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COMPOUND SYNCHRONIZATION USING DISTURBANCE OBSERVER BASED ADAPTIVE SLIDING MODE CONTROL TECHNIQUE

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Abstract. This manuscript addresses a methodology for investigating compound synchronization in a class of commensurate FO chaotic Genesio-Tesi system using DOB adaptive sliding mode control technique among fractional order chaotic systems with unknown bounded disturbances. The unknown disturbances are estimated using the nonlinear fractional order disturbance observer. Sliding mode technique has been employed by considering a simple sliding surface among four identical fractional order chaotic systems to achieve the desired synchronization which is further based on Lyapunov stability theory. The obtained results have been compared with prior published literature to realize the robustness of the proposed strategy. Finally, some numerical results using MATLAB are illustrated for visualizing the effectiveness and the correctness of the developed approach on the considered system in the presence of external disturbances.

Keywords: compound synchronization; adaptive sliding mode control; unknown bounded disturbances; fractional order disturbance observer.

2010 AMS Subject Classification: 37D45, 37N10, 37E99, 37F99.

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1. INTRODUCTION:

Interestingly, chaos theory has become the most significant research field encompassing the biological, chemical, physical, meta-physical aspects of science. It involves the qualitative investigations of immensely complex non linear systems found in nature. Chaotic systems are dynamical systems that are extremely sensitive to small perturbation in the initial conditions and system parameters. A small change in the chaotic system can lead to broadly diverging consequences.Recently, several researches have introduced new chaotic fractional order chaotic dynamical system involving fractional differential equations. Numerous systems are famously known to exhibit FO dynamics, for example, dielectric polarization, visco-elastic system, electromagnetic waves, electrode-electrolyte polarization. Primarily, fractional calculus [18] [6] [1] [7] is an exciting field of mathematics dealing with derivatives and intgration of fractional order. The evolutionary record of FC dates back to Leibnitz, a great German mathematician's research article in his findings to L'Hospital dated September 30, 1695.

In recent years, the synchronization [17] [15] among FO chaotic system [9] [13] [19] [24] has been widely explored as a potential emerging research field. Several researchers have recently proposed very effective and robust techniques of control and synchronization [10] [11] [16] of chaotic models.Fundamentally, synchronization is an important phenomenon which occurs when atleast two identical or non-identical chaotic systems are adjusted in a way such that both exhibit the similar behavior owing to pairing to attain stability. Several techniques for synchronization and control of chaos have been introduced and studied.It is observed that in nearly all the previously published literature, the authors have discussed FO chaotic system without considering unknown parameters and uncertainties whereas in real scenario these are known to perturb the system.

Essentially SMC [5] [3] [2] approach is a powerful procedure to design control inputs for both the linear and non-linear dynamical models with unknown inputs. SMC is basic in designing and has remarkable features such as quick response, good stability, tracking potential and is capable to check model uncertainties and external disturbances.Considering the robustness of SMC method we have investigated compound synchronization using disturbance based adaptive SMC technique among FO Genesio Tesi systems with unknown bounded disturbance. To attain the primal goal of compound synchronization [20] [21] [23] [22] [12] among four identical FO Genesio Tesi systems with unknown bounded disturbance, FO disturbance observer based adaptive SMC technique has been adopted. It is noted that the investigated approach may also compensate powerful disturbance signals. We also have compared over attained results concerning synchronization with the prior published results in order to demonstrate the robustness and effectiveness of our considered scheme. Finally, feasibility of the considered technique is also ensured by performing numerical simulations using MATLAB.

This paper is framed as:
Section 2 presents preliminaries which include basic concepts, definition and some results from fractional calculus that will be used throughout the paper. Section 3 deals with the description of chaotic FO Genesio-Tesi system. Section 4 contains the formulation of FODO based for the FO chaotic Genesio Tesi system including unknown external disturbance. In Section 5, we first describe an elementary SMS and further we design the sliding mode synchronization controllers depend on investigated non linear FODO. In Section 6, numerical simulations and discussions are also presented to show the effectiveness of the proposed strategy using MATLAB. Further, Section 7 consists of the comparative analysis of the proposed technique. Conclusions are given in Section 8.

2. PRELIMINARIES:

We here state a few preliminaries that will be used throughout the paper:
Definition 1: The Caputo's derivative of fractional order 'α' on function f(t) is given by:

$$cD_t^{\alpha} f(y) = \frac{1}{\gamma(n-\alpha)} \int_c^y \frac{f^n(x)}{(y-x)^{\alpha-n+1}} dx$$

where $n-1 < \alpha < n$ and $\gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is the Gamma function.

Property 1: The Caputo's fractional derivative satisfies the following condition if function g is constant

$$D^{\alpha}g(t) = 0$$

Property 2: The Caputo's fractional derivative satisfies the linear property:
$D^{\alpha}[ag_1(t) + bg_2(t)] = aD^{\alpha}g_1(t) + bD^{\alpha}g_2(t)$, where a & b are constants.

Lemma 1: Suppose $\Upsilon \in R$ be a continuously derivable function and $0 < \alpha < 1$.Then, for any time $t \geq t_0$.

$$\frac{1}{2}D^{\alpha}\Upsilon(t)^2 \leq \Upsilon(t)D^{\alpha}\Upsilon(t)$$

Lemma 2: For the equation

$$D^{\alpha}H(t) \leq -b_0H(t) + b_1$$

there exists a constant $t_1 > 0$ for which for all $t \in (t_1, \infty)$ satisfies the condition

$$H(t) \leq \frac{2b_1}{b_0}$$

where H(t) is state variable of the system,$b_0 > 0$ and $b_1 > 0$ are constants

*Assumption 1:*In this paper we assumed the the Caputo's derivative of the unknown external disturbances to be bounded i.e $|D^{q}\Upsilon_i(t)| \leq |a_i|$ where $\Upsilon_i(t)$ are unknown external disturbances and $a_i > 0$ are positive constant.

3. System Description:

Here Compound synchronization has been performed where we have taken 3 master systems and 1 slave system described as follows:

First Master System:

$$
\begin{aligned}
&D^{\alpha}u_{11}(t) = u_{12} \\
(1) \qquad &D^{\alpha}u_{12}(t) = u_{13} \\
&D^{\alpha}u_{13}(t) = -\beta_1u_{11} - \beta_2u_{12} - \beta_3u_{13} + \beta_4u_{11}^2
\end{aligned}
$$

This system shows chaotic behaviour for different set of parameter values and initial conditions.In [14] author shows the chaotic behaviour for parameter values $(\beta_1,\beta_2,\beta_3,\beta_4)$=$(1.0,1.1,0.4,1.0)$ with initial conditions $(-0.3,0.,-0.2)$ and $(-0.5,0.4,-0.3)$ and fractional order $\alpha = 0.99$.Also in [4] author describes the chaotic behaviour and dynamical properties for the set of parameter values $(\beta_1,\beta_2,\beta_3,\beta_4)$=$(6.0,2.92,1.2,1.0)$ with initial conditions $(0.7,1.3,1)$ and $(0.2,-0.3,0.1)$ for fractional order $\alpha = 0.94$. Also we have seen that in [8] the system (1) shows the chaotic behaviour for two set of parameter values $(\beta_1,\beta_2,\beta_3,\beta_4)$=$(1.0,1.1,0.15,1.0)$ for fractional order $\alpha = 0.9$ and $(\beta_1,\beta_2,\beta_3,\beta_4)$=$(1.0,1.1,-0.232,1.0)$ for fractional order $\alpha = 0.8$. The Fig.1 and Fig.2 shows the chaotic attractors for two set of parameter values.

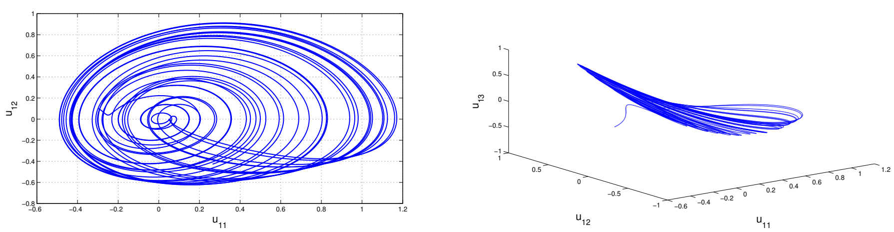

Fig.1: Chaotic attractors for parameter values $(\beta_1,\beta_2,\beta_3,\beta_4)$=$(1.0,1.1,0.4,1.0)$ with initial conditions $(-0.3,0.,-0.2)$ and $\alpha = 0.99$

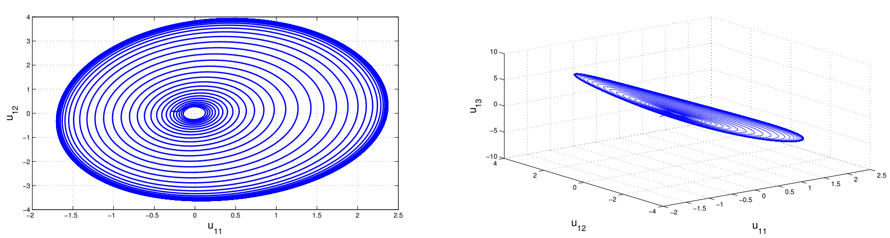

Fig.2: Chaotic attractors for parameter values $(\beta_1,\beta_2,\beta_3,\beta_4)$=$(6.0,2.92,1.2,1.0)$ for $\alpha = 0.94$

Second Master System:

$$
(2)\qquad
\begin{aligned}
D^{\alpha}u_{21}(t) &= u_{22}\\
D^{\alpha}u_{22}(t) &= u_{23}\\
D^{\alpha}u_{23}(t) &= -\beta_1 u_{21} - \beta_2 u_{22} - \beta_3 u_{23} + \beta_4 u_{21}^2
\end{aligned}
$$

Third Master System:

$$
\begin{aligned}
D^{\alpha}u_{31}(t) &= u_{32} \\
D^{\alpha}u_{32}(t) &= u_{33} \\
D^{\alpha}u_{33}(t) &= -\beta_1 u_{31} - \beta_2 u_{32} - \beta_3 u_{33} + \beta_4 u_{31}^2
\end{aligned} \tag{3}
$$

Slave System:

$$
\begin{aligned}
D^{\alpha}v_1(t) &= v_2 + cos3t + C_1 \\
D^{\alpha}v_2(t) &= v_3 + sin5t + C_2 \\
D^{\alpha}v_3(t) &= -\beta_1 v_{11} - \beta_2 v_{12} - \beta_3 v_{13} + \beta_4 v_{11}^2 + sin4t + C_3
\end{aligned} \tag{4}
$$

Here $\psi_1 = cos3t, \psi_2 = sin5t, \psi_3 = sin4t$ are the external bounded disturbance of the system and C_1, C_2, C_3 are the controllers of the system.

4. PROBLEM FORMULATION:

In this section we design the FODO based adaptive sliding mode compound synchronization method. Next we design a Fractional Order Disturbance Observer to estimate the unknown bounded disturbances and using the sliding mode control technique to design the controllers to achieve the desired synchronization. To estimate the external unknown disturbances in the response system, we design the non linear fractional order disturbance observer as:

$$
\begin{aligned}
\omega_1(t) &= \psi_1(t) - \rho_1 y_1(t) \\
\omega_2(t) &= \psi_2(t) - \rho_2 y_2(t) \\
\omega_3(t) &= \psi_3(t) - \rho_3 y_3(t)
\end{aligned} \tag{5}
$$

where ρ_1, ρ_2, ρ_3 are non zero positive constants to be determined later.

Differentiating the above system and using (4), we get:

$$
\begin{aligned}
&D^{\alpha}\omega_1(t) = D^{\alpha}\psi_1(t) - \rho_1(v_2+\omega_1+\rho_1 v_1) - \rho_1 C_1\\
&D^{\alpha}\omega_2(t) = D^{\alpha}\psi_2(t) - \rho_2(v_3+\omega_2+\rho_2 v_2) - \rho_2 C_2 \qquad (6)\\
&D^{\alpha}\omega_3(t) = D^{\alpha}\psi_3(t) - \rho_3(-\beta_1 v_1-\beta_2 v_2-\beta_3 v_3+\beta_4 v_1^2+\omega_3+\rho_3 v_3) - \rho_3 C_3
\end{aligned}
$$

To calculate the disturbance estimates, the estimates of $\omega_i(t)(i=1,2,3)$ are described as:

$$
\begin{aligned}
&D^{\alpha}\hat{\omega}_1(t) = -\rho_1(v_2+\rho_1 v_1) - \rho_1\hat{\omega}_1(t) - \rho_1 C_1\\
&D^{q}\hat{\omega}_2(t) = -\rho_2(v_3+\rho_2 v_2) - \rho_2\hat{\omega}_2(t) - \rho_2 C_2 \qquad (7)\\
&D^{\alpha}\hat{\omega}_3(t) = -\rho_3(-\beta_1 v_1-\beta_2 v_2-\beta_3 v_3+\beta_4 v_1^2+\rho_3 v_3) - \rho_3\hat{\omega}_3(t) - \rho_3 C_3
\end{aligned}
$$

where $\hat{\omega}_i(t)$ are the estimates of $\omega_i(t)$.
From (5) the disturbances $\psi_i(t)$ can be estimated as

$$\hat{\psi}_i(t) = \hat{\omega}_i(t) + \rho_i v_i \qquad (8)$$

Consider

$$\tilde{\psi}_i(t) = \psi_i - \hat{\psi}_i, (i=1,2,3)$$

Using equations (5) and (8), we have

$$\tilde{\omega}_i(t) = \omega_i(t) - \hat{\omega}_i(t) = \psi_i(t) - \hat{\psi}_i(t) = \tilde{\psi}_i(t) \qquad (9)$$

Using equations (8) and (9), the Caputo fractional derivatives of $\tilde{\omega}_i(t), (i=1,2,3)$ can be written as

$$D^{\alpha}\tilde{\omega}_i(t) = -\rho_i\tilde{\omega}_i(t) + D^{\alpha}\psi_i(t) \qquad (10)$$

To analyse the convergence of disturbance estimate error $\tilde{\psi}_i(t)(i=1,2,3)$, we consider the Lyapunov function as

$$V_{\psi_i}(t) = \frac{1}{2}\tilde{\psi}_i{}^2(t) = \frac{1}{2}\tilde{\omega}_i{}^2(t), (i=1,2,3)$$

Using Lemma 1, the Caputo's derivative of V_{ψ_i} can be written as

$$D^{\alpha}V_{\psi_i}(t) < \tilde{\omega}_i(t)D^{\alpha}\tilde{\omega}_i(t) \tag{11}$$

After substituting (10) in (11),we obtain

$$\begin{aligned} D^{\alpha}V_{\psi_i}(t) &\leq \tilde{\omega}_i(t)(-\rho_i\tilde{\omega}_i(t)+D^{\alpha}\psi_i(t)) \\ &\leq -\rho_i(\tilde{\omega}_i(t))^2+\tilde{\omega}_i(t)D^{\alpha}\psi_i(t)) \end{aligned} \tag{12}$$

Using $(\tilde{\omega}_i - D^{\alpha}\psi_i(t))^2 \geq 0$ we have

$$\tilde{\omega}_i(t)D^{\alpha}\psi_i(t)) < 1/2(\tilde{\omega}_i)^2 + 1/2(D^{\alpha}\psi_i(t))^2 \tag{13}$$

Using (13) & Assumption 1 in equation (12) we obtain

$$\begin{aligned} D^{\alpha}V_{\psi_i}(t) &\leq -\rho_i\tilde{\omega}_i{}^2(t)+\frac{1}{2}\tilde{\omega}_i{}^2(t)+\frac{1}{2}\vartheta_i{}^2 \\ &\leq -(\rho_i-\frac{1}{2})\tilde{\omega}_i{}^2(t)+\frac{1}{2}\vartheta_i{}^2 \\ &= N_0V_{\psi_i}(t)+N_1 \end{aligned} \tag{14}$$

where $N_0 = 2\rho_i - 1$ and $N_1 = \frac{1}{1}\vartheta_i^2$. We choose control gain ρ_i such that $\rho_i > 0.5$, to ensure the estimation of error bounded. Using Lemma 2 and (14),we have

$$|V_{\psi_i}(t)| \leq \frac{\vartheta_i{}^2}{2(\rho_i - 0.5)} \tag{15}$$

$$|\tilde{\psi}_i(t)| \leq \sqrt{\frac{\vartheta_i{}^2}{(\rho_i - 0.5)}} \tag{16}$$

From (16) we have the disturbance estimation error $\tilde{\omega}_i(t)$ bounded above. Therefore the external disturbances $\psi_i(t)(i = 1,2,3)$ and the disturbance approximation error $\psi_i(t)$ satisfy $|\tilde{\psi}_i(t)| \leq k_i$, where $k_i > 0$ is unknown positive constant. As in actual practice it is not easy to determine the upper bound of $|\ \tilde{\psi}_i(t)\ |$, therefore we introduce the estimated value $\tilde{\eta}_i$ of $\eta_i(i = 1,2,3)$.

From the above we have that the disturbance estimated error of the proposed fractional order Genesio-Tesi chaotic system is bounded using FODO.

5. Adaptive Sliding Mode Compound Synchronization

To achieve the compound synchronization between the identical fractional order Genesio Tesi systems with different initial conditions in presence of unknown bounded disturbance, we define the error as:

$$\begin{aligned} e_1(t) &= v_1 - u_{11}(u_{21}+u_{31}) \\ e_2(t) &= v_2 - u_{12}(u_{22}+u_{32}) \\ e_3(t) &= v_3 - u_{13}(u_{23}+u_{33}) \end{aligned} \tag{17}$$

The error dynamics can then be written as:

$$\begin{aligned} D^\alpha e_1(t) &= D^\alpha v_1 - D^\alpha u_{11}(u_{21}+u_{31}) - u_{11}(D^\alpha u_{21}+D^\alpha u_{31}) \\ D^\alpha e_2(t) &= D^\alpha v_2 - D^\alpha u_{12}(u_{22}+u_{32}) - u_{12}(D^\alpha u_{22}+D^\alpha u_{32}) \\ D^\alpha e_3(t) &= D^\alpha v_3 - D^\alpha u_{13}(u_{23}+u_{33}) - u_{13}(D^\alpha u_{23}+D^\alpha u_{33}) \end{aligned} \tag{18}$$

Substituting the values of the derivatives, we get:

$$\begin{aligned} D^\alpha e_1(t) &= e_2 + (u_{22}+u_{32})(u_{12}-u_{11}) - u_{12}(u_{21}+u_{31}) + cos3t + C_1 \\ D^\alpha e_2(t) &= e_3 + (u_{23}+u_{33})(u_{13}-u_{12}) - u_{13}(u_{22}+u_{32}) + sin5t + C_2 \\ D^\alpha e_3(t) &= \beta_1(-v_1 + u_{11}(u_{23}+u_{33}) + u_{13}(u_{21}+u_{31})) + \beta_2(-v_2 + u_{12}(u_{23}+u_{33}) \\ &\quad + u_{13}(u_{22}+u_{32})) - \beta_3 e_3 + \beta_3 u_{13}(u_{23}+u_{33}) + \beta_4(v_1^2 - u_{11}^2(u_{23}+u_{33}) \\ &\quad - u_{13}(u_{21}^2+u_{31}^2)) + sin4t + C_3 \end{aligned} \tag{19}$$

To study the stability of fractional order synchronization error dynamical system, we introduce a simple sliding mode surface as:

$$s_i(t) = e_i(t) \tag{20}$$

implying

$$D^\alpha s_i(t) = D^\alpha e_i(t)$$

Using the adaptive sliding mode approach, we design the controllers as:

$$\begin{aligned}
C_1(t) &= -e_2-(u_{22}+u_{32})(u_{12}-u_{11})+u_{12}(u_{21}+u_{31})-\sigma_1 s_1-\hat{\eta}_1 sign(s_1(t))-\hat{\psi}_1\\
C_2(t) &= -e_3-(u_{23}+u_{33})(u_{13}-u_{12})+u_{13}(u_{22}+u_{32})-\sigma_2 s_2-\hat{\eta}_2 sign(s_2(t))-\hat{\psi}_2\\
C_3(t) &= \beta_3 e_3-\beta_1(-v_1+u_{11}(u_{23}+u_{33})+u_{13}(u_{21}+u_{31}))-\beta_2(-v_2+u_{12}(u_{23}+u_{33})\\
&\quad +u_{13}(u_{22}+u_{32}))-\beta_3 u_{13}(u_{23}+u_{33})-\beta_4(v_1^2-u_{11}^2(u_{23}+u_{33})-u_{13}(u_{21}^2+u_{31}^2))\\
&\quad -\sigma_3 s_3-\hat{\eta}_3 sign(s_3(t))-\hat{\psi}_3
\end{aligned} \tag{21}$$

Here σ_i are non zero positive constants. Substituting the controllers we get the error dynamical system as:

$$\begin{aligned}
D^{\alpha}e_1(t) &= -\sigma_1 s_1-\hat{\eta}_1 sign(s_1(t))+\psi_1-\hat{\psi}_1\\
D^{\alpha}e_2(t) &= -\sigma_2 s_2-\hat{\eta}_2 sign(s_2(t))+\psi_2-\hat{\psi}_2\\
D^{\alpha}e_3(t) &= -\sigma_3 s_3-\hat{\eta}_3 sign(s_3(t))+\psi_3-\hat{\psi}_3
\end{aligned} \tag{22}$$

The update laws of the estimated value $\hat{\eta}_i$ are given by:

$$\begin{aligned}
D^{\alpha}\hat{\eta}_1 &= \phi_1(|s_1(t)|-\hat{\eta}_1)\\
D^{\alpha}\hat{\eta}_2 &= \phi_2(|s_2(t)|-\hat{\eta}_2)\\
D^{\alpha}\hat{\eta}_3 &= \phi_3(|s_3(t)|-\hat{\eta}_3)
\end{aligned} \tag{23}$$

where $\phi_i > 0$ are constants.

The sliding mode surface $s_i(t)$ is stable and bounded for the designed controllers (21) and hence the error system is bounded and stable.

For $\alpha \in$ (0,1),the sliding surface as in (20) in presence of external bounded disturbance approximated by using the designed non linear fractional order disturbance observer (5) and (7), the compound synchronization error is ultimately bounded and stable under the adaptive sliding scheme.

We now summarize the above in the form of the following theorem:

Theorem 1: For the compound synchronization error system (19) with $0 < \alpha_i < 1$, if the sliding mode surface is designed according to (20) and external unknown bounded disturbance is approximated by using the scheme non-linear FODO (7) and (8). Then, compound error is bounded and stable under the adaptive sliding mode control scheme as (21) and (23).

Proof The Lyapunov function V(t) is selected for the convergence of synchronization error e(t) as:

$$V(t)=\sum_{i=1}^{3}\frac{1}{2}s_i(t)^2+\sum_{i=1}^{3}\frac{1}{2}\tilde{\omega}_i(t)^2+\sum_{i=1}^{3}\frac{1}{2}\Big(\frac{1}{\sqrt{\phi_i}}(\hat{\eta}_i-\eta_i)\Big)^2 \tag{24}$$

Differentiating (24), we get

$$D^{\alpha}V(t)=\frac{1}{2}\Big(\sum_{i=1}^{3}D^{\alpha_i}s_i(t)^2+\sum_{i=1}^{3}D^{\alpha_i}\tilde{\omega}_i(t)^2+\sum_{i=1}^{3}D^{\alpha_i}\Big(\frac{1}{\sqrt{\phi_i}}(\hat{\eta}_i-\eta_i)\Big)^2\Big) \tag{25}$$

using $\tilde{\eta}_i=\hat{\eta}_i-K_i$ and Lemma 1 in equation (25) can be written as

$$\begin{aligned}D^{\alpha}V(t)\le&\sum_{i=1}^{3}\frac{1}{2}s_i(t)D^{\alpha_i}s_i(t)+\sum_{i=1}^{3}\frac{1}{2}D^{\alpha_i}\tilde{\omega}_i^{\,2}(t)\\&+\sum_{i=1}^{3}\frac{1}{\sqrt{\phi_i}}\tilde{\eta}_iD^{\alpha_i}\Big(\frac{1}{\sqrt{\phi_i}}\tilde{\eta}_i\Big)\end{aligned} \tag{26}$$

On applying property 2 in equation (26), we obtain

$$D^{\alpha}V(t)\le\sum_{i=1}^{3}s_i(t)D^{\alpha_i}s_i(t)+\sum_{i=1}^{3}\frac{1}{2}D^{\alpha_i}\tilde{\omega}_i(t)^2+\sum_{i=1}^{3}\frac{1}{\phi_i}\tilde{\eta}_iD^{\alpha_i}\tilde{\eta}_i \tag{27}$$

Using (20) and substituting (22) into (27), we obtain

$$\begin{aligned}D^{\alpha}V(t)\le&\sum_{i=1}^{3}s_i(t)(-\sigma_is_i+\tilde{\omega}_i(t)-\hat{\eta}_isign(s_i(t)))+\sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\omega}_i^{\,2}(t)\\&+\sum_{i=1}^{3}\frac{1}{\phi_i}\tilde{\eta}_iD^{\alpha_i}\tilde{\eta}_i\end{aligned} \tag{28}$$

Applying Property 1 and using $\tilde{\eta}_i = \hat{\eta}_i - \eta_i$,(i=1,2,3), we obtain

$$D^{\alpha}\tilde{\eta}_i = D^{\alpha_i}\hat{\eta}_i \tag{29}$$

where K_j is a constant parameter

Using (23) and (29), we have

$$\begin{aligned}\sum_{i=1}^{3}\frac{1}{\phi_i}\tilde{\eta}_i D^{\alpha_i}\tilde{\eta}_i &= \sum_{i=1}^{3}\tilde{\eta}_i(|s_i(t)|-\hat{\eta}_i)\\ &= \sum_{i=1}^{3}\tilde{\eta}_i|s_i(t)| - \sum_{i=1}^{3}\tilde{\eta}_i(\tilde{\eta}_i+\eta_i)\\ &= \sum_{i=1}^{3}\tilde{\eta}_i|s_i(t)| - \frac{1}{2}\sum_{i=1}^{3}\tilde{\eta}_i^2 - \frac{1}{2}\sum_{i=1}^{3}\tilde{\eta}_i{}^2 - \sum_{i=1}^{3}\tilde{\eta}_i\eta_i\\ &\leq \sum_{i=1}^{3}\tilde{\eta}_i|s_i(t)| - \frac{1}{2}\sum_{i=1}^{3}\tilde{\eta}_i{}^2 + \frac{1}{2}\sum_{i=1}^{3}\eta_i{}^2\end{aligned} \tag{30}$$

After substituting (30) into (28), we get

$$\begin{aligned}D^{\alpha}V(t) \leq &\sum_{i=1}^{3}s_i(t)(\sigma_i s_i(t) + \tilde{\omega}_i(t) - \hat{\eta}_i sign(s_i(t))) + \frac{1}{2}\sum_{i=1}^{3}D^{\alpha_i}\tilde{\omega}_i{}^2(t)\\ &+ \sum_{i=1}^{3}\tilde{\eta}_i|s_i(t)| - \frac{1}{2}\sum_{i=1}^{3}\tilde{\eta}_i{}^2 + \frac{1}{2}\sum_{i=1}^{3}\eta_i{}^2\end{aligned} \tag{31}$$

Equation (31) can be rewritten as

$$\begin{aligned}D^{\alpha}V(t) \leq &-\sum_{j=1}^{3}\sigma_i s_i^2(t) + \sum_{i=1}^{3}|s_i(t)|\tilde{\omega}_i| + \sum_{i=1}^{3}\tilde{\eta}_i|s_i(t)| - \sum_{i=1}^{3}\frac{1}{2}\tilde{\eta}_i{}^2 + \sum_{i=1}^{3}\frac{1}{2}\eta_i{}^2\\ &-\sum_{i=1}^{3}\hat{\eta}_i|s_i(t)| + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha_i}\tilde{\omega}_i{}^2(t)\end{aligned} \tag{32}$$

Using $|\tilde{\omega}_i(t)| < \eta_i$ and $\sum_{i=1}^{3}\tilde{\eta}_i|s_i(t)| - \sum_{i=1}^{3}\hat{\eta}_i|s_i(t)| = -\sum_{i=1}^{3}\eta_i|s_i(t)|$, equation (32) can be written as:

$$D^{\alpha}V(t) \leq -\sum_{j=1}^{3}\sigma_i s_i^2(t) - \sum_{i=1}^{3}\frac{1}{2}\tilde{\eta}_i{}^2 + \sum_{i=1}^{3}\frac{1}{2}\eta_i^2 + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\omega}_i{}^2(t) \tag{33}$$

From equation (14) and (33), we have

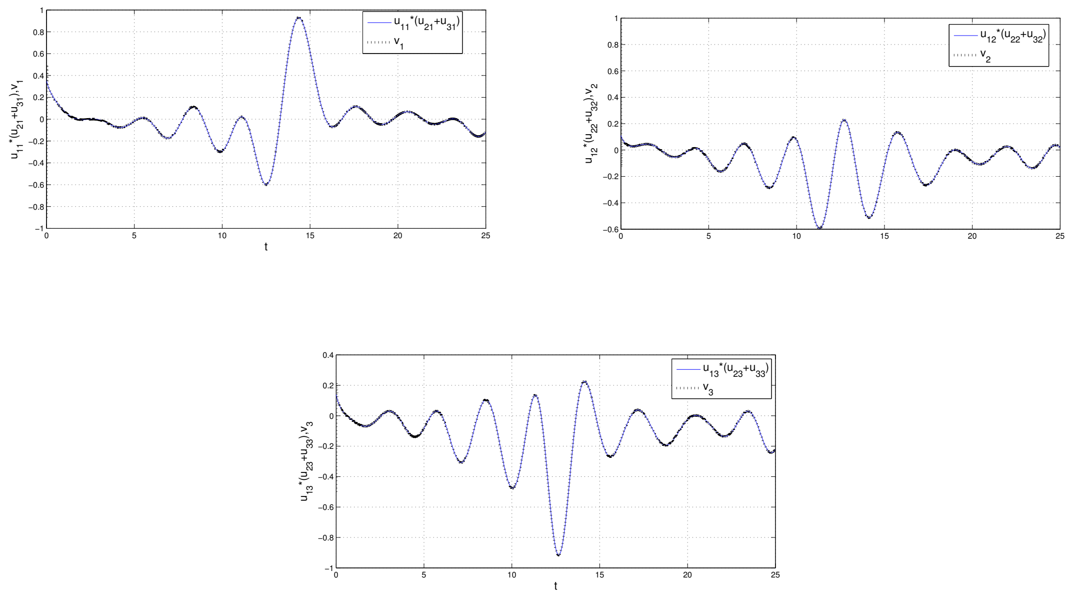

Fig. 3: The synchronized trajectories of the master and slave systems

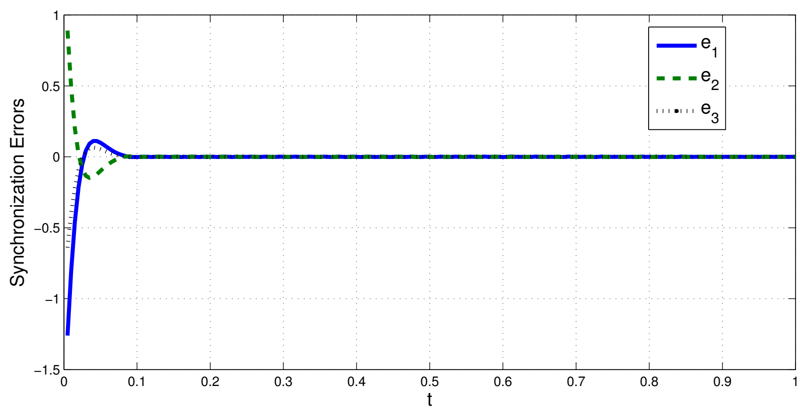

Fig. 4: The synchronization error shows the convergence at time t=0.1second

$$D^{\alpha}V(t) \leq -\sum_{i=1}^{3}\sigma_i s_i^2(t) - \sum_{i=1}^{3}\frac{1}{2}\tilde{\eta_i}^2 + \sum_{i=1}^{3}\frac{1}{2}\eta_i^2$$

$$+\sum_{i=1}^{3}-(\sigma_i - \frac{1}{2})\tilde{\sigma_i}^2(t) + \sum_{i=1}^{3}\frac{1}{2}\vartheta_i^2 \leq -N_2V(t) + N_3 \tag{34}$$

where $N_2 = min(2\sigma_i, 1, 2\sigma_i - 1)$ and $N_3 = \sum_{i=1}^{3}\frac{1}{2}\vartheta_i^2 + \sum_{i=1}^{3}\frac{1}{2}\eta_i^2$.

On selecting the value of parameters $\sigma_i > 0$ and $\rho_i > 0.5$, we have the error bounded. Using Lemma 2 in (34),we get

$$|V(t)| \leq \frac{2N_3}{N_2}$$
$$= \frac{\sum_{i=1}^{3} \vartheta_i^2 + \sum_{i=1}^{3} \eta_i^2}{N_2} \tag{35}$$

(35) implies that

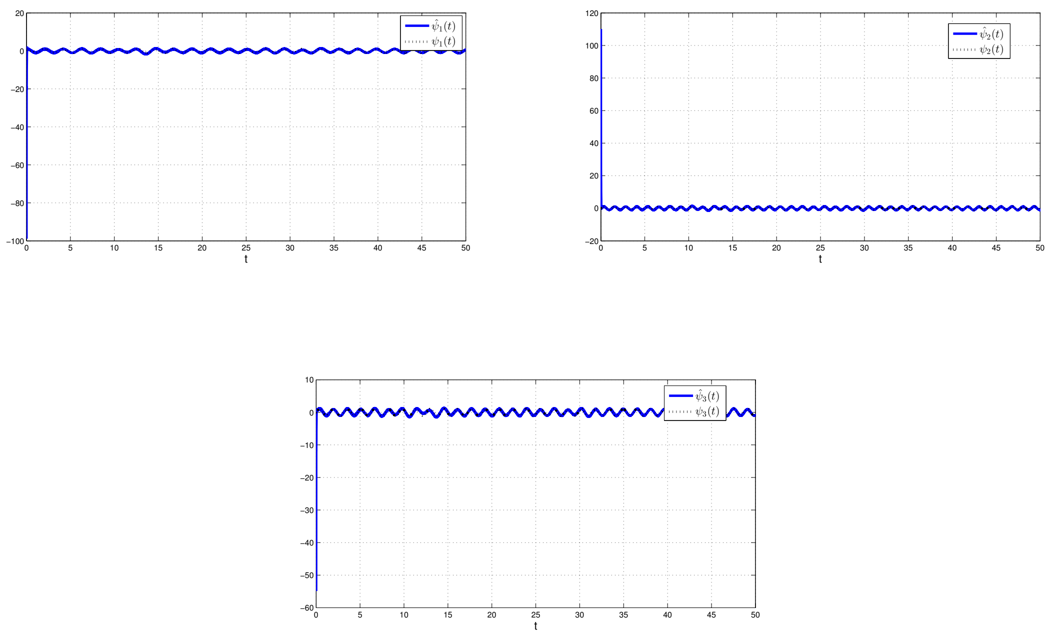

Fig. 5: Estimates of External Disturbances

$$\|s(t)\| \leq \sqrt{\frac{2(\sum_{i=1}^{3} \vartheta^2 + \sum_{i=1}^{3} \eta_i^2)}{N_2}} \tag{36}$$

From (35) and (36), it is clear that the sliding surface $s_i(t)$ and synchronization error $e_i(t)$ are bounded as $t \to \infty$.Thus error dynamical system (19) is bounded and stable implying that the synchronization between master systems and slave system has been achieved.

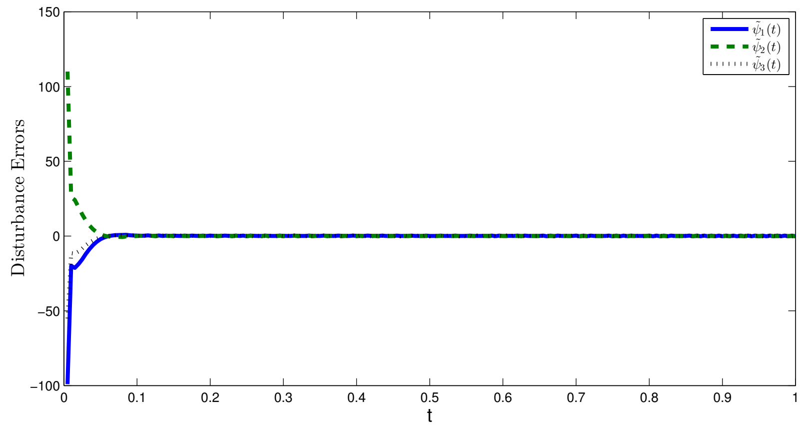

Fig. 6: Synchronized Disturbance observer errors shows the convergence at time t=0.1second

6. Numerical Simulations & Discussions

Here we have considered the values of the parameters $\hat{\omega_1}(0), \hat{\omega_2}(0), \hat{\omega_3}(0)) = (.2,.2,.2)$,$(\hat{\eta_1}(0)$, $\hat{\eta_2}(0), \hat{\eta_3}(0)) = (.1,.1,.1)$,$(\rho_1, \rho_2, \rho_3) = (110, 110, 110)$,$(\phi_1, \phi_2, \phi_3) = (.1,.1,.1)$,$(\sigma_1, \sigma_2, \sigma_3) = (80, 80, 80)$. We have considered the disturbance $\psi_1 = cos3t, \psi_2 = sin5t, \psi_3 = sin4t$. The synchronized trajectories of the master and slave systems are displayed in Fig. 3 and Fig.4 represents the synchronization error. Also Fig.5 and Fig 6 represents the disturbance observer estimates and disturbance error converging which is converging to zero.

7. Comparison with the Previous Published Literature

In this manuscript, we compare our obtained results with the already published work for compound synchronization of chaotic systems using a different technique. In [22] the author has investigated compound synchronization of Lorenz and Chen chaotic systems and they achieved the error tending to zero at t= 4 sec (approx). In [21], the author has investigated compound synchronization of identical memristor chaotic system and its error converges to zero at t=3 sec (approx.). In [23], the author has investigated compound synchronization of the same identical memristor chaotic systems using different technique technique and error converges to zero at t=4 sec (approx.).

However in our case we have also considered external disturbance in the slave system and using the technique of disturbance observer based sliding mode control and got our synchronization error at t=0.1 sec(approx.) which is far better than the previously published literature. Thus our technique and designed controllers prove more efficient than the previous one.

8. Conclusion

In this paper compound synchronization among four identical fractional order Genesio Tesi chaotic systems arising from different initial conditions has been achieved in presence of unknown bounded disturbances. A non linear fractional order disturbance observer has been used to estimate the disturbances. The synchronization has been achieved using the adaptive sliding mode control technique.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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