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ON THE MIDDLE CN-DOMINATING GRAPHS

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Abstract. Let $G = (V, E)$ be a graph and $A(G)$ is the collection of all minimal CN-dominating sets of G. The middle CN-dominating graph of G is the graph denoted by $M_{cnd}(G)$ with vertex set the disjoint union of $V \cup A(G)$ and (u, v) is an edge if and only if $u \cap v \neq \phi$ whenever $u, v \in A(G)$ or $u \in v$ whenever $u \in v$ and $v \in A(G)$. In this paper, characterizations are given for graphs whose middle CN-dominating graph is connected and $K_p \subseteq M_{cnd}(G)$. Other properties of middle CN-dominating graphs are also obtained.

Keywords: CN-Dominating Graph; Minimal CN-Dominating set; CN-Domination number.

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1. Introduction

All the graphs considered here are finite, undirected with no loops and multiple edges. A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. The connectivity $k = k(G)$ of a graph G is the minimum number of vertices whose removal results a disconnected or trivial graph. Analogously the line-connectivity

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$\lambda = \lambda(G)$ is the minimum number of lines whose removal results a disconnected or trivial graph. For terminology and notations not specifically defined here we refer reader to [2]. For more details about domination number and neighbourhood number and their related parameters, we refer to [3], [4], and [9].

Let G be simple graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ for $i \neq j$, the common neighborhood of the vertices v_i and v_j, denoted by $\Gamma(v_i, v_j)$, is the set of vertices, different from v_i and v_j , which are adjacent to both v_i and v_j. A subset D of V is called common neighbourhood dominating set (CN-dominating set) if for every vertex $v \in V - D$ there exists at least one vertex $u \in D$ such that $uv \in E(G)$ and $|\Gamma(u, v)| \geq 1$, where $|\Gamma(u, v)|$ is the number of common neighbourhood between the vertices u and v. The minimum cardinality of such CN-dominating set denoted by γ_{cn} and is called common neighbourhood domination number (CN-domination number) of G. It is clear that CN-domination number is defined for any graph. A common neighbourhood dominating set D is said to be minimal common neighbourhood dominating set if no proper subset of D is common neighbourhood dominating set. A minimal common neighbourhood dominating set D of maximum cardinality is called Γ_{cn}-set and its cardinality is denoted by Γ_{cn}. Let $u \in V$. The CN-neighbourhood of u denoted by $N_{cn}(u)$ is defined as $N_{cn}(u) = \{v \in N(u) : |\Gamma(u, v)| \geq 1\}$. The cardinality of $N_{cn}(u)$is denoted by $d_{cn}(u)$ in G, and $N_{cn}[u] = N_{cn}(u) \cup \{u\}$. The maximum and minimum common neighbourhood degree of a point in G are denoted respectively by $\Delta_{cn}(G)$ and $\delta_{cn}(G)$. That is $\Delta_{cn}(G) = max_{u \in V}|N_{cn}(u)|$, $\delta_{cn}(G) = min_{u \in V}|N_{cn}(u)|$. A subset S of V is called a common neighbourhood independent set (CN-independent set), if for every $u \in S, v \notin N_{cn}(u)$ for all $v \in S - \{u\}$. It is clear that every independent set is CN-independent set. An CN-independent set S is called maximal if any vertex set properly containing S is not CN-independent set, The maximum cardinality of CN-independent set is denoted by β_{cn}, and the lower CN-independence number i_{cn} is the minimum cardinality of the CN-maximal independent set. An edge $e = uv \in E(G)$ is said to be common neighbourhood edge(CN-edge) if $|\Gamma(u, v)| \geq 1$. A subset S of V is called Common neighbourhood vertex covering (CN-vertex covering) of G if for CN-edge $e = uv$ either $u \in S$ or $v \in S$. The minimum cardinality of CN-vertex covering of G is called the CN-covering number of G and denoted by $\alpha_{cn}(G)$.

For more details about CN-dominating set see [1].

Let S be a finite set and $F = \{S_1, S_2, ..., S_n\}$ be a partition of S. Then the intersection graphs $\Omega(F)$ of F is the graph whose vertices are the subsets in F and in which two vertices S_i and S_j are adjacent if and only if $S_i \cap S_j \neq \phi$. Kulli and Janakiram introduced many classes of intersection graphs in the field of domination theory see [5-8]. The purpose of this paper is to introduce a new class of intersection graphs called The middle CN-dominating graph $M_{cnd}(G)$ of a graph G, we get characterizations for graphs whose middle CN-dominating graph is connected and $K_p \subseteq M_{cnd}(G)$. Other interesting properties of middle CN-dominating graphs are also obtained.

3. Main results

Let $G = (V, E)$ be a graph and S be the collection of minimal CN-dominating set of G. The middle CN-dominating graph of G is the graph denoted by $M_{cnd}(G)$ with vertex set the disjoint union $V(G) \cup S$ and uv is an edge if and only if $u \cap v \neq \phi$ whenever $u, v \in S$ or $u \in v$ whenever $u \in V(G)$ and $v \in S$.

Example. In figure 1a we have the graph G the middle CN-dominating sets are $A = \{1, 3, 5, 6\}, B = \{1, 3, 5, 7\}, C = \{3, 4, 5\}, D = \{2, 3, 5, 6\}, E = \{2, 3, 5, 7\}$ and the graph $M_{cnd}(G)$ is showen in Figure 1b.

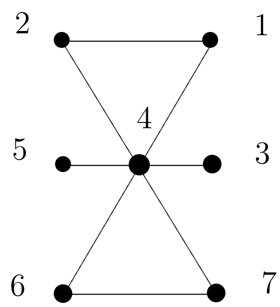

Figure 1a

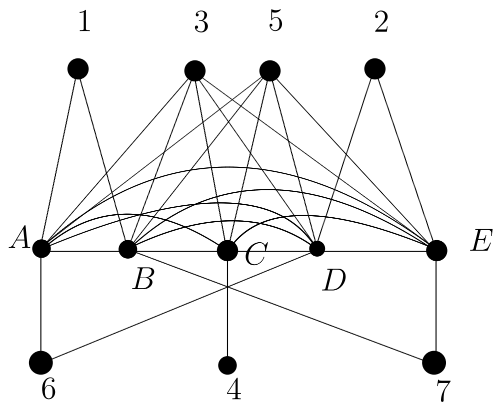

Figure 1b

Proposition 2.1. *$M_{cnd}(G) \cong K_{1,p}$ if and only if G is triangle-free graph.*

Proof. Suppose that G is triangle-free graph. Then there exists only one minimal CN-dominating set and from the definition of $M_{cnd}(G)$ every vertex in G is adjacent to this minimal CN-dominating set in $M_{cnd}(G)$, that is $M_{cnd}(G)$ is isomorphic to $K_{1,p}$.
Conversely, assume that $M_{cnd}(G)$ is isomorphic to $K_{1,p}$ and if it is possible G is not triangle-free graph, then there exists at least two minimal CN-dominating set S and $S\prime$ and $\Delta_{cn}(M_{cnd}(G)) < p-1$, a contradiction. Hence G triangle-free graph.

Proposition 2.2. *$M_{cnd}(G) = pK_2$ if and only if $G = K_p$.*

Proof. Suppose that $G = K_p$. Then clearly each vertex in G will form a minimal CN-dominating Set. Hence $M_{cnd}(G) = pK_2$.
Conversely, Suppose $M_{cnd}(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal CN-dominating set S containing two vertices of G. Then S will form p_3 in $M_{cnd}(G)$, a contradiction. Hence $G = K_p$.

Theorem 2.3. *Let G be a graph with p vertices and q edges. $M_{cnd}(G)$ is a graph with $2P$ vertices and p edges if and only if $G \cong K_p$.*

Proof. Let $M_{cnd}(G)$ be $(2p, p)$-graph. Since only the the graph pK_2 is of $2p$ vertices and p edges, then by Proposition 2.2. $G \cong K_p$.

Theorem 2.4. *Let G be a graph with p vertices and q edges. $M_{cnd}(G)$ is a graph with $P+1$ vertices and p edges if and only if G is triangle-free graph.*

Proof. Let $M_{cnd}(G)$ be a graph with $P+1$ vertices and p edges that means $M_{cnd}(G) \cong K_{1,p}$, and by Proposition 2.1. G is triangle-free graph. Conversely, Let G be triangle-free graph . Then by Proposition 2.1. $G \cong K_{1,p}$ which is of $p+1$ vertices and P edges. In the next theorem characterizations are given for graphs whose middle CN-dominating graph is connected.

Theorem 2.5. *For any graph G with at least two vertices $M_{cnd}(G)$ is connected if and only if $\Delta_{cn}(G) \leq p-1$.*

Proof. Let $\Delta_{cn}(G) \leq p-1$ and suppose there is no minimal CN-dominating containing both u and v. Then there exists a vertex w in $V(G)$ such that w is neither adjacent to u nor v. Let D and $D\prime$ be two maximal CN-independent set containing u, w and v, w respectively since every maximal CN-independent set is minimal CN-dominating set, u and v are connected by the path $uDD'v$. Hence $M_{cnd}(G)$ is connected.
Conversely, suppose $M_{cnd}(G)$ is connected. let $\Delta_{cn}(G) = p-1$ and let u is a vertex such that $d_{cn}(u) = p-1$. Then $\{u\}$ is minimal CN-dominating set and G has at least two vertices , it is clear that G has no CN-isolated vertices, then $V-D$ containing minimal CN-dominating set say D'. Since $D \cup D' = \phi$ in $M_{cnd}(G)$ there is one path joining u to any vertex of $V-D$, this implies that $M_{cnd}(G)$ is disconnected, a contradiction. Hence $\Delta_{cn}(G) \leq p-1$.

Corollary 2.6. *Let $G = (V, E)$ be a graph and u, v any two vertices in $V(G)$. Then $d(u, v)_{M_{cnd}(G)} \leq 3$, where $d(u, v)_{M_{cnd}(G)}$ is the distance between the vertex u and v in the graph $M_{cnd}(G)$.*

Theorem 2.7. *For any graph G, $K_t \subseteq M_{cnd}(G)$ if and only if G contains at least one CN-isolated vertex, where t is the number of minimal CN-dominating set in G.*

Proof. If G has CN-isolated vertex then the subgraph of $M_{cnd}(G)$ which induced by the set of all minimal CN-dominating sets of G is complete graph K_t. Hence $K_p \subseteq M_{cnd}(G)$.

Conversely, Suppose $K_t \subseteq M_{ed}(G)$, then it is clear that the vertices of K_t are the minimal CN-dominating sets of G. Therefor there exists at least one CN-isolated vertex in G.

Theorem 2.8. *For any graph G, $M_{cnd}(G)$ is either connected or it has at least one component which is K_2.*

Proof. If $\Delta_{cn}(G) \leq p-1$, then by Theorem 2.5. M_{cnd} is connected, and by Proposition 2.2. If G is complete clear M_{cnd} is connected. Hence we must prove the case $\delta_{cn}(G) < \Delta_{cn}(G) = p-1$.

Let $\{u_1 ... u_s\}$ is the set of vertices in G such that $d_{cn}(u_i) = p-1$, where $i = 1, ..., t$, then it is clear $\{u_i\}$ is minimal CN-dominating set. Then each vertex u_i with the minimal corresponding CN-dominating set $\{u_i\}$ $i = 1, ...s$ form component isomorphic to K_2. Hence has at least one component which is K_2.

Theorem 2.9. *For any graph G with $\Delta_{cn}(G) < p-1$, $diam(M_{cnd}(G)) \leq 5$.*

Proof. Let G be any graph and $\Delta_{cn}(G) < p-1$. Then by Theorem 2.5. $M_{ed}(G)$ is connected, suppose u, v be any two vertices in $M_{cnd}(G)$, then we have the following cases:

Case 1: Let u, v be any two vertices in $V(G)$. Then by Corollary 2.6. $d(u,v)_{M_{cnd}(G)} \leq 3$.

Case 2: Suppose $u \in V(G)$ and $v = D$ be a minimal CN-dominating set in G, If $u \in D$ then $d(u,v) = 1$ and if $u \notin D$ then there exists a vertex $w \in D$ adjacent to u. Hence $d(u,v)_{M_{cnd}(G)} = d(u,w)_{M_{cnd}(G)} + d(w,v)_{M_{cnd}(G)}$ and from corollary 2.6. We have $d(u,w)_{M_{cnd}(G)} \leq 3$. Hence $d(u,v)_{M_{cnd}(G)} \leq 4$

Case3: Suppose that both u and v are not in $V(G)$, and $u = D$, $v = D'$ are two minimal CN-dominating sets if D and D' are adjacent,then $d(u,v)_{M_{cnd}(G)} = 1$, suppose that D and D' are not adjacent then every vertex $w \in D$ is adjacent to some vertex $x \in D'$ and vice versa. Hence $d(u,v) = d(u,w) + d(w,x) + d(x,v) \leq 5$.

Hence $diam(M_{ed}(G)) \leq 5$.

Theorem 2.10. *Let G be a graph. Then $d_{cn}(M_{cnd}(G)) = 2$ if and only if $G = K_p$ or $G = \overline{K_2}$, where $d_{cn}(M_{cnd}(G))$ is the CN-domatic number of the graph $M_{cnd}(G)$.*

Proof. If $G = K_p$ or $G = \overline{K_2}$, then $M_{cnd}(G) = pK_2$ or $G = K_{1,2}$ by Proposition 2.2. Hence $d_{cn}(M_{cnd}(G)) = 2$. The converse is obvious.

Theorem 2.11. *For any graph G with at least one CN-isolated vertex,*

$$\beta_{cn}(G) = \beta(G) = p.$$

Proof. Let G be a graph of p vertices, and has at least one CN-isolated vertex w. Then from the definition of $M_{cnd}(G)$ if u and v any vertices in G then u and v can not be adjacent in $M_{cnd}(G)$, that is there is CN-independent containing w and of size p,and this CN-independent will be the maximal CN-independent because w is adjacent for all the minimal CN-dominating sets. Therefore $\beta_{cn}(G) = \beta(G) = p$.

Corollary 2.12.*For any graph G with at least one CN-isolated vertex, $\alpha_{cn}(G) = \alpha(G) = |S|$, where $|S|$ is the number of minimal CN-dominating set in G.*

Proof. We have for any graph G with p vertices $\alpha_{cn}(G) + \beta_{cn}(G) = \alpha(G) + \beta(G) = p$ see [1], and from Theorem 2.11. Corollary is proved.

Observation 2.13. *Let G is complete multi bipartite graph $K_{n_1,n_2,\ldots n_t}$, then $M_{cnd}(G) \cong K_{1,\sum_{i=1}^{t}} n_i$.*

Theorem 2.14. *For any graph G, $d_{cn}(G) = \beta_{cn}(M_{cnd}(G))$, if and only if G is complete graph.*

Proof. Let G be complete graph K_p. Then from Theorem 2.11. $\beta_{cn}(M_{cnd}(G)) = p$, and we have $d_{cn}(G) = p$. Hence $d_{cn}(G) = \beta_{cn}(M_{ed}(G))$.
Conversely, suppose $d_{cn}(G) = \beta_{cn}(M_{ed}(G))$. From Theorem 2.11. $\beta_{cn}(M_{cnd}(G)) = p$ implies that $d_{cn}(G) = p$. Hence $G = K_p$.

Theorem 2.15. *If $G = K_p \cup K_1$ for every $p \geq 3$, then the corona graph $K_p o K_1$ and K_p are edge-disjoint subgraphs of the graph $M_{cnd}(G)$.*

Proof. Let $G = K_p \cup K_1$. Then there exists one CN-isolated vertex in G say x and this vertex will be exist in every minimal CN-dominating set of G. Hence the vertices in $M_{cnd}(G)$ other than x which belong to $V(G)$ with the minimal CN-dominating set induced the corona subgraph $K_p o K_1$ of $M_{cnd}(G)$, and the minimal CN-dominating sets of G induced the subgraph K_p of $M_{cnd}(G)$, clearly the two subgraphs are edge-disjoint.

The vertices of the graph $M_{cnd}(G)$ can be labeling as following $v_1, ..., v_p, S_1, ..., S_n$, where $v_i, i = 1, 2, , ..., p$ are the vertices in the graph G and $S_j, j = 1, 2, ..., n$ where n is number of minimal CN-dominating sets are the minimal CN-dominating set of G. By using this labeling we can get the connectivity and line-connectivity of the graph $M_{cnd}(G)$ in the following theorem.

Theorem 2.16. *For any graph G,*

$$k(M_{cnd}(G)) = \lambda(M_{cnd}(G)) = min\{min(d_{M_{cnd}(G)}(v_i)), min|S_j|\}.$$

Proof. We consider two cases:

Case 1: Let $u \in \{v_1, v_2, ..., v_p\}$ and u have the minimum degree among all the vertices v_i in $M_{cnd}(G)$. If the degree of u is less than any vertex in $M_{cnd}(G)$. Then by deleting those vertices of $M_{cnd}(G)$ which are adjacent with u, we get disconnected graph, similarly by deleting those edges of $M_{cnd}(G)$ which are incident with u, results in a disconnected graph.

Case 2: If $u \in \{S_1, S_2, ..., S_n\}$ and u have the minimum degree among all the vertices S_i in $M_{cnd}(G)$. If the degree of u is less than any vertex in $M_{cnd}(G)$. Then by deleting those vertices of $M_{cnd}(G)$ which are adjacent with u, we get disconnected graph, similarly by deleting those edges of $M_{cnd}(G)$ which are incident with u, results in a disconnected graph.

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