such that

$$a_{n+1} \leq [1-b_n+c_n]a_n + d_n + e_n,\ n \geq 1. \tag{3.1}$$

Let $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$. Then we have the following results:

(i) If $d_n \leq Mb_n$ for some $M > 0$, then $\{a_n\}_{n=1}^{\infty}$ is bounded.

(ii) If $\lim\limits_{n\to\infty} b_n = 0$; $\sum_{n=1}^{\infty} b_n = \infty$, and $\limsup\limits_{n\to\infty} \frac{d_n}{b_n} \leq 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Suppose (i) holds, then

$$\begin{aligned}
a_{n+1} &\leq [1-b_n+c_n]a_n + d_n + e_n \\
&\leq [1-b_n+c_n]a_n + Mb_n + e_n \\
&\leq [1-b_n+c_n][(1-b_{n-1}+c_{n-1})a_{n-1} + Mb_{n-1} + e_{n-1}] + Mb_n + e_n \\
&\leq [1-b_n+c_n][1-b_{n-1}+c_{n-1}]a_{n-1} + M[(1-b_n+c_n)b_{n-1} + b_n] \\
&\quad +(1-b_n+c_n)e_{n-1} + e_n \\
&= [1-b_n+c_n][1-b_{n-1}+c_{n-1}]a_{n-1} \\
&\quad +M[1-(1-b_n+c_n)(1-b_{n-1}+c_{n-1}) + (1-b_n+c_n)c_{n-1} + c_n] \\
&\quad +(1-b_n+c_n)e_{n-1} + e_n \\
&\leq [1-b_n+c_n][1-b_{n-1}+c_{n-1}]\big[(1-b_{n-2}+c_{n-2})a_{n-2} \\
&\quad +Mb_{n-2} + e_{n-2}\big] \\
&\quad +M[1-(1-b_n+c_n)(1-b_{n-1}+c_{n-1}) + (1-b_n+c_n)c_{n-1} + c_n] \\
&\quad +(1-b_n+c_n)e_{n-1} + e_n \\
&= [1-b_n+c_n][1-b_{n-1}+c_{n-1}][1-b_{n-2}+c_{n-2}]a_{n-2} \\
&\quad +M[(1-b_n+c_n)(1-b_{n-1}+c_{n-1})b_{n-2} \\
&\quad +1-(1-b_n+c_n)(1-b_{n-1}+c_{n-1}) + (1-b_n+c_n)c_{n-1} + c_n] \\
&\quad +(1-b_n+c_n)(1-b_{n-1}+c_{n-1})e_{n-2}
\end{aligned}$$

$$
\begin{aligned}
& +(1-b_n+c_n)e_{n-1}+e_n \\
= \; & [1-b_n+c_n][1-b_{n-1}+c_{n-1}][1-b_{n-2}+c_{n-2}]a_{n-2} \\
& +M[1-(1-b_n+c_n)(1-b_{n-1}+c_{n-1})(1-b_{n-2}+c_{n-2}) \\
& +(1-b_n+c_n)(1-b_{n-1}+c_{n-1})c_{n-2}+(1-b_n+c_n)c_{n-1}+c_n] \\
& +(1-b_n+c_n)(1-b_{n-1}+c_{n-1})e_{n-2} \\
& +(1-b_n+c_n)e_{n-1}+e_n \\
\vdots & \\
\leq \; & \prod_{j=1}^{n}(1-b_j+c_j)a_1+M(1-\prod_{j=1}^{n}(1-b_j+c_j)) \\
& +M\prod_{j=1}^{n}(1+c_j)[\sum_{k=1}^{n}c_k]+\prod_{j=1}^{n}(1+c_j)[\sum_{k=1}^{n}e_k] \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
\leq \; & \prod_{j=1}^{n}(1+c_j)a_1+M(1-\prod_{j=1}^{n}(1-b_j+c_j)) \\
& +\prod_{j=1}^{n}(1+c_j)[M\sum_{k=1}^{n}c_k]+\prod_{j=1}^{n}(1+c_j)[\sum_{k=1}^{n}e_k]. \qquad (3.3)
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}c_n<\infty$, then $\prod_{j=1}^{\infty}(1+c_j)<\infty$. Also $\sum_{n=1}^{\infty}e_n<\infty$ and hence it follows from (3.3) that $\{a_n\}$ is bounded.

Suppose (ii) holds. Let $\varepsilon>0$ be arbitrary and let N be a positive integer such that:

$$
d_n\leq \varepsilon b_n,\ \forall n\geq N;\ \sum_{n=N}^{\infty}c_n<\frac{\varepsilon}{M};\ \sum_{n=N}^{\infty}e_n<\varepsilon,
$$

then it follows from (3.2) that

$$
\begin{aligned}
a_{n+1} \; \leq \; & \prod_{j=N}^{n}(1-b_j+c_j)a_N+\varepsilon(1-\prod_{j=N}^{n}(1-b_j+c_j))+\prod_{j=N}^{n}(1+c_j)[M\sum_{k=N}^{n}c_k+\sum_{k=N}^{n}e_k] \\
\leq \; & \prod_{j=N}^{n}(1-b_j+c_j)a_N+\varepsilon(1-\prod_{j=N}^{n}(1-b_j+c_j))+2\varepsilon\prod_{j=N}^{n}(1+c_j). \qquad (3.4)
\end{aligned}
$$

Observe that

$$\begin{aligned}\prod_{j=N}^{n}(1-b_j+c_j) &\leq \prod_{j=N}^{n}[(1-b_j)(1+Dc_j)], (\text{ since } \frac{1}{(1-b_n)} \leq D \text{ for some } D>0)\\ &\leq \exp(D\sum_{j=N}^{n}c_j)\exp(-\sum_{j=N}^{n}b_j) \to 0 \text{ as } n\to\infty.\end{aligned}$$

This together with (3.4) yields

$$\limsup_{n\to\infty} a_n \leq 3\varepsilon.$$

Hence $\lim_{n\to\infty} a_n = 0$. □

Theorem 3.1 Let H be a real Hilbert space and let $T : H \to H$ be an asymptotically nonexpansive mapping with a nonempty fixed point set $F(T)$ and with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Let $\{x_n\}$ be the sequence generated from arbitrary $x_0, x_1 \in H$ by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}),\ n \geq 1 \\ x_{n+1} = \alpha_n x_0 + \beta_n y_n + \gamma_n T^n y_n + e_n,\ n \geq 1, \end{cases} \tag{3.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0,1)$; $\{\varepsilon_n\}$ is a positive sequence and $\{e_n\} \subseteq H$ is a sequence of errors satisfying the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\varepsilon_n = o(\alpha_n)$, where $\varepsilon_n = o(\alpha_n)$ means $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

(ii) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$ and $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

(iii) Either $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

(iv) $\theta \in (0,1)$, $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, x_n \neq x_{n-1} \\ \theta,\ \text{otherwise.} \end{cases}$$

(vi) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$.

Then the algorithm (3.5) converges strongly to $z = P_{F(T)}x_0$.

Proof. Let $p \in F(T)$. Then

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n x_0+\beta_n y_n+\gamma_n T^n y_n+e_n-p\| \\
&= \|\alpha_n(x_0-p)+\beta_n(y_n-p)+\gamma_n(T^n y_n-p)+e_n\| \\
&\le \alpha_n\|x_0-p\|+\beta_n\|y_n-p\|+\gamma_n\|T^n y_n-p\|+\|e_n\| \\
&\le \alpha_n\|x_0-p\|+\beta_n\|y_n-p\|+\gamma_n k_n\|y_n-p\|+\|e_n\| \\
&= \alpha_n\|x_0-p\|+(\beta_n+\gamma_n)\|y_n-p\|+\gamma_n(k_n-1)\|y_n-p\|+\|e_n\| \\
&= [1-\alpha_n+\gamma_n(k_n-1)]\|y_n-p\|+\alpha_n\|x_0-p\|+\|e_n\| \\
&\le [1-\alpha_n+\gamma_n(k_n-1)][\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|]+\alpha_n\|x_0-p\|+\|e_n\| \\
&= [1-\alpha_n+\gamma_n(k_n-1)]\|x_n-p\|+[1-\alpha_n+\gamma_n(k_n-1)]\theta_n\|x_n-x_{n-1}\|] \\
&\quad +\alpha_n\|x_0-p\|+\|e_n\| \\
&\le [1-\alpha_n+\gamma_n(k_n-1)]\|x_n-p\|+\alpha_n[(1-\alpha_n+\gamma_n(k_n-1))\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| \\
&\quad +\|x_0-p\|+\frac{\|e_n\|}{\alpha_n}] \qquad (3.6)
\end{aligned}$$

If $\lim_{n\to\infty}\frac{\|e_n\|}{\alpha_n}=0$, then there exists $D>0$ such that

$$[(1-\alpha_n+\gamma_n(k_n-1))\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|x_0-p\|+\frac{\|e_n\|}{\alpha_n}]\le D,\ \forall n.$$

Thus we obtain from (3.6) that

$$\|x_{n+1}-p\|\le[1-\alpha_n+\gamma_n(k_n-1)]\|x_n-p\|+M_1\alpha_n,$$

and it follows from Lemma 3.1 that $\{x_n\}$ is bounded.

If $\sum_{n=1}^{\infty}\|e_n\|<\infty$, from (3.6) we have that

$$\begin{aligned}
\|x_{n+1}-p\| &\le [1-\alpha_n+\gamma_n(k_n-1)]\|x_n-p\| \\
&\quad +\alpha_n[1-\alpha_n+\gamma_n(k_n-1)]\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|x_0-p\|+\|e_n\| \\
&\le [1-\alpha_n+\gamma_n(k_n-1)]\|x_n-p\|+M\alpha_n+\|e_n\|,\ \forall n \text{ and for some } M>0. \qquad (3.7)
\end{aligned}$$

It follows from Lemma 3.1 and inequality (3.7) that $\{x_n\}$ is bounded. Using Lemma 2.1 and the fact that $\theta \in (0,1)$ yields:

$$\begin{aligned}
\|y_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - x_{n-1}, x_n - p\rangle \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + \theta_n[\|x_n - p\|^2 + \|x_n - x_{n-1}\|^2 - \|x_{n-1} - p\|^2] \\
&= \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 + \theta_n[\|x_n - p\|^2 - \|x_{n-1} - p\|^2]. \qquad (3.8)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n x_0 + \beta_n y_n + \gamma_n T^n y_n + e_n - p\|^2 \\
&= \|\alpha_n(x_0 - p) + \beta_n(y_n - p) + \gamma_n(T^n y_n - p) + e_n\|^2 \\
&= \|\alpha_n(x_0 - p + \frac{e_n}{\alpha_n}) + \beta_n(y_n - p) + \gamma_n(T^n y_n - p)\|^2 \\
&\leq \|\beta_n(y_n - p) + \gamma_n(T^n y_n - p)\|^2 + 2\langle \alpha_n(x_0 - p + \frac{e_n}{\alpha_n}), x_{n+1} - p\rangle \\
&= \beta_n(\beta_n + \gamma_n)\|y_n - p\|^2 + \gamma_n(\beta_n + \gamma_n)\|T^n y_n - p)\|^2 - \beta_n\gamma_n\|y_n - T^n y_n\|^2 \\
&\quad +2\alpha_n\langle (x_0 - p + \frac{e_n}{\alpha_n}), x_{n+1} - p\rangle \\
&\leq \beta_n(\beta_n + \gamma_n)\|y_n - p\|^2 + \gamma_n(\beta_n + \gamma_n)k_n\|y_n - p\|^2 - \beta_n\gamma_n\|y_n - T^n y_n\|^2 \\
&\quad +2\alpha_n\langle (x_0 - p + \frac{e_n}{\alpha_n}), x_{n+1} - p\rangle \\
&= (\beta_n + \gamma_n)^2\|y_n - p\|^2 + \gamma_n(\beta_n + \gamma_n)(k_n - 1)\|y_n - p)\|^2 - \beta_n\gamma_n\|y_n - T^n y_n\|^2 \\
&\quad +2\alpha_n\langle (x_0 - p + \frac{e_n}{\alpha_n}), x_{n+1} - p\rangle \\
&\leq (\beta_n + \gamma_n)\|y_n - p\|^2 + \gamma_n(\beta_n + \gamma_n)(k_n - 1)\|y_n - p)\|^2 - \beta_n\gamma_n\|y_n - T^n y_n\|^2 \\
&\quad +2\alpha_n\langle (x_0 - p + \frac{e_n}{\alpha_n}), x_{n+1} - p\rangle \\
&= (1 - \alpha_n)\|y_n - p\|^2 + \gamma_n(1 - \alpha_n)(k_n - 1)\|y_n - p)\|^2 - \beta_n\gamma_n\|y_n - T^n y_n\|^2 \\
&\quad +2\alpha_n\langle (x_0 - p + \frac{e_n}{\alpha_n}), x_{n+1} - p\rangle \\
&= (1 - \alpha_n)[1 + \gamma_n(k_n - 1)]\|y_n - p\|^2 - \beta_n\gamma_n\|y_n - T^n y_n\|^2 \\
&\quad +2\alpha_n\langle (x_0 - p + \frac{e_n}{\alpha_n}), x_{n+1} - p\rangle. \qquad (3.9)
\end{aligned}$$

From (3.8) and (3.9), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)[1+\gamma_n(k_n-1)][\|x_n-p\|^2+\theta_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2)\\ &\quad +2\theta_n\|x_n-x_{n-1}\|^2]-\beta_n\gamma_n\|y_n-T^ny_n\|^2+2\alpha_n\langle(x_0-p+\frac{e_n}{\alpha_n}),x_{n+1}-p\rangle\\ &= (1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-p\|^2\\ &\quad +(1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2)\\ &\quad -\beta_n\gamma_n\|y_n-T^ny_n\|^2+2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-x_{n-1}\|^2\\ &\quad +2\alpha_n\langle(x_0-p+\frac{e_n}{\alpha_n}),x_{n+1}-p\rangle \qquad (3.10)\end{aligned}$$

Setting $\Gamma_n=\|x_n-p\|^2\ \forall\ n\ \geq\ 1$ in (3.10) gives

$$\begin{aligned}\Gamma_{n+1} &\leq (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n+(1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n(\Gamma_n-\Gamma_{n-1})\\ &\quad -\beta_n\gamma_n\|y_n-T^ny_n\|^2+2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-x_{n-1}\|^2\\ &\quad +2\alpha_n\langle(x_0-p+\frac{e_n}{\alpha_n}),x_{n+1}-p\rangle. \qquad (3.11)\end{aligned}$$

We now consider the following two cases:

Case I: Suppose $\exists$ an $n_0\in\mathbb{N}$ such that $\Gamma_n\geq\Gamma_{n+1},\ \forall n\geq n_0$, then $\lim_{n\to\infty}\Gamma_n$ exists and it follows from (3.11) that

$$\begin{aligned}\beta_n\gamma_n\|y_n-T^ny_n\|^2 &\leq \Gamma_n-\Gamma_{n+1}+(1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n(\Gamma_n-\Gamma_{n-1})\\ &\quad +2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-x_{n-1}\|^2\\ &\quad +2\alpha_n\langle(x_0-p+\frac{e_n}{\alpha_n}),x_{n+1}-p\rangle\\ &= \Gamma_n-\Gamma_{n+1}+\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)](\Gamma_n-\Gamma_{n-1})\\ &\quad +2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-x_{n-1}\|^2+2\alpha_n\langle(x_0-p,x_{n+1}-p\rangle\\ &\quad +2\langle e_n,x_{n+1}-p\rangle. \qquad (3.12)\end{aligned}$$

It now follows from (3.12) that

$$\lim_{n\to\infty}\beta_n\gamma_n\|T^ny_n-y_n\|=0. \qquad (3.13)$$

Since $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, it follows from (3.13) that

$$\lim_{n\to\infty} \|T^n y_n - y_n\| = 0. \tag{3.14}$$

Observe that $\|y_n - x_n\| = \theta_n\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Furthermore,

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|\alpha_n x_0 + \beta_n y_n + \gamma_n T^n y_n + e_n - y_n\| \\
&= \|\alpha_n(x_0 - y_n + \frac{e_n}{\alpha_n}) + \gamma_n(T^n y_n - y_n)\| \\
&\le \alpha_n\|x_0 - y_n + \frac{e_n}{\alpha_n}\| + \gamma_n\|T^n y_n - y_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\|y_n - y_{n-1}\| \le \|y_n - x_n\| + \|x_n - y_{n-1}\| \to 0$ as $n \to \infty$, and it follows that

$$\begin{aligned}
\|y_n - Ty_n\| &\le \|y_n - T^n y_n\| + \|T^n y_n - Ty_n\| \\
&\le \|y_n - T^n y_n\| + L\|T^{n-1} y_n - y_n\| \\
&\le \|y_n - T^n y_n\| + L\|T^{n-1} y_n - T^{n-1} y_{n-1}\| + L\|T^{n-1} y_{n-1} - y_n\| \\
&\le \|y_n - T^n y_n\| + L(1+L)\|y_n - y_{n-1}\| + L\|T^{n-1} y_{n-1} - y_{n-1}\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since $\{x_n\}_{n=1}^{\infty}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \rightharpoonup q \in H$ and

$$\limsup_{n\to\infty} \langle x_0 - p, x_n - p\rangle = \limsup_{k\to\infty} \langle x_0 - p, x_{n_k} - p\rangle = \langle x_0 - p, q - p\rangle. \tag{3.15}$$

Using $y_n = x_n + \theta_n(x_n - x_{n-1})$ gives $\|y_n - x_n\| = \theta_n\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. (3.16)

Since $x_{n_k} \rightharpoonup q$ as $k \to \infty$, then $y_{n_k} \rightharpoonup q$ as $k \to \infty$, and it follows from the demiclosedness property of $(I - T)$ at zero that $q \in F(T)$. Furthermore, from $p = P_{F(T)}x_0$ we obtain

$$\limsup_{n\to\infty} \langle x_0 - p, x_n - p\rangle \le 0. \tag{3.17}$$

From (3.10), we have

$$\begin{aligned}
\Gamma_{n+1} &\le (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n + (1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n(\Gamma_n - \Gamma_{n-1}) \\
&\quad + 2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_0 - p + \frac{e_n}{\alpha_n}, x_{n+1} - p\rangle \\
&= (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n + (1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\
&\quad + 2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_0 - p + \frac{e_n}{\alpha_n}, x_{n+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n + (1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n[(\|x_n-p\|-\|x_{n-1}-p\|) \\
&\quad \times(\|x_n-p\|+\|x_{n-1}-p\|)] + 2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-x_{n-1}\|^2 \\
&\quad +2\alpha_n\langle x_0-p+\frac{e_n}{\alpha_n}, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n \\
&\quad +(1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n[(\|x_n-p+p-x_{n-1}\|)(\|x_n-p\|+\|x_{n-1}-p\|)] \\
&\quad +2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-x_{n-1}\|^2 + 2\alpha_n\langle x_0-p+\frac{e_n}{\alpha_n}, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n + (1-\alpha_n)[1+\gamma_n(k_n-1)]\theta_n\|x_n-x_{n-1}\|(\sqrt{\Gamma}_n+\sqrt{\Gamma}_{n-1}) \\
&\quad +2\theta_n(1-\alpha_n)[1+\gamma_n(k_n-1)]\|x_n-x_{n-1}\|^2 + 2\alpha_n\langle x_0-p+\frac{e_n}{\alpha_n}, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n + \theta_n\|x_n-x_{n-1}\|K_1 + 2\alpha_n\langle x_0-p+\frac{e_n}{\alpha_n}, x_{n+1}-p\rangle, \qquad (3.18)
\end{aligned}
$$

where $K_1 = \sup_{n\geq 1}\{(1-\alpha_n)[1+\gamma_n(k_n-1)][\sqrt{\Gamma}_n+\sqrt{\Gamma}_{n-1}+2\|x_n-x_{n-1}\|]\}$.
Since $\sum_{n=1}^{\infty}\|e_n\| < \infty$, then (3.18) gives

$$
\Gamma_{n+1} \leq (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n + \theta_n\|x_n-x_{n-1}\|K_1 + \alpha_n u_n + g_n \qquad (3.19)
$$

with $u_n := 2\langle x_0-p, x_{n+1}-p\rangle$, $g_n := K_2\|e_n\|,\ K_2 > 0$. Using Lemma 3.1 and conditions (i) and (iii) of the Theorem we obtain $\Gamma_n = \|x_n-p\| \to 0$ as $n\to\infty$.
From the fact that $\lim_{n\to\infty}\frac{\|e_n\|}{\alpha_n} = 0$, (3.18) gives

$$
\Gamma_{n+1} \leq (1-\alpha_n)[1+\gamma_n(k_n-1)]\Gamma_n + \theta_n\|x_n-x_{n-1}\|K_1 + 2\alpha_n u_n \qquad (3.20)
$$

$u_n = \langle x_0-p+\frac{e_n}{\alpha_n}, x_{n+1}-p\rangle$.

Observe from (3.18) that $\limsup_{n\to\infty} u_n \leq 0$. Hence by Lemma 2.3 and conditions of Theorem 3.1 we obtain $x_n \to p$ as $n\to\infty$.

Case II: Assume that $\{\|x_n-p\|\}$ is not a monotone decreasing sequence.
Following the method of proof in ([18, 35]) we set $\Gamma_n = \|x_n-p\|^2$ and let $\tau : N \longrightarrow N$ be a mapping for all $n \geq n_0$ for some n_0 large enough by
$\tau(n) = max\{k\in N : k\leq n,\ \Gamma_k \leq \Gamma_{k+1}\}$.
Clearly, $\{\tau(n)\}$ is a non decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and

$$
0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}\ \forall n \geq n_0. \qquad (3.21)
$$

With similar arguments as in (3.19), we easily obtain

$$\lim_{n\to\infty}\|T^{\tau(n)}y_{\tau(n)}-y_{\tau(n)}\|=0,\ \lim_{n\to\infty}\|Ty_{\tau(n)}-y_{\tau(n)}\|=0,\ \lim_{n\to\infty}\|y_{\tau(n)}-x_{\tau(n)}\|=0,$$

$$\lim_{n\to\infty}\|T^n y_{\tau(n)}-x_{\tau(n)}\|=0. \tag{3.22}$$

Using the boundedness of $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and assumptions and conditions of Theorem 3.1 we have that

$$\begin{aligned}\|x_{\tau(n+1)}-x_{\tau(n)}\| \leq\ & \alpha_{\tau(n)}\|x_0-x_{\tau(n)}\|+\beta_{\tau(n)}\|y_{\tau(n)}-x_{\tau(n)}\| \\ & +\gamma_{\tau(n)}\|T^n y_{\tau(n)}-x_{\tau(n)}\|+\|e_{\tau(n)}\|\to 0 \text{ as } n\to\infty. \end{aligned}\tag{3.23}$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence $\{x_{\tau(n_k)}\}$ of $\{x_{\tau(n)}\}$ such that $\{x_{\tau(n_k)}\}$ converges weakly to $q\in F(T)$. Similar to Case I above, it can be shown that $\limsup_{n\to\infty}\langle x_0-p,x_{\tau(n)+1}-p\rangle\leq 0$. Using (3.18) we have that

$$\begin{aligned}\alpha_{\tau(n)}\Gamma_{\tau(n)} \leq\ & (1-\alpha_{\tau(n)})\gamma_{\tau(n)}(k_{\tau(n)}-1)\Gamma_{\tau(n)}+\theta_{\tau(n)}\|x_{\tau(n)}-x_{\tau(n)-1}\|K_1 \\ & +2\alpha_{\tau(n)}\langle x_0-p+\frac{e_{\tau(n)}}{\alpha_{\tau(n)}},x_{\tau(n)+1}-p\rangle. \end{aligned}\tag{3.24}$$

Thus

$$\begin{aligned}\Gamma_{\tau(n)} \leq\ & (1-\alpha_{\tau(n)})\gamma_{\tau(n)}\frac{(k_{\tau(n)}-1)}{\alpha_{\tau(n)}}\Gamma_{\tau(n)}+\frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)}-x_{\tau(n)-1}\|K_1 \\ & +2\langle x_0-p+\frac{e_{\tau(n)}}{\alpha_{\tau(n)}},x_{\tau(n)+1}-p\rangle. \end{aligned}\tag{3.25}$$

From (3.25) we obtain $\lim_{n\to\infty}\|x_{\tau(n)}-p\|=\lim_{n\to\infty}\Gamma_{\tau(n)}=0$ and it follows that $\lim_{n\to\infty}\|x_{\tau(n)+1}-p\|=0$. Clearly $\Gamma_n\leq\Gamma_{\tau(n)+1}$ for $n\geq n_0$. We note also that $\tau(n)\leq n$ for $n\geq n_0$, and consider the three cases namely: $\tau_n=n,\tau_n=n-1$, and $\tau(n)<n-1$. Obviously $\Gamma_n\leq\Gamma_{\tau(n)+1}$ for $\tau(n)=n$ and $\tau(n)=n-1$. For $\tau(n)\leq n-2$ and for any integer $n\geq n_0$, it follows from the definition of $\tau(n)$ that $\Gamma_i\geq\Gamma_{i+1}$, for $\tau(n)+1\leq i\leq n-1$. Hence $\Gamma_{\tau(n)+1}\geq\Gamma_{\tau(n)+2}\geq\cdots\geq\Gamma_{n-1}\geq\Gamma_n$. Thus for all sufficiently large n we obtain $0\leq\Gamma_n\leq\Gamma_{\tau(n)+1}$, from which it follows that $\lim_{n\to\infty}\Gamma_n=0$. Thus $\{x_n\}_{n=1}^{\infty}$ converges strongly to p. □

For arbitrary $\beta \in (0,1)$, we can consider the following algorithm:

Algorithm 3.2 With $\{\alpha_n\}, \{\varepsilon_n\}, \{e_n\}, \theta, \{\theta_n\}$ and $\{\bar{\theta}_n\}$ as in Theorem 3.1, let $\{x_n\}$ be generated from arbitrary starting points $x_0, x_1 \in H$ by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)T_{(\beta,n)}(x_n + \theta_n(x_n - x_{n-1})) + e_n, \tag{3.26}$$

where $T_{(\beta,n)} = (1-\beta)I + \beta T^n$.

We obtain the following Corollary:

Corollary 3.1 Let H be a real Hilbert space and let $T : H \to H$ be an asymptotically nonexpansive mapping with a nonempty fixed point set $F(T)$ and with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence generated from arbitrary $x_0, x_1 \in H$ by Algorithm 3.26. Then the algorithm (3.26) converges strongly to $z = P_{F(T)}x_0$.

Proof. With $y_n = x_n + \theta_n(x_n - x_{n-1})$, we obtain

$$\begin{aligned} x_{n+1} &= \alpha_n x_0 + (1-\alpha_n)T_{(\beta,n)}y_n + e_n, \\ &= \alpha_n x_0 + (1-\alpha_n)(1-\beta)y_n + (1-\alpha_n)\beta T^n y_n + e_n \\ &= \alpha_n x_0 + \beta_n y_n + \gamma_n T^n y_n + e_n, \end{aligned}$$

where $\beta_n = (1-\alpha_n)(1-\beta)$ and $\gamma_n = (1-\alpha_n)\beta$. Since $\alpha_n + \beta_n + \gamma_n = 1$ and $\liminf_{n\to\infty} \beta_n\gamma_n = \beta(1-\beta) > 0$, then the results follows from Theorem 3.1. □

4. NUMERICAL EXAMPLES

Example 4.1 Let X denote the real Hilbert space ℓ_2 and B the unit closed ball in X. Define $T : B \to B$ by

$$Tx = T(x_1, x_2, x_3, \ldots) = (0, x_1^2, A_2x_2, A_3x_3, \ldots),$$

where A_i is a sequence of numbers such that $0 < A_i < 1$ and $\prod_{i=2}^{\infty} A_i = \frac{1}{2}$. Then T is Lipschitzian and $||Tx - Ty|| \le 2||x-y||$, $\forall x, y \in B$. Furthermore, $||T^n x - T^n y|| \le 2\prod_{i=2}^{n} A_i ||x-y|| = k_n||x-y||$, for $n = 2, 3, \ldots$. Since $\lim_{n\to\infty} k_n = 1$, we have that T is asymptotically nonexpansive. If $x = (\frac{3}{4}, 0, 0, 0, \ldots)$ and $y = (\frac{1}{2}, 0, 0, 0, \ldots)$, then $||Tx - Ty|| = \frac{5}{16} > \frac{1}{4} = ||x-y||$. This example has served as standard example for various works on asymptotically nonexpansive maps and its generalizations.

In particular we can take $A_i = \frac{i^2-1}{i^2}$, $i > 1$. Then $k_n = 2\prod_{i=2}^{n} A_i$, and we further choose $\alpha_n = \sqrt{k_n - 1} + \frac{1}{n+1}$, $\beta_n = \gamma_n = \frac{1}{2}(\frac{n}{n+1} - \sqrt{k_n - 1})$, $\varepsilon_n = \frac{1}{(n+1)^2}$, $e_n = \frac{c}{(n+1)^2}$, where $c \in H$ is any fixed vector. Choosing $x_1 = (\frac{1}{2}, 0, 0, 0, \ldots)$, $x_2 = (\frac{2}{5}, 0, 0, 0, \ldots)$, $\theta = 0.5$, and $c = (\frac{1}{3}, 0, 0, 0, \ldots)$ in H, algorithm (3.5) and algorithm (3.26) with $\beta = 0.5$ and 0.9 converge to zero as shown in Figure 1 and table 1 below:

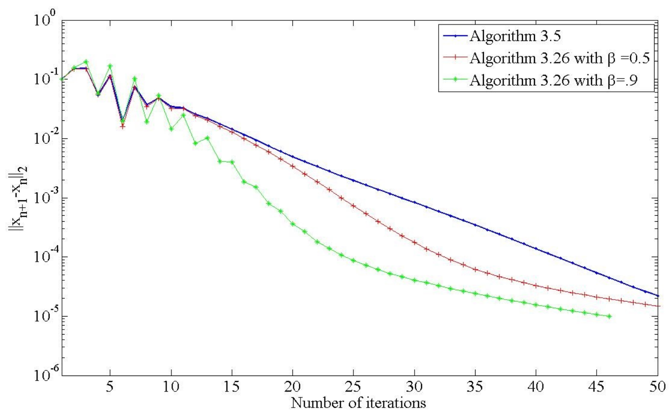

FIGURE 1. Graph showing the convergence of iterative algorithms

Number of Iterates	Algorithm 3.5	Algorithm 3.26 with $\beta = 0.5$	Algorithm 3.26 with $\beta = 0.9$
2	0.1	0.1	0.1
3	0.149702	0.1497	0.1558
4	0.151111	0.1485	0.1957
5	0.0538255	0.0553	0.0578
⋮	⋮	⋮	⋮
29	0.00116518	0.0003	0.0001
30	0.000983747	0.0002	0
31	0.000830058	0.0002	0
32	0.000699556	0.0001	0
⋮	⋮	⋮	⋮
57	0.000010828	0	0
58	0.000010008	0	0
Elapsed time	0.155271 seconds	0.471473 seconds	0.430826 seconds.

TABLE 1. Showing the numerical values of the iterates for the two algorithms

Example 4.2 Let $\Re$ denote the reals with the usual norm and define $T:\Re\to\Re$ by

$$Tx = \begin{cases} -3x, \ x\in(-\infty,0] \\ 0, \ x\in(0,\infty). \end{cases}$$

Then

$$k_n = \begin{cases} \frac{1+\sqrt{2}}{\sqrt{2}-1}, \ n=1, \\ 1, \ n\geq 2, \end{cases}$$

and we can choose $\alpha_n=\sqrt{k_n-1}+\frac{1}{n+1}$, $\beta_n=\gamma_n=\frac{1}{2}(\frac{n}{n+1}-\sqrt{k_n-1})$, $\varepsilon_n=\frac{1}{(n+1)^2}$, $e_n=\frac{c}{(n+1)^2}$, where $c\in H$ is any fixed vector. Choosing $x_1=0.6$, $x_2=2.5$, $\theta=0.5$, and $c=0.9$ in H, then the algorithm (3.5) and algorithm (3.26) with $\beta=0.5$ and 0.9 converge to zero as shown in Figure 2 and table 2:

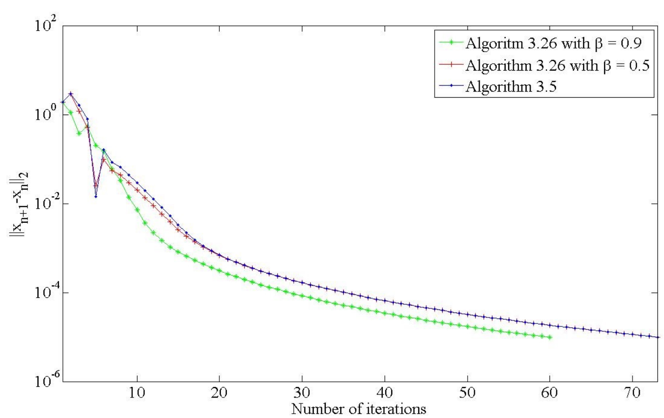

FIGURE 2. Graph showing the convergence of iterative algorithms

Number of Iterates	Algorithm 3.5	Algorithm 3.26 with $\beta = 0.5$	Algorithm 3.26 with $\beta = 0.9$
1	1.9	1.9	1.9
2	2.8844	2.8844	1.1021
3	1.611	1.1711	0.375
4	0.7984	0.5198	0.5427
5	0.0142	0.0252	0.2027
⋮	⋮	⋮	⋮
27	0.0002	0.0002	0.0001
28	0.0002	0.0002	0.0001
29	0.0002	0.0002	0
⋮	⋮	⋮	⋮
33	0.0001	0.0001	0
34	0.0001	0.0001	0
35	0.0001	0.0001	0
Elapsed Time	0.169970 seconds	0.496013 seconds	0.483270 seconds

TABLE 2. Showing the numerical values of the iterates for the two algorithms

5. CONCLUSION

Two Halpern-type averaging algorithm (algorithms 3.5 and 3.26) with both inertial and error terms were introduced and studied in this paper. Both algorithms were employed in the approximation of fixed points of asymptotically nonexpansive maps in real Hilbert spaces. Asymptotically nonexpansive maps are more general than nonexpansive maps and as such the results presented here generalize and extend some existing results in this area. Strong convergence results were obtained for both algorithms. The validity of the algorithms is illustrated using numerical examples in both finite and infinite dimensional real Hilbert spaces. From the numerical experiment, algorithm 3.5 converges faster than algorithm 3.26. Although algorithm 3.26 has fewer number of iterations, it took more time than algorithm 3.5 to complete the iterative process. In practical application of the results to real world problems, it is advisable to implement algorithm 3.5.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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