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SOLVING CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS USING PICARD'S ITERATION METHOD

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Abstract: In this paper we will generalize Picard's iterated approximation, to solve the conformable fractional differential equations with an initial condition. This will be by proving the uniqueness, convergence and existence of the solution under the definition and properties of the conformable fractional derivative and integral. Besides the Lipschitz condition and the Gronwall's inequality after generalizing it to the conformable fractional case. Also, we will show some CFDE examples and their solution besides of the graphs to show the convergence of the approximation solutions to the exact one and their applications.

Keywords: conformable derivative; integral; Lipschitz function; Picard's method; uniformly convergence; existence; uniqueness.

2010 AMS Subject Classification: 34A08.

1. INTRODUCTION AND PRELIMINARIES

Fractional Calculus has become more interesting for many researchers because it plays an important role in several applications from engineering and science problems to physics, economics and chemistry. Few years ago it has been found that Fractional calculus use in studies of viscoelastic materials, as well as in many fields of science and engineering including fluid flow, diffusive transport, electrical networks, electromagnetic theory and probability [1-3]. Particularly

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in engineering, the fractional calculus arises in the time-dependent viscous-diffusion fluid mechanics problems, in addition to the classical transient viscous-diffusion equation in a semi-infinite space to yield explicit analytical (fractional) solutions for the shear stress and fluid speed anywhere in the domain [4].

The theory of conformable fractional calculus was first introduced in 2014 by Dr. Rushdi Khalil [5]. It is one of the most useful and easy to understand theories for fractional differential equations. This is because the new definition introduced by Dr. Khalil and all subsequent properties and results coincide with the definition and properties of the normal derivative more so than other definitions presented by Riemann-Liouville or Caputo [6,7].

The purpose of this paper is to improve on Picard's successive iterative approximation [10]. and develop a solution to the Conformable initial value problem:

$T_\alpha(y)(x) = f(x,y), y(0) = y_0$, If $f(x,y)$ satisfies the Lipschitz condition. It will also prove the existence and convergence of the solution through the use of Gromwall's inequality to prove uniqueness.

Gromwall's inequality plays an important role in solving integral equations and proving some related inequalities, that why the scientists nowadays interest on it and generalize to the fractional calculus.

Definition 1.1: (Riemann-Liouville) [6]. For $\alpha \in [n-1, n)$ the α −derivative of f is:

$$D_a^\alpha(f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt$$

Definition 1.2: *(Caputo) [7]. For $\alpha \in [n-1, n]$ the α −derivative of f is:*

$$D_a^\alpha(f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt.$$

Barriers to these definitions are as follows:

(1) They don't satisfy that $D_a^\alpha(c) = 0$ for any Constant c.

(2) They don't satisfy the known formulas for the derivative of the product and quotient for any two $\alpha -$ differentiable functions $f(x) and\ g(x)$:

$D_a^\alpha(f.g) \neq D_a^\alpha(f).g + D_a^\alpha(g).f$ And $D_a^\alpha\left(\frac{f}{g}\right)(x) \neq \frac{gD_a^\alpha(f)-f.D_a^\alpha(g)}{g^2}$.

(3) They don't satisfy the chain rule for any two $\alpha -$ Differentiable functions $f(x) and\ g(x)$ then: $D_a^\alpha(fog)(x) \neq D_a^\alpha(f(g(x)).D_a^\alpha(g)(x)$

(4) They don't satisfy $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$.

Definition 1.3: (Conformable fractional derivative) [5]: Let $f(x)$: defined from $[0,\infty) to\ \mathbb{R}$ then for all $x > 0$ and $\alpha \in (0,1)$:

$$T_{\alpha}(f)(x) = \lim_{\epsilon \to 0} \frac{f\left(x+\epsilon x^{1-\alpha}\right)-f(x)}{\epsilon}$$

T_{α} Is the conformable fractional derivative of f of order α.

If f is α-differentiable in some interval $(0,a), a > 0$ and $\lim_{x\to 0+} f^{(\alpha)}(x)$ exist, then we define:

$T_{\alpha}\big(f(0)\big) = \lim_{x\to 0+} f^{(\alpha)}(x)$.

From definition 1.3 the following formulas can be demonstrated:

1) $T_{\alpha}(c) = 0$ For any constant $c \in \mathbb{R}$.
2) $T_{\alpha}(x^{q}) = qx^{q-\alpha}$
3) $T_{\alpha}(sinax) = ax^{1-\alpha}cosax\ a \in \mathbb{R}$
4) $T_{\alpha}(cosax) = -ax^{1-\alpha}sinax$
5) $T_{\alpha}(e^{ax}) = ax^{1-\alpha}e^{ax}$
6) $T_{\alpha}(f)(x) = x^{1-\alpha}\frac{df}{dx}$ Is direct from the definition see [5].

In addition, some functions arise in solving CFDE:

1) $T_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right) = 1$
2) $T_{\alpha}\left(e^{\frac{x^{\alpha}}{\alpha}}\right) = e^{\frac{x^{\alpha}}{\alpha}}$
3) $T_{\alpha}\left(\sin\left(\frac{x^{\alpha}}{\alpha}\right)\right) = \cos\left(\frac{x^{\alpha}}{\alpha}\right)$
4) $T_{\alpha}\left(\cos\left(\frac{x^{\alpha}}{\alpha}\right)\right) = -\sin\left(\frac{x^{\alpha}}{\alpha}\right)$

Definition 1.4: (Conformable fractional integral) [5]. Let $a \geq 0$ And $x \geq a$ let f be a function defined on $(a,x]$,Then the α-fractional integral of f is defined by: $I_{\alpha}^{a}f(x) = \int_{a}^{x}\frac{f(s)}{s^{1-\alpha}}ds = \int_{a}^{x}s^{\alpha-1}f(s)ds$.

Proposition: $T_{\alpha}I_{\alpha}(f)(x) = f(x)$ comes by:

$T_{\alpha}I_{\alpha}(f)(x) = x^{1-\alpha}\frac{d}{dx}\int_{a}^{x}\frac{f(s)}{s^{1-\alpha}}ds = x^{1-\alpha}\frac{f(x)}{x^{1-\alpha}} = f(x)$.

2. MAIN RESULTS

In this section the proof of the convergence and continuity of the solution of the conformable fractional differential equation with initial condition in (1) can be demonstrated as:

$$T_\alpha(y)(x) = f(x,y), y(x_0) = y_0 \tag{1}$$

Definition 2.1: (Lipschitz function) [5]. A function $f(x,y) \in C[[x_0, x_0+T] \times \mathbb{R}, \mathbb{R}]$ is said to be a Lipschitz function in y if for any y_1, y_2 there exists $L > 0$ such that: $|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|$.

Theorem 1: (Gromwall's inequality) [6-9]: If $f(x)$ and $g(x)$ are a nonnegative functions and continuous for all $x > 0$ where $k > 0$ is a constant and let: $f(x) \leq k + \int_0^x s^{\alpha-1} g(s) f(s) ds$, for $x \geq 0$.

Then: $f(x) \leq k e^{\int_0^x s^{\alpha-1} g(s) ds}$.

Proof: we will start by: $\dfrac{f(x)}{k + \int_0^x s^{\alpha-1} g(s) f(s)} \leq 1$

$\dfrac{x^{\alpha-1} f(x).g(x)}{k + \int_0^x s^{\alpha-1} g(s) f(s)} \leq x^{\alpha-1} g(x)$ Now we will integrate both sides normally to get:

$$\ln\left(k + \int_0^x s^{\alpha-1} g(s) f(s) ds\right)\Big|_0^x \leq \int_0^x s^{\alpha-1} g(s) ds,$$

$$\ln\left(k + \int_0^x s^{\alpha-1} g(s) f(s) ds\right) - \ln k \leq \int_0^x s^{\alpha-1} g(s) ds.$$

Then: $\ln\left(k + \int_0^x s^{\alpha-1} g(s) f(s) ds\right) \leq \ln k + \int_0^x s^{\alpha-1} g(s) ds$

Hence $k + \int_0^x s^{\alpha-1} g(s) f(s) ds \leq k e^{\int_0^x s^{\alpha-1} g(s) ds}$.

Then: $f(x) \leq k e^{\int_0^x s^{\alpha-1} g(s) ds}$ The proof now completed.

Result 1: if $g(x) = f(x)\ then\ f(x) \equiv 0$.

Result 2: if $g(x) = 1$, then $f(x) \leq ke^{\frac{x^\alpha}{\alpha}}$

Lemma 1: (Differential type) if $f(x)$ and $g(x)$ are a nonnegative functions and continuous for all $x > 0$ where $k > 0$ is a constant and assume: $T_\alpha(f)(x) \leq g(x)f(x)$, then: $f(x) \leq f(0)e^{\int_0^x s^{\alpha-1}g(s)ds}$.

Theorem 2: Assume that $f(x,y)$ is a continuous function in the region D of xy plan and M be a constant where:

$|f(x,y)| < M, \forall(x,y) \in D$ And satisfy Lipschitz condition in y where: $|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|$ where L is constant, let R be the rectangle defined by:

$R = \{(x,y): |x - x_0| \leq a, |y - y_0| \leq b\}$ Where $R \subset D$, let $h = \min\left\{a, \frac{b}{M}\alpha^{\frac{1}{\alpha}}\right\}$:

Then the CFDE in (1) has a unique solution $y = y(x)$ where $y(x_0) = y_0, \forall|x - x_0| < h$. and hence the iterations:

$y_n(x) = y_0 + \int_{x_0}^x s^{\alpha-1}f\big(s, y_{n-1}(s)\big)ds$, will converge uniformly to the solution of the CFIVP (1).

Proof: Now the proof of theorem 2 is divided into 3 steps and fix the initial condition on $x_0 = 0$ mainly $y(0) = y_0$ then we can generalize for any x_0.

Step (1): Continuity: to show that $\forall n \in \mathbb{N}, y_n(x)$ is continuous for all $[0,h] = I$.This can be done through induction.

Because $y_0(x) = y(0) = y_0$ is continuous on I since it is constant.

Now $y_1(x) = y_0 + \int_0^x s^{\alpha-1}f\big(s, y_0(s)\big)ds$ is also continuous because f is continuous, this means that all terms of the sequence $\{y_n\}$ are continuous. Mainly:

$y_{n+1}(x) = y_0 + \int_0^x s^{\alpha-1}f\big(s, y_n(s)\big)ds$ is continuous.

Step (2): Convergence, it can be shown that the sequence $\{y_n(x)\}$ is convergent uniformly.

We will start by the relation: $y_n = y_0 + \sum_{i=1}^{n}(y_i - y_{i-1})$. It is clear that this relation is true.

We aim to show that $|y_n(x)|$ converges uniformly which is equal $\left|y_0 + \sum_{i=1}^{n}(y_i - y_{i-1})\right|$.

To show that we will use the continuity of f and the fact $|f(x,y)| < M$ in addition to the fact that $f(x,y)$ is Lipschitzian in y. From this we aim to show that: $|y_{n+1}(x) - y_n(x)| \le \frac{ML^n x^{(n+1)\alpha}}{(n+1)!\alpha^{(n+1)}}$.

For $n = 0$:

$$|y_1(x) - y_0(x)| = \left|\int_0^x s^{\alpha-1} f(s, y_0(s))ds\right| \quad \le \int_0^x s^{\alpha-1} |f(s, y_0(s))|ds \quad \le M\int_0^x s^{\alpha-1}ds = \frac{Mx^\alpha}{\alpha}$$

Now assume it is true for $n = k$:

$$|y_{k+1}(x) - y_k(x)| \le \frac{ML^k x^{(k+1)\alpha}}{(k+1)!\alpha^{(k+1)}} = \frac{ML^k}{(k+1)!}\left(\frac{x^\alpha}{\alpha}\right)^{k+1}.$$ Then

$$|y_{k+2}(x) - y_{k+1}(x)| = \left|\int_0^x s^{\alpha-1}[f(s, y_{k+1}(s)) - f(s, y_k(s))]ds\right|$$

$$\le \int_0^x s^{\alpha-1} |f(s, y_{k+1}(s)) - f(s, y_k(s))|ds$$

$$\le L\int_0^x s^{\alpha-1} |y_{k+1}(s) - y_k(s)|ds \le \frac{L \cdot L^k M}{(k+1)!\alpha^{k+1}}\int_0^x s^{\alpha-1} s^{(k+1)\alpha}ds$$

$$= \frac{L^{k+1}M}{(k+1)!\alpha^{k+1}}\int_0^x s^{(k+2)\alpha-1}ds$$

$$= \frac{L^{k+1}M}{(k+1)!\alpha^{k+1}} \frac{x^{(k+2)\alpha}}{(k+2)\alpha} = \frac{L^{k+1}M.x^{(k+2)\alpha}}{(k+2)!\alpha^{k+2}}$$

So $|y_{k+2}(x) - y_{k+1}(x)| \le \frac{L^{k+1}M.x^{(k+2)\alpha}}{(k+2)!\alpha^{k+2}}$

The induction now is completed. Now to show that $\{y_n\}_{n\in\mathbb{N}}$ is convergent we need to find a bound for $|y_n(x)|$.

$$|y_n| = \left|y_0 + \sum_{i=1}^n (y_i - y_{i-1})\right| \le |y_0| + \sum_{i=1}^n |y_i - y_{i-1}| \le |y_0| + \sum_{i=1}^n \frac{ML^{i-1}x^{i\alpha}}{i!\alpha^i} = |y_0| + \frac{M}{L}\left[\sum_{i=1}^n \frac{(L\frac{x^\alpha}{\alpha})^i}{i!} - 1\right]$$

$$\le |y_0| + \frac{M}{L}\left(e^{L\frac{x^\alpha}{\alpha}} - 1\right) \le |y_0| + \frac{M}{L}\left(e^{L\left(\frac{h^\alpha}{\alpha}\right)} - 1\right)$$

Hence, the term $|y_0| + \frac{M}{L}\left(e^{L\left(\frac{h^\alpha}{\alpha}\right)} - 1\right)$ is an upper bound for the series $\{y_n(x)\}_{n\in N}$. By this, we demonstrated that the series converges uniformly on I to a continuous function $y(x)$.

As $\{y_n(x)\}_{n\in N}$ converges to a continuous function $y(x)$ then take the limit to both sides of:

$$y_n(x) = y_0 + \int_0^x s^{\alpha-1} f(s, y_{n-1}(s))ds$$

$\lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty} \int_0^x s^{\alpha-1} f(s, y_{n-1}(s))ds$, then $y(x) = y_0 + \int_0^x s^{\alpha-1} f(s, y(s))ds$

This means there is a solution $y(x)$ on I where $\{y_n(x)\}_{n\in N}$ converges uniformly to $y(x)$ which is a solution of the CIVB (1).

Step(3) (uniqueness): Assuming there is another solution that satisfies the CIVP in (1), the other solution is $w(x)$.

Now letting $u(x) = |y(x) - w(x)|$, it is clear that:

$u(0) = 0$, Because $y(x)$ and $w(x)$ are both solutions to the same CIVP.

$$u(x) = |y(x) - w(x)| = \left|\int_0^x s^{\alpha-1}\left[f(s, y(s)) - f(s, w(s))\right]ds\right|$$

$$\le \int_0^x s^{\alpha-1} \left| f(s, y(s)) - f(s, w(s))\right| ds \le \int_0^x s^{\alpha-1} L|y(s) - w(s)|ds = \int_0^x s^{\alpha-1} L u(s)ds ,$$

Then we get: $u(x) \le \int_0^x L s^{\alpha-1} u(s)ds$

From Gromwall's inequality (Theorem1) and because, $L > 0$ and $u(x) > 0$ We get $u(x) \le u(0)e^{L\frac{x^\alpha}{\alpha}} = 0$

This shows that $u(x) = 0$. Then $|y(x) - w(x)| = 0$. Then $y(x) = w(x)$.

From this, we conclude that the solution of the CIVP (1) is unique and exists.

At last the proof of theorem 2 is completed.

3. APPLICATIONS AND EXAMPLES

In this section, we post some numerical examples to illustrate our method using the Picard's iteration to find up to three approximations of the solution. We plot the solutions as well as the approximations for comparison, these three examples are played an important role in the engineering and their applications, the first and second are modeling of the exponential growth or decay depending on the y coefficient. While the third example shows a non linear CFDE and its solution the fractional tan function.

Example 1: Consider the CFIVP: $T_\alpha(t)(x) = y, y(0) = 1$

Here $f(x, y) = y$

$$y_1 = y_0 + \int_0^x s^{\alpha-1}.f(s, y_0)ds = 1 + \int_0^x s^{\alpha-1}.1ds = 1 + \frac{x^\alpha}{\alpha}.$$

$$y_2 = y_0 + \int_0^x s^{\alpha-1} f(s, y_1(s))ds = 1 + \int_0^x s^{\alpha-1}\left[1 + \frac{s^\alpha}{\alpha}\right]ds \; y_2(x) = 1 + \frac{x^\alpha}{\alpha} + \frac{x^{2\alpha}}{2\alpha^2}$$

$$y_3 = 1 + \int_0^x s^{\alpha-1} f(s, y_2(s))ds = 1 + \int_0^x s^{\alpha-1}\left[1 + \frac{s^\alpha}{\alpha} + \frac{s^{2\alpha}}{2\alpha^2}\right]ds \; y_3(x) = 1 + \frac{x^\alpha}{\alpha} + \frac{x^{2\alpha}}{2\alpha^2} + \frac{x^{3\alpha}}{6\alpha^3}$$

From the above series of functions $y_1, y_2, y_3, ...$ we find that $y_n = \sum_{i=0}^{n} \frac{(\frac{x^\alpha}{\alpha})^i}{i!}$.

Then $y(x) = \lim_{x\to\infty} y_n(x) = e^{\frac{x^\alpha}{\alpha}}$. See figure 1, we take $\alpha = 0.25$ as an example.

Example 2: Solve the conformable fractional differential equation: $T_\alpha(y)(x) - x^\alpha y = 0, y(0) = 1$.

In this example, $f(x, y) = x^\alpha y \quad x_0 = 0$ and $y_0 = 1$.

$$y_1 = y_0 + \int_0^x s^{\alpha-1} f(s, y_0(s))ds = 1 + \int_0^x s^{\alpha-1}s^\alpha ds = 1 + \frac{x^{2\alpha}}{2\alpha}.$$

$$y_2 = 1 + \int_0^x s^{\alpha-1} f(s, y_1)ds = 1 + \int_0^x s^{\alpha-1}\left[s^\alpha(1 + \frac{s^{2\alpha-1}}{2\alpha})\right]ds$$

$$y_2(x) = 1 + \frac{x^{2\alpha}}{2\alpha} + \frac{x^{4\alpha}}{8\alpha^2}.$$

$$y_3 = 1 + \int_0^x s^{\alpha-1} f(s, y_2) ds = 1 + \int_0^x s^{\alpha-1} \left[s^{\alpha} \left(1 + \frac{s^{2\alpha}}{2\alpha} + \frac{s^{4\alpha}}{8\alpha^2} \right) \right] ds$$

$$y_3(x) = 1 + \frac{x^{2\alpha}}{2\alpha} + \frac{x^{4\alpha}}{8\alpha^2} + \frac{x^{6\alpha}}{48\alpha^3} = 1 + \frac{(\frac{x^{\alpha}}{2\alpha})^1}{1!} + \frac{(\frac{x^{\alpha}}{2\alpha})^2}{2!} + \frac{(\frac{x^{\alpha}}{2\alpha})^3}{3!}.$$

According to the series of functions $y_1, y_2, y_3, \ldots$ it can be concluded that, $y_n(x) = \sum_{i=0}^{n} \frac{(\frac{x^{2\alpha}}{2\alpha})^i}{i!}$.

So $y(x) = \lim_{x \to \infty} y_n(x) = e^{\frac{x^{(2\alpha)}}{(2\alpha)}}$. See figure 2 we take $\alpha = 0.3$ as an example.

Example 3: The nonlinear CFDE. $T_{\alpha}(y)(x) - y^2 = 1,\ y(0) = 0$.

Her $f(x, y) = 1 + y^2$, and $x_0 = 0\ and\ y_0 = 0$.

$$y_1 = y_0 + \int_0^x s^{\alpha-1} f(s, y_0(s)) ds = \int_0^x s^{\alpha-1} ds = \frac{x^{\alpha}}{\alpha}$$

$$y_2 = y_0 + \int_0^x s^{\alpha-1} f(s, y_1(s)) ds = \int_0^x s^{\alpha-1} (1 + (\frac{x^{\alpha}}{\alpha})^2) ds = \frac{x^{\alpha}}{\alpha} + \frac{x^{3\alpha}}{3\alpha^3}$$

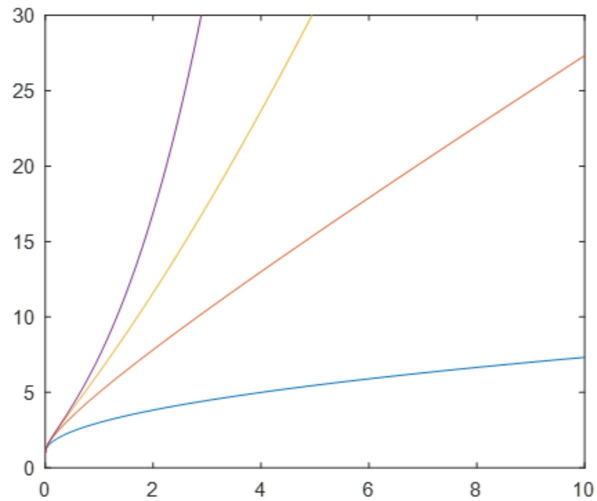
$$y_3 = y_0 + \int_0^x s^{1-\alpha} f(s, y_2(s)) ds = \int_0^x s^{1-\alpha} \left[1 + (\frac{s^{\alpha}}{\alpha} + \frac{s^{3\alpha}}{3\alpha^3})^2 \right] ds = \frac{x^{\alpha}}{\alpha} + \frac{x^{3\alpha}}{3\alpha^3} + \frac{2x^{5\alpha}}{3 \cdot 5\alpha^5} + \frac{x^{7\alpha}}{7 \cdot 9\alpha^7}.$$

$$y_4 = y_0 + \int_0^x s^{\alpha-1} f(s, y_3) ds$$

$$= \int_0^x s^{\alpha-1} (1 + \left(\frac{s^{\alpha}}{\alpha} + \frac{s^{3\alpha}}{3\alpha^3} + \frac{2s^{\alpha}}{3 \cdot 5\alpha^5} + \frac{s^{7\alpha}}{7 \cdot 9\alpha^7} \right)^2) ds = \frac{x^{\alpha}}{\alpha} + \frac{x^{3\alpha}}{3\alpha^3} + \frac{2x^{5\alpha}}{3 \cdot 5\alpha^5} + \frac{17x^{7\alpha}}{5 \cdot 7 \cdot 9\alpha^7} + \frac{38x^{9\alpha}}{5 \cdot 7 \cdot 9 \cdot 9\alpha^9}$$

From the results of the series of functions: $y_1, y_2, y_3, y_4, \ldots$ we get that this is the McLaurin series of $\tan(\frac{x^{\alpha}}{\alpha})$

Then:$y(x) = \tan(\frac{x^{\alpha}}{\alpha})\ where\ |x| \leq (\frac{\alpha\pi}{2})^{\frac{1}{\alpha}}$. See figure 3, we take $\alpha = 0.5$ as an example.



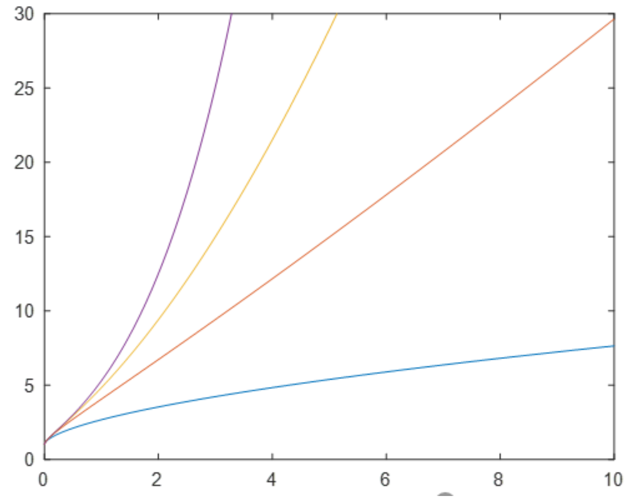


Fig. 1 : $y(x) = e^{\frac{x^{\alpha}}{\alpha}}$, $\alpha = 0.5$

Fig.2. $y(x) = e^{\frac{x^{2\alpha}}{2\alpha}}$, $\alpha = 0.3$.

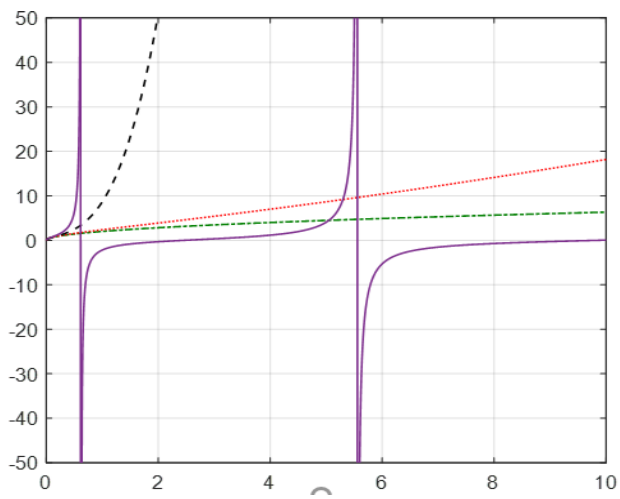


Fig. 3 : $y(x) = tan\ (e^{\frac{x^{\alpha}}{\alpha}})$, $\alpha = 0.5$

4. CONCLUSION

Conformable fractional calculus is becoming one of the leading strategies to deal with fractional differential or integral equations because the kind of relationship between the integer-order and the fractional-order relation $T_{\alpha}(y)(x) = x^{1-\alpha}\dfrac{dy}{dx}$. In this paper, we rewrote Gromwall's inequality in the case of CFC and the proof of the existence and uniqueness of the CFIVP with help of the Lipschitz condition.

Furthermore, in this paper, we established Picard's iteration of approximation solutions of the CFIVP because it gives us a manageable way to find an approximate solution for the CFIVP which always converges to the unique solution if the correct conditions are applied.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] B. Ross, ed., Fractional calculus and its applications: proceedings of the international conference held at the University of New Haven, June 1974, Springer, Berlin, 1975.

[2] K. Diethelm, The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type, Springer-Verlag, Heidelberg ; New York, 2004.

[3] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic Press, San Diego, 1999.

[4] V.V. Kulish, J.L. Lage, Application of Fractional Calculus to Fluid Mechanics, J. Fluids Eng. 124 (2002), 803–806.

[5] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014), 65–70.

[6] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, J. Math. Anal. Appl. 328 (2007), 1075–1081.

[7] K.S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley, New York, 1993.

[8] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, 1st ed, Elsevier, Amsterdam ; Boston, 2006.

[9] Z. Denton, A.S. Vatsala, Fractional integral inequalities and applications, Computers & Mathematics with Applications. 59 (2010), 1087–1094.

[10] Rainey Lyons, Aghalaya Vatsala, Ross Chiquet, Picard's Iterative Method for Caputo Fractional Differential Equations with Numerical Results, Mathematics. 5 (2017), 65.