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# FIXED POINT THEOREM FOR $P_\alpha$-NONEXPANSIVE WRT ORBITS IN LOCALLY CONVEX SPACE

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**Abstract.** In this paper, first we establish a fixed point theorem for a $p_\alpha$-nonexpansive wrt orbits mapping in a locally convex space, then we apply it to get a fixed point theorem in probabilistic normed spaces.

**Keywords:** fixed point; $p_\alpha-$nonexpansive wrt orbits; weak $p_\alpha-$normal structure; locally convex space; probabilistic normed spaces.

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## 1. Introduction

Some results concerning fixed theorems for nonexpansive mappings on locally convex topological vector spaces have been obtained in Taylor (1972) and also in Tarafdar (1974). In Kakutani (1938), Markov (1936), and Day (1961) fixed point theorems for commutative family of linear contnuous of self mappings on a compact convex subset of a topological vector space have been investigated. In DeMarr (1963), Belluce and Kirk (1966), Hong (1968) and in many

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other places the fixed point theorems for a commutative family of nonexpansive mappings on a Banach space have been considered. These theorems have been proved in locally convex topological vector spaces in Tarafdar (1975) which will be the subject matter of the next two subsections. In the present paper, the concept of $p_\alpha-$nonexpansive wrt orbits and of $p_\alpha$-normal structure in a locally convex space have been introduced

In the sequel, $(X,\tau)$ will be a Hausdorff locally convex topological vector space. A famliy $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms defined on $X$ is said to be an associatd family of seminorms for $\tau$ if the family $[\rho U : \rho > 0]$, where $U = \cap_{i=1}^{n} U_{\alpha i}$ and $U_{\alpha i} = \{x \in X : p_{\alpha_i}(x) < 1\}$ forms a base of neighbourhoods of zero for $\tau$ .A family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms defined on $X$ is called an augmented associated family for $\tau$ if $\{p_\alpha : \alpha \in \Lambda\}$ is an associated family with the property that the seminorm $\max\left\{p_\alpha, p_\beta\right\} \in \{p_\alpha : \alpha \in \Lambda\}$ for any $\alpha, \beta \in \Lambda$ .The associated and augmented

families of seminorms shall be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space $(X,\tau)$, there always exists a family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms defined on $X$ such that $\{p_\alpha : \alpha \in \Lambda\} = A^*(\tau)$ $(see, [10, p, 203])$ . A subset $M$ of $X$ is $\tau$-bounded if and only if each $p_\alpha$, is bounded on $M$.

## 2. $p_\alpha-$NORMAL STRUCTURE

In the setting of locally convex spaces, the fixed point theory of nonexpansive mappings has been extensively studied by many authors; see [1, 2, 3] and the references therein

Let $A$ be a nonempty subset of a locally convex space $X$, and let $T : A \to A$ be a mapping. We say that T is $p_\alpha$-nonexpansive if

$$p_\alpha(Tx - Ty) \leq p_\alpha(x - y) \text{ for all } p_\alpha \in A^*(\tau) \text{ and } x, y \in X$$

The concept of normal structure of a bounded convex set in a Banach space was first introduced in Brodskii and Milman (1948). We have introduced below the same concept for a bounded convex subset of $X$. A point $x$ of a bounded subset $K$ of $X$ is said to be a $p_\alpha$-diametral point of $K$ if $\delta(K,\alpha) = \sup\{p_\alpha(x-y) : y \in K\}$, as before $\delta(K,\alpha)$ is the $p_\alpha$-diameter of $K$, i.e. $\delta(K,\alpha) = \sup\{p_\alpha(x-y) : x, y \in K\}$ . A point $y \in K$ which is not a $p_\alpha$-diametral point of $K$ is called a $p_\alpha$-nondiametral point of $K$

For any subset $D$ of $X$ and for $x \in X$, $r_x(D,\alpha) = \sup_{y\in D} p_\alpha(x-y)$.

**Definition 1.** *A point $x \in D$ is said to be $p_\alpha-$nondiametral if.* $r_x(D,\alpha) < \delta(D,\alpha)$

**Definition 2.** *A convex subset A of X is said to have $p_\alpha-$normal structure if each convex, bounded subset K of A with $\delta(K,\alpha) > 0$ contains a $p_\alpha-$nondiametral point*

**Definition 3.** *A Hausdorff locally convex space $(X,\tau)$ has weak $p_\alpha-$normal structure if every weakly compact, convex subset A of X with $\delta(K,\alpha)$ contains a $p_\alpha$-nondiametral point*

## 3. Fixed Point Theorems

**Definition 4.** *A mapping $T : A \to A$ is said to be $p_\alpha-$nonexpansive wrt orbits if*

$$p_\alpha(Tx-Ty) \leq r_x(O_T(y),\alpha) \text{ for all } x,y \in A$$

*where* $O_T(y) := \{T^n y, n \geq 0\}$

It is clear that every $p_\alpha-$nonexpansive mapping $T : A \to A$ is $p_\alpha-$nonexpansive wrt orbits. However, there are mappings which are $p_\alpha-$nonexpansive wrt orbits but fail to be $p_\alpha-$nonexpansive

**Theorem 5.** *Let A be a nonempty weakly compact convex subset of a Hausdorff locally convex space $(X,\tau)$ which has weak $p_\alpha-$normal structure, and $T : A \to A$ is said to be $p_\alpha-$nonexpansive wrt orbits. Then T has a fixed point.*

*Proof.* Let $\mathscr{F} = \{K \subset A : K$ is a nonempty closed convex set and $T(K) \subset K\}$

since $C \subset \mathscr{F}$ then $\mathscr{F}$ is a nonempty family . $\mathscr{F}$ is partially ordered by

$$K_1, K_2 \in \mathscr{F}\ ,\ K_1 \prec K_2 \Leftrightarrow K_1 \subset K_2$$

By weakly compactnees of $A$ and Zorn's Lemma, $\mathscr{F}$ has a minimal element $H$.

Now we shall show that $H$ consists of a single point. Assume on the contrary that there exists $p_\beta \in A^*(\tau)$ such that $\delta(H,\beta) = d > 0$ and for each $x,y \in H$,we have

$$p_\beta(Tx-Ty) \leq r_x(O_T(y)) \leq r_x(H,\beta) \quad (2.1)$$

Since $TH \subset H$, this implies $\overline{co}(TH) \subset \overline{co}(H) = H$ ( $\overline{co}(TH)$ denotes the closed convex hull of $TH$ ). Hence $T(\overline{co}(TH)) \subset TH \subset \overline{co}(TH)$. Thus $\overline{co}(TH) \in \mathscr{F}$ . From this and the minimality of $H$ we get $\overline{co}(TH) = H$.and so by (2.1), we get

$$r_{Tx}(H) = r_{Tx}(\overline{co}(TH), \beta) = r_{Tx}(coTH, \beta) \leq r_x(H, \beta) \quad (2.2)$$

for each $x \in H$. Since $H$ has $p_\alpha-$normal structure, then there exist $x_1, x_2 \in H$ such that $r_1 = r_{x_1}(H, o) < r_{x_2}(H, \beta) = r_2$ . Denoting

$$H_0 = \left\{ y \in H : r_y(H, \beta) \leq \frac{r_1 + r_2}{2} \right\}$$

it is easy to prove that $H_0$ is a nonempty closed bounded and convex subset in $H$, since $x_1, x_2 \in H$ and $p_\beta$ is a convex continuous function.

Now we show that $H_0$ is invariant under $T$ by (2.2), we get $r_{Tx}(H_0) = r_{Tx}(\overline{co}(TH_0), \beta) = r_{Tx}(coTH_0, \beta) \leq r_x(H_0, \beta)$ so $TH_0 \subset H_0$. Hence $H_0 \in \mathscr{F}$. By the minimality of $H$ in $\mathscr{F}$ we get $H = H_0$. Thus $T : A \to A$ has a fixed point. □

## 4. Application to Probabilistic Normed Spaces

**Definition 6.** $[4-5]$ *A probabilistic normed space is a triple $(X, F, min)$, where $X$ is a linear space $F = \{F_x : x \in X\}$ is a family of distribution functions satisfying:*

*(1)* $F_x(0) = 0$ *for all* $x \in X$

*(2)* $F_x(t) = 1$ *for all* $t > 0 \Longleftrightarrow x = 0$

*(3)* $F_{\alpha x}(t) = F\left(\frac{t}{|\alpha|}\right), \forall t \geq 0, \forall \alpha \in \mathbb{C}$ *or* $\mathbb{R}, \alpha \neq 0, \forall x \in X$

*(4)* $F_{x+y}(s+t) \geq \min(F_x(s), F_y(t)), \forall x, y \in X, \forall t, s \geq 0.$

The topology in $X$ is defined by the system of neighborhoods of $0 \in X$ :

$$U(0, \varepsilon, \alpha) = \{x \in X : F_x(\varepsilon) > 1 - \alpha\}, \varepsilon > 0, \alpha \in (0,1).$$

This is a locally convex Hausdorff topology, called the $(\varepsilon, \alpha)$-topology .

To see this we define for each $\alpha \in (0,1)$

(1) $$p_\alpha(x) = \sup\{t \in \mathbb{R} : F_x(t) \leq 1 - \alpha\}$$

From properties 1) - 4) of $F_x$ one can verify that $p_\alpha$ is a seminorm on $X$ and $p_\alpha(0) = 0$ , $\forall\alpha \in (0,1) \Rightarrow x = 0$, and the topology on $X$ defined by the family of seminorms$\{p_\alpha : \alpha \in (0,1)\}$ coincides with the $(\varepsilon,\alpha)$-topology. In particular, we have

$$F_x(p_\alpha(x)) \le 1-\alpha, \forall x \in X, \forall\alpha \in (0,1) \tag{2}$$

and

$$p_\alpha(x) < \varepsilon \Longleftrightarrow F_x(\varepsilon) > 1-\alpha \tag{3}$$

(For details, see [5]) In the sequel all topological notions (boundedness, compactness, weak compactness,...) in a probabilistic normed space are understood as those in the corresponding locally convex space

**Definition 7.** *A mapping $T$ in $(X,F,min)$ is said to be probabilistic nonexpansive wrt orbitsif if for all for all $x,y \in X$ and $t \in IR$*

$$F_{Tx-Ty}(t) \ge r_x(O_T(y))$$

*where* $O_T(y) := \{T^n y, n \ge 0\}$ *and* $r_x(A) = \sup\left\{t : \sup_{y\in A} F_{x-y}(t) \le 1-\alpha\right\}, (t \in IR)$

**Definition 8.** *a subset A of a probabilistic normed space $(X,F,min)$ is said to*

*have probabilistic uniformly normal structure if for every convex closed bounded subset H of A containing more than one nondiametral point, there exists $x_0 \in H$ and $0 < k < 1$ such that*

$$\inf_{y\in H} F_{x_0-y}(kt) \geqslant \inf_{x,y\in H} F_{x-y}(t) \tag{4}$$

*for all $t \geqslant 0$*

Before stating another fixed point theorem we establish three following lemmas.

**Lemma 9.** *Every probabilistic nonexpansive wrt orbits if mapping in a probabilistic normed space $(X,F,min)$ is $p_\alpha-$nonexpansive wrt in the corresponding locally convex space $(X,\{p_\alpha\})$*

*Proof.* Since $F_{Tx-Ty}(t) \ge r_x(O_T(y))$ .We have for each $\alpha \in (0,1)$

$$\{t : F_{Tx-Ty}(t) \le 1-\alpha\} \subset \left\{t : \sup_{u\in O_T(y)} F_{x-u}(t) \le 1-\alpha\right\} \tag{5}$$

This implies

$$\sup\left\{t : F_{Tx-Ty}(t) \leq 1-\alpha\right\} \leq \sup\left\{t : \sup_{u\in O_T(y)} F_{x-u}(t) \leq 1-\alpha\right\}$$

and finally

$$p_\alpha\left({}_{Tx-Ty}\right) \leq r_x\left(O_T\left(y\right)\right)$$

then $T$ is $p_\alpha-$nonexpansive wrt in the corresponding locally convex space $(X, \{p_\alpha\})$ □

**Lemma 10.** *Let a probabilistic normed space $(X, F, min)$ satisfy the following condition: For each fixed $t \in IR$, the function $F_x(t) : X \to [0,1]$ is weakly lower semicontinuous in $x \in X$ $(C)$.*
*Then every weakly compact set $A \subset X$ having probabilistic uniformly normal structure has weak $p_\alpha-$normal structure in the corresponding locally convex space $(X, \{p_\alpha\})$*

*Proof.* Let $D$ be any closed convex subset of $A$, then $D$ is also weakly compact. We show that for each $\alpha \in (0,1)$

$$\sup_{x\in D}\sup\{t : F_x(t) \leq 1-\alpha\} = \sup\left\{t : \inf_{x\in D} F_x(t) \leq 1-\alpha\right\} \tag{6}$$

Since $F(t) = \inf_{x\in D} F_x(t) \leq F_x(t)$ for each $x \in D$, we have

$$a = \sup\{t : F(t) \leq 1-\lambda\} \geq \sup_{x\in D}\sup\{t : F_x(t) \leq 1-\alpha\} = b$$

If $a > b$, then we have $F_x(a) > 1-\alpha$ for each $x \in D$.The condition $(C)$ shows that $F(a) > 1-\alpha$ this implies $a > a$, a contradiction. Thus $a = b$, so (4.6) is proved.

Now we prove the assertion of the lemma. From the inequality

$$\inf_{y\in D} F_{x_0-y}(kt) \geqslant \inf_{x,y\in D} F_{x-y}(t)$$

we get

$$\left\{t : \inf_{y\in H} F_{x_0-y}(kt) \leq 1-\alpha\right\} \subseteq \left\{\inf_{x,y\in H} F_{x-y}(t) \leq 1-\alpha\right\}$$

hence

$$\frac{1}{k}\left\{t : \inf_{y\in H} F_{x_0-y}(t) \leq 1-\alpha\right\} \subseteq \left\{\inf_{x,y\in H} F_{x-y}(t) \leq 1-\alpha\right\}$$

so

$$\left\{t : \inf_{y\in H} F_{x_0-y}(t) \leq 1-\alpha\right\} \subseteq k\left\{\inf_{x,y\in H} F_{x-y}(t) \leq 1-\alpha\right\}$$

This implies

$$\sup\left\{t : \inf_{y\in H} F_{x_0-y}(t) \leq 1-\alpha\right\} \leq k \sup\left\{\inf_{x,y\in H} F_{x-y}(t) \leq 1-\alpha\right\}$$

From this and (6) we get

$$\sup_{y\in H}\sup\left\{t : F_{x_0-y}(t) \leq 1-\alpha\right\} \leq k \sup_{x,y\in H}\sup\left\{F_{x-y}(t) \leq 1-V\right\}$$

and finally

$$\begin{aligned} \sup_{y\in H} p_\alpha(x_0-y) &\leq k \sup_{x,y\in H} p_\alpha(x-y) = k\delta(H,\alpha) < \delta(H,\alpha) \\ r_{x_0}(x,\alpha) &\leq \delta(H,\alpha) \end{aligned}$$

if $\delta(H,\alpha) > 0$, as desired. The proof is complete □

**Theorem 11.** *Let A be a nonempty weakly compact convex set having probabilistic uniformly normal structure in a probabilistic normed space $(X, F, min)$ satisfying condition $(C)$. Let $T$ be a probabilistic nonexpansive wrt orbits mapping from A into A. Then T has a fixed point*

*Proof.* By Lemmas 9 and 10, $T$ satisfies all conditions in Theorem 1 with $E = (X, \{p_\alpha\})$ corresponding to $(X, F, min)$, so $T$ has a fixed point □

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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