Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 5, 1620-1632
https://doi.org/10.28919/jmcs/4700
ISSN: 1927-5307

RELATION-THEORETIC ψ-CONTRACTIVE MAPPINGS IN CONE PENTAGONAL METRIC SPACES

SHISHIR JAIN, POOJA CHAUBEY*

Department of Mathematics, Shri Vaishnav Vidyapeeth Vishwavidyalaya, Sanwer Road, Indore (M.P.) 453331, India

Abstract. The presented paper introduces the concept of relation-theoretic ψ-contractive mappings in cone pentagonal metric space. Some fixed point results for this new class of mappings are proved and illustrated with suitable examples. The results proved herein generalize the results of Abba Auwalu [2] and Garg et.al. [4].

Keywords: cone pentagonal metric space; fixed point; relation-theoretic contraction principle.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

Ran and Reurings[6] in 2003 established a new version of BCP (Banach contraction principle) in partially ordered metric spaces. In 2007, Huang and Zhang[5] defined cone metric space. Inspired by the work of Huang and Zhang[5], Azam et. al.[3] constructed a new space known as cone rectangular metric space and the famous Banach contraction principle (BCP) is proved. In 2012, Rashwan and Saleh [7] improved and extended the result of Azam et. al.[3] by eliminating the settings of normality. Alam and Imdad [2] established BCP on a complete metric

*Corresponding author
E-mail address: chaubey.pooja8991@gmail.com
Received May 11, 2020

space equipped with a binary relation and gave a new approach to prove fixed point theory. After that Garg et.al.[4] defined cone pentagonal metric spaces and established the proof of BCP by supposing the condition of normality of cone. In this paper, we prove Banach contraction principle using relation theoretic contraction without assuming the normality of cone.
Now recall some definitions from literature which are useful in our work.

2. PRELIMINARIES

Definition 1. [5] *Let E be a real Banach space and $P \subset E$ is a cone if and only if:*

- **(I):** $P \neq \phi$ *, closed and* $P \neq \{\theta\}$.
- **(II):** *if* $c,d \in \mathbb{R}, c,d \geq 0$ *and* $y,z \in P$ *implies that* $cy + dz \in P$.
- **(III):** $y \in P$ *and* $-y \in P$ *implies that* $y = \theta$.

For a given cone P which is a subset of E, we defined a partial ordering $\preceq$ with respect to P by $y \preceq z \Longleftrightarrow z - y \in P$. We could write $y \prec z$ which implies $y \preceq z$ but $y \neq z$ while $y \ll z$ will imply $z - y \in int(P)$ where the $int(P)$ denotes the interior of P. If in a cone P $\exists$ number $k > 0$ with $\forall\, y,z \in E$

$$\theta \preceq y \preceq z \Rightarrow \parallel y \parallel \leq k \parallel z \parallel$$

then P is called normal cone and normal constant of P is the least positive number k which satisfy the above.
In our paper we consistently assumed E is a real Banach space, P be a cone with non-empty interior (solid cone) and $\preceq$ be a partial order with respect to P.

Definition 2. [5] *A cone metric space (X,d) over Banach space E is an ordered pair where X is a non-empty set. Let $d : X \times X \to E$ is a map with $\forall\, r,s,t \in X$,*

- **(i):** $\theta \preceq d(r,s)$ *and* $d(r,s) = \theta \Longleftrightarrow r = s$*;*
- **(ii):** $d(r,s) = d(s,r)$*;*
- **(iii):** $d(r,s) \preceq d(r,t) + d(t,s)$.

Definition 3. [3] *A cone rectangular metric space (X,d_1) over Banach space E where X is a non-empty set. Let $d_1 : X \times X \to E$ is a map with $\forall\, r,s \in X$,*

- **(R1):** $\theta \preceq d_1(r,s)$ *and* $d_1(r,s) = \theta \Longleftrightarrow r = s$*;*

(R2): $d_1(r,s) = d_1(s,r)$;

(R3): $d_1(r,s) \preceq d_1(r,w) + d_1(w,t) + d_1(t,s) \forall r,s \in X$ *and* $\forall$ *distinct point* $w,t \in X - \{r,s\}$ *this characteristic is known as rectangular.*

Definition 4. [4] *A cone pentagonal metric space* (X,ρ) *over Banach space E where X is a non-empty set. Let* $\rho : X \times X \to E$ *is a map with* $\forall r,s \in X$,

(i): $\theta \preceq \rho(r,s)$ *and* $\rho(r,s) = \theta \iff r = s$;

(ii): $\rho(r,s) = \rho(s,r)$;

(iii): $\rho(r,s) \preceq \rho(r,t) + \rho(t,w) + \rho(w,u) + \rho(u,s) \quad \forall r,s,t,w,u \in X$ *and* $\forall$ *distinct point* $t,w,u \in X - \{r,s\}$ *this characteristic is known as pentagonal.*

3. Criteria of Convergence [4]

Definition 5. *Let* (X,ρ) *be a cone pentagonal metric space, a sequence* $\{u_m\} \in X$ *is convergent and converge to u if for every* $c \in int(P)$ $\exists$ *a positive integer* m_0 *with* $\rho(u_m,u) \ll c \; \forall m > m_0$. *In this case u is called the limit of* u_m, *denoted by* $\lim_{m\to\infty} u_m = u$.

Definition 6. *Let* (X,ρ) *be a cone pentagonal metric space, a sequence* $\{u_m\} \in X$ *is called Cauchy sequence if* $\rho(u_m,u_n) \ll c$, $\forall m,n > m_0$ *and* (X,ρ) *is called complete cone pentagonal metric space if every Cauchy sequence* $\{u_m\}$ *converges to* $u \in X$.

Definition 7. [7] *Let* $P \subset E$ *is a cone defined as above and* φ *be the set of non decreasing continuous functions* $\psi : P \to P$ *such that*

(a1): $\theta \preceq \psi(r) \prec r \; \forall \; r \in P \setminus \{\theta\}$.

(a2): *The series* $\sum_{n\geq 0} \psi^n(r)$ *converge* $\forall \; r \in P \setminus \{\theta\}$.

From (a1), we have $\psi(\theta) = \theta$ *and from (a2), we have* $\lim_{n\longrightarrow 0} \psi^n(r) = \theta \; \forall \; r \in P \setminus \{\theta\}$.

Definition 8. [5] *Let* (X,d) *be a cone metric space and* $P \subset E$ *is a cone (not necessarily normal) and for* $b,d,r,s,t \in E$, *then if*

(a1): $b \preceq kb$ *and* $k \in [0,1)$ *implies* $b = \theta$;

(a2): $\theta \preceq r \preceq d$ *for each* $\theta \preceq d$, *implies* $r = \theta$;

(a3): $r \preceq s$ *and* $s \preceq t$ *implies* $r \preceq t$;

(a4): *$d \in int(P)$ and $b_n \to \theta$ implies $\exists\, n_0 \in N$: $\forall n > n_0$, $b_n \ll d$.*

Definition 9. [1] *Let (X,ρ) be a complete cone pentagonal metric space and $\{u_n\} \in X$ be a Cauchy sequence. Assume that $\exists$ a natural number $\mathbb{N}$ with*

- **(a):** *$u_n = u_m \; \forall\, n,m > \mathbb{N}$.*
- **(b):** *u_n, u are distinct points in $X \; \forall\, n > \mathbb{N}$.*
- **(c):** *v_n, v are distinct points in $X \; \forall\, n > \mathbb{N}$.*
- **(d):** *$u_n \longrightarrow u$ and $v_n \longrightarrow v$ as $n \longrightarrow \infty$ implies $u = v$.*

Now recall some useful terminologies about relations given by A.Alam. et.al.[2].

Definition 10. *Let $X \neq \phi$ and $\mathscr{T}$ be a binary relation on X. Let R be a self map on X.*

(1) s and t are $\mathscr{T}$-comparative if $(s,t) \in \mathscr{T}$ or $(t,s) \in \mathscr{T}$. This is shown by $[s,t] \in \mathscr{T}$.

(2) Sequence $\{s_n\} \in X$ is said to be $\mathscr{T}$-preserving if $(s_n, s_{n+1}) \in \mathscr{T} \; \forall\, n \in \mathbb{N}$.

(3) Relation $\mathscr{T}$ is said to be R-closed if for every $(s,t) \in \mathscr{T}$, we have $(Rs, Rt) \in \mathscr{T}$.

(4) Let $C \subset X$ is said to be $\mathscr{T}$-directed if for each $s,t \in E$, $\exists\, w \in X$ with $(s,w) \in \mathscr{T}$ and $(t,w) \in \mathscr{T}$.

(5) In $\mathscr{T}$ a path of length k (where $k \in \mathbb{N}$) for $s,t \in X$, from s to t is a finite sequence $\{w_j\}_{j=0}^{k} \subset X$ satisfies the following conditions:

- *$w_0 = r$ and $w_r = s$;*
- *$(w_j, w_{j+1}) \in \mathscr{T}$ for each $0 \leq j \leq k-1$.*

4. Main Results

Definition 11. *Let (X,ρ) be a complete cone pentagonal metric space, $\mathscr{T}$ be binary relation on X is said to be d-self closed if for every $\mathscr{T}$- preserving sequence such that $u_n \longrightarrow u \in X$ as $n \longrightarrow \infty$, then $\exists$ a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathscr{T} \; \forall\, k \in \mathbb{N}$.*

Definition 12. *Let (X,ρ) is a complete cone pentagonal metric space over Banach space E, φ be the collection of all cone comparison function on P and $\mathscr{T}$ is a binary relation on X. Let R is the self map on X. Then R is said to be relation theoretic ψ-contractive mapping if $\exists$ a comparison function $\psi \in \varphi$ such that*

$$\rho(Ru,Rv) \preceq \psi(\rho(u,v)) \quad forall \;\; u,v \in X \;\; with \;\; (u,v) \in \mathscr{T} \tag{1}$$

Theorem 1. *Let (X,ρ) be a complete cone pentagonal metric space over Banach space E. Let R be a relation theoretic ψ-contractive map such that the following conditions are satisfied:*

(I): *$\mathscr{T}$ is R-closed.*

(II): *$\exists\, u_0 \in X$ with $(u_0, R^r u_0) \in \mathscr{T} \;\forall\; r \in \mathbb{N}$*

(III): *R is continuous on X.*

Then R has a fixed point in X.

Proof. Consider $u_0 \in X$ with $(u_0, R^r u_0) \in \mathscr{T}$ for all $r \in \mathbb{N}$. Take a sequence $\{u_n\}$ in X as

$$u_{n+1} = Ru_n \;\forall\; n = 0,1,2,\ldots$$

Therefore $(u_0,u_r) \in \mathscr{T}$ for all $r \in \mathbb{N}$. Therefore, from (I) we get $(Ru_0, Ru_r) = (u_1, u_{r+1}) \in \mathscr{T} \;\forall\; r \in \mathbb{N}$. By induction we obtain,

$$(u_n, u_{r+n}) \in \mathscr{T} \text{ for all } r,n \in \mathbb{N}$$

Assume $u_n = u_{n+1}$ for some $n \in \mathbb{N}$, it implies that T has a fixed point $u = u_n$. Therefore suppose $u_n \neq u_{n+1} \;\forall\; n \in \mathbb{N}$.

As R is a relation theoretic ψ-contractive mapping, then by (1), it follows,

$$\begin{aligned} \rho(u_n,u_{n+1}) &= \rho(Ru_{n-1},Ru_n) \\ &\preceq \psi(\rho(u_{n-1},u_n)) \\ &\preceq \psi^2(\rho(u_{n-2},u_{n-1})) \\ &\;\;\vdots \\ &\preceq \psi^n(\rho(u_0,u_1)) \end{aligned} \tag{2}$$

Here, we can also consider u_0 is not a periodic point. Otherwise, when $u_0 = u_n$ for any $n \geq 2$, then

$$\begin{aligned} \rho(u_0, Ru_0) &= \rho(u_n, Ru_n) \\ &= \rho(u_n, u_{n+1}) \\ &= \psi^n(\rho(u_0, Ru_0)) \end{aligned}$$

Since $\psi \in \varphi$ therefore $\lim_{n\to\infty} \psi^n(r) = \theta$ for all $r \in P \setminus \{\theta\}$. So, by above we get that $\rho(u_0, Ru_0) = \theta$, so u_0 is a fixed point of R. Therefore, consistently considering $u_n \neq u_m$ $\forall$ distinct $n, m \in \mathbb{N}$.

It again follows,

$$\begin{aligned} \rho(u_n, u_{n+2}) &= \rho(Ru_{n-1}, Ru_{n+1}) \\ &\preceq \psi(\rho(u_{n-1}, u_{n+1})) \\ &\preceq \psi^2(\rho(u_{n-2}, u_n)) \\ &\vdots \\ &\preceq \psi^n(\rho(u_0, u_2)) \end{aligned} \tag{3}$$

Again we have,

$$\begin{aligned} \rho(u_n, u_{n+3}) &= \rho(Ru_{n-1}, Ru_{n+2}) \\ &\preceq \psi(\rho(u_{n-1}, u_{n+2})) = \rho(Ru_{n-2}, Ru_{n+1}) \\ &\preceq \psi^2(\rho(u_{n-2}, u_{n+1})) \\ &\vdots \\ &\preceq \psi^n(\rho(u_0, u_3)) \end{aligned} \tag{4}$$

In a similar manner, for $k = 1, 2, 3, \ldots,$

$$\rho(u_n, u_{n+3k+1}) = \psi^n(\rho(u_0, u_{3k+1})) \tag{5}$$

$$\rho(u_n, u_{n+3k+2}) = \psi^n(\rho(u_0, u_{3k+2})) \tag{6}$$

$$\rho(u_n, u_{n+3k+3}) = \psi^n(\rho(u_0, u_{3k+3})) \tag{7}$$

By using (2) and pentagonal property,we have

$$\begin{aligned}
\rho(u_0,u_4) &\preceq \rho(u_0,u_1)+\rho(u_1,u_2)+\rho(u_2,u_3)+\rho(u_3,u_4)\\
&\preceq \rho(u_0,u_1)+\psi(\rho(u_0,u_1))+\psi^2(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_1))\\
&\preceq \sum_{i=0}^{3}\psi^i(\rho(u_0,u_1))
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho(u_0,u_7) &\preceq \rho(u_0,u_1)+\rho(u_1,u_2)+\rho(u_2,u_3)+\rho(u_3,u_4)+\rho(u_4,u_5)+\rho(u_5,u_6)+\rho(u_6,u_7)\\
&\preceq \rho(u_0,u_1)+\psi(\rho(u_0,u_1))+\psi^2(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_1))\\
&+ \psi^4(\rho(u_0,u_1))+\psi^5(\rho(u_0,u_1))+\psi^6(\rho(u_0,u_1))\\
&\preceq \sum_{i=0}^{6}\psi^i(\rho(u_0,u_1))
\end{aligned}$$

Now, from induction for each $k=1,2,3,\ldots,$

$$\rho(u_0,u_{3k+1}) \preceq \sum_{i=0}^{3k}\psi^i(\rho(u_0,u_1)) \tag{8}$$

Also, by using (2), (3) and by pentagonal property, we have

$$\begin{aligned}
\rho(u_0,u_5) &\preceq \rho(u_0,u_1)+\rho(u_1,u_2)+\rho(u_2,u_3)+\rho(u_3,u_5)\\
&\preceq \rho(u_0,u_1)+\psi(\rho(u_0,u_1))+\psi^2(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_2))\\
&\preceq \sum_{i=0}^{2}\psi^i(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_2))
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho(u_0,u_8) &\preceq \rho(u_0,u_1)+\rho(u_1,u_2)+\rho(u_2,u_3)+\rho(u_3,u_4)+\rho(u_4,u_5)+\rho(u_5,u_6)+\rho(u_6,u_8)\\
&\preceq \rho(u_0,u_1)+\psi(\rho(u_0,u_1))+\psi^2(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_1))\\
&+ \psi^4(\rho(u_0,u_1))+\psi^5(\rho(u_0,u_1))+\psi^6(\rho(u_0,u_2))\\
&\preceq \sum_{i=0}^{5}\psi^i(\rho(u_0,u_1))+\psi^6(\rho(u_0,u_2))
\end{aligned}$$

Again from induction for each $k=1,2,3,\ldots,$

$$\rho(u_0,u_{3k+2}) \preceq \sum_{i=0}^{3k-1}\psi^i(\rho(u_0,u_1))+\psi^{3k}(\rho(u_0,u_2)) \tag{9}$$

Also, by using (2), (4) and by pentagonal property, we have

$$\begin{aligned}
\rho(u_0,u_6) &\preceq \rho(u_0,u_1)+\rho(u_1,u_2)+\rho(u_2,u_3)+\rho(u_3,u_6)\\
&\preceq \rho(u_0,u_1)+\psi(\rho(u_0,u_1))+\psi^2(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_3))\\
&\preceq \sum_{i=0}^{2}\psi^i(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_3))
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho(u_0,u_9) &\preceq \rho(u_0,u_1)+\rho(u_1,u_2)+\rho(u_2,u_3)+\rho(u_3,u_4)+\rho(u_4,u_5)+\rho(u_5,u_6)+\rho(u_6,u_9)\\
&\preceq \rho(u_0,u_1)+\psi(\rho(u_0,u_1))+\psi^2(\rho(u_0,u_1))+\psi^3(\rho(u_0,u_1))\\
&+ \psi^4(\rho(u_0,u_1))+\psi^5(\rho(u_0,u_1))+\psi^6(\rho(u_0,u_3))\\
&\preceq \sum_{i=0}^{5}\psi^i(\rho(u_0,u_1))+\psi^6(\rho(u_0,u_3))
\end{aligned}$$

From induction for each $k=1,2,3,\ldots$,

$$\rho(u_0,u_{3k+3}) \preceq \sum_{i=0}^{3k-1}\psi^i(\rho(u_0,u_1))+\psi^{3k}(\rho(u_0,u_3)) \tag{10}$$

Now, from (5)and (8)for $k=1,2,3,\ldots$, implies

$$\begin{aligned}
\rho(u_n,u_{n+3k+1}) &\preceq \psi^n(\rho(u_0,u_{3k+1}))\\
&\preceq \psi^n\sum_{i=0}^{3k}\psi^i(\rho(u_0,u_1))\\
&\preceq \psi^n[\sum_{i=0}^{3k}\psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))]\\
&\preceq \psi^n[\sum_{i=0}^{\infty}\psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \qquad (11)
\end{aligned}$$

Similarly for $k = 1,2,3,\ldots$, using inequalities (6)and (9)

$$\begin{aligned}
\rho(u_n, u_{n+3k+2}) &\preceq \psi^n(\rho(u_0, u_{3k+2})) \\
&\preceq \psi^n[\sum_{i=0}^{3k-1} \psi^i(\rho(u_0,u_1)) + \psi^{3k}(\rho(u_0,u_2))] \\
&\preceq \psi^n[\sum_{i=0}^{3k} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3)) \\
&+ \psi^{3k}(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \\
&\preceq \psi^n[\sum_{i=0}^{3k} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \\
&\preceq \psi^n[\sum_{i=0}^{\infty} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \qquad (12)
\end{aligned}$$

Again for $k = 1,2,3,\ldots$, using inequalities (7)and (10)

$$\begin{aligned}
\rho(u_n, u_{n+3k+3}) &\preceq \psi^n(\rho(u_0, u_{3k+3})) \\
&\preceq \psi^n[\sum_{i=0}^{3k-1} \psi^i(\rho(u_0,u_1)) + \psi^{3k}(\rho(u_0,u_3))] \\
&\preceq \psi^n[\sum_{i=0}^{3k} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3)) \\
&+ \psi^{3k}(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \\
&\preceq \psi^n[\sum_{i=0}^{3k-1} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \\
&\preceq \psi^n[\sum_{i=0}^{\infty} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \qquad (13)
\end{aligned}$$

Thus, by inequality (11), (12), (13) we have, for each m,

$$\rho(u_n, u_{n+m}) \preceq \psi^n[\sum_{i=0}^{\infty} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \qquad (14)$$

As from definition 6, $\sum_{i=0}^{\infty} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))$ converges, where $\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3) \in P\setminus\{\theta\}$. Definition of P implies that $\sum_{i=0}^{\infty} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3)) \in P\setminus\{\theta\}$. Hence

$$\lim_{n\longrightarrow\infty} \psi^n[\sum_{i=0}^{\infty} \psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \qquad (15)$$

Then, for given $c \gg 0$, $\exists$ positive number N_1 with,

$$\psi^n[\sum_{i=0}^{\infty}\psi^i(\rho(u_0,u_1)+\rho(u_0,u_2)+\rho(u_0,u_3))] \ll c, \forall n \geq N_1 \tag{16}$$

Thus, from (14) and (16), we have

$$\rho(u_n,u_{n+m}) \quad \ll \quad c \;\; forall \;\; n \geq N_1$$

Therefore $\{u_n\} \in X$ is a Cauchy sequence. By the completeness of X, $\exists$ a point $u \in X$ with $\lim_{n\longrightarrow\infty}\{u_n\} = \lim_{n\longrightarrow\infty} Ru_{n-1} = u$.

Now to prove that fixed point of R is u.

From the continuity of R, we have $u_{n+1} = Ru_n \longrightarrow Ru$ as $n \longrightarrow \infty$.. Since limit is unique in cone pentagonal metric space, so it implies that $u = Ru$, i.e., fixed point of R is u. □

The following theorem is independent from continuity of R.

Theorem 2. *Let (X,ρ) is a complete cone pentagonal metric space over Banach space E. Let R (self map on X) be a relation theoretic ψ-contractive map satisfying following settings:*

- **(I'):** *$\mathscr{T}$ is R-closed.*
- **(II'):** *$\exists\, u_0 \in X$ with $(u_0, R^r u_0) \in \mathscr{R}\ \forall\, r \in \mathbb{N}$*
- **(III'):** *$\mathscr{T}^s$ is d-self-closed.*

As a result, $\exists\, u \in X$ with $Ru = u$.

Proof. By using the similar proof as in above theorem, implies $\{u_n\} \in X$ is a Cauchy sequence. And $(u_n, u_{r+n}) \in \mathscr{T}\ \forall\, r,n \in \mathbb{N}$. By completeness of X, $\exists\, u \in X$ with $u_n \longrightarrow u$ as $n \longrightarrow \infty$. From hypothesis $\exists$ a subsequence $\{u_{n_k}\}$ with $(u_{n_k}, u) \in \mathscr{T}^s\ \forall\, k \in \mathbb{N}$.

Now using (1), (2) and hypothesis (III'), we obtain

$$\begin{aligned}
\rho(Ru,u) &= \rho(Ru,Ru_{n_k}) + \rho(Ru_{n_k},Ru_{n_k-1}) + \rho(Ru_{n_k-1},Ru_{n_k-2}) + \rho(Ru_{n_k-2},u) \\
&\preceq \psi(\rho(u,u_{n_k}) + \rho(u_{n_k+1},u_{n_k}) + \rho(u_{n_k},u_{n_k-1}) + \rho(u_{n_k-1},u)) \\
&\prec \rho(u,u_{n_k}) + \rho(u_{n_k+1},u_{n_k}) + \rho(u_{n_k},u_{n_k-1}) + \rho(u_{n_k-1},u)
\end{aligned}$$

Since $u_n \longrightarrow u$ as $n \longrightarrow \infty$ for every $c \in P$ such that $\theta \ll c$. We can choose natural numbers N_2, N_3, N_4, with $\rho(u,u_{n_k}) \ll \frac{c}{4}, \forall n \geq N_2$, $\rho(u_{n_k+1},u_{n_k}) \ll \frac{c}{4}, \forall n \geq N_3$ and $\rho(u_{n_k-1},u) \ll \frac{c}{4}, \forall n \geq$

N_4. As $u_n \neq u_m \; \forall \; n \neq m$, Hence,

$$\rho(Ru,u) \ll \frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}=c, \forall n \geq N,$$

Where $N = max\{N_2,N_3,N_4\}$. As c is arbitrary therefore $\rho(Ru,u) \ll \frac{c}{m}, \forall m \in N$. As $\frac{c}{m} \longrightarrow \theta$ as $m \longrightarrow \infty$. It implies that $\frac{c}{m} - \rho(Ru,u) \longrightarrow -\rho(Ru,u)$ as $m \longrightarrow \infty$. Since P is closed, $-\rho(Ru,u) \in P$. Hence $\rho(Ru,u) \in P \bigcap -P$, by definition of cone we get $\rho(Ru,u) = \theta$, and so $Ru = u$. Therefore, R has a fixed point $u \in X$.

□

Now to prove that fixed point is unique, introduced given settings:

(A): $\forall \; u,v \in X$, $\exists \; w \in X$ with $(u,w) \in \mathscr{T}^s$ and $(v,w) \in \mathscr{T}^s$.

Theorem 3. *By including (A) in the settings of theorem 1(and theorem 2) uniqueness of the fixed point of R is obtained.*

Proof. Assume u^* and v^* are two distinct fixed point of R. By (A), $\exists \; w \in X$ with $(u^*,w) \in \mathscr{T}^s$ and $(v^*,w) \in \mathscr{T}^s$. Assume $u^* \neq w$ and $v^* \neq w$. As R is a relation theoretic ψ-contractive mapping, using R-closedness of $\mathscr{T}$, we obtain

$$\begin{aligned} \rho(u^*,R^n w) &= \rho(Ru^*,R^n w) \\ &= \rho(Ru^*,RR^{n-1}w) \\ &\preceq \psi(\rho(u^*,R^{n-1}w)) \end{aligned}$$

Repetition the process as above and by the characteristic of ψ, implies

$$\rho(u^*,R^n w) \preceq \psi^n(\rho(u^*,R^{n-1}w))$$

Since $\lim_{n\longrightarrow\infty} \psi^n(\rho(u^*,R^{n-1}w)) = \theta$, by definition of cone, for every $c \in int(P)$, $\exists \; n_0 \in \mathbb{N}$ with $\rho(u^*,R^n w) \ll c \; \forall \; n > n_0$. Therefore, $R^n w \longrightarrow u^*$ as $n \longrightarrow \infty$.

With a similar process we can obtain $R^n w \longrightarrow v^*$ as $n \longrightarrow \infty$. So we obtain $u^* = u^*$. This contradiction proves the uniqueness of fixed point of R. □

The below corollary is revised form of [4].

Corollary 1. *Let (X,d) be a complete cone pentagonal metric space and $P \subset E$ be a cone (solid). Assume that the self map R on X fascinate given settings*

$$\rho(Ru,Rv) \ll s\rho(u,v)\ \forall\ u,v \in X.$$

with $s \in P$ and $s \in (0,1]$, implies that R has a unique fixed point in X and furthermore, the iterative sequence $\{R^n u\}$ converges to the fixed point.

Proof. Define $\mathscr{T} = X \times X$, that is the universal relation and $\psi(u) = su\ \forall\ s \in P$ then by theorem 1 corollary is follow. □

Example 1. *Assume $X = \{5,6,7,8,9\}$, $E = \mathbb{R}^2$, $P = \{(u,v) : u,v \geq 0\}$ cone in E. Assign $\rho : X \times X \to E$ such that:*

$$\rho(5,7) = \rho(7,5) = \rho(7,8) = \rho(8,7) = \rho(6,7) = \rho(7,6) = \rho(6,8) = \rho(8,6) = \rho(5,8) = \rho(8,5) = (1,2)$$

$$\rho(5,6) = \rho(6,5) = (4,8)$$

$$\rho(5,9) = \rho(9,5) = \rho(6,9) = \rho(9,6) = \rho(7,9) = \rho(9,7) = \rho(8,9) = \rho(9,8) = (3,6).$$

Clearly, (X,ρ) is a cone pentagonal metric space. Assign R a self map on X with:

$$R(u) = \begin{cases} 8, & \text{if } u \neq 9; \\ 6, & \text{if } u = 9; \end{cases}$$

A binary relation $\mathscr{T}$ on X is defined as $\mathscr{T} = \{(5,5),(8,8),(8,6),(6,8),(7,6),(9,6)\}$. It implies that R is a relation theoretic ψ contractive map where $\psi(u,v) = (u+1,v+1)\ \forall\ (u,v) \in P$ and by $\mathscr{T}$ is R-closed. Here R is not continuous. And for each $u \in X \Longrightarrow (u_0, R^r u_0) \in \mathscr{T}$. Since X is a closed subset of $\mathbb{R}$, therefore $\mathscr{T}$ is d-self-closed. Clearly, it can be see that 8 is the unique fixed point.

Conclusion

The present work shows that we can prove Banach contraction principle in cone pentagonal metric space by another approach i.e. using realtion theoretic contraction principle. By this approach the contractions condition becomes weakend than the ordinary contraction condition, since it requires to keep only on those point which are belong to the relation. We belief that outcomes of present work will be useful in fixed point theory.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Auwalu, Banach fixed point theorem in a Cone pentagonal metric spaces, J. Adv. Stud. Topol. 7 (2016), 60-67.

[2] A. Alam, M. Imdad, Relation-theoretic contraction principle. J. Fixed Point Theory Appl. 17 (2015), 693–702.

[3] A. Azam, M. Arshad, I. Beg, Banach contraction principle on cone rectangular metric spaces, Appl. Anal. Discrete Math. 3 (2009), 236-241.

[4] M. Garg, S. Agarwal, Banach Contraction Principle on Cone Pentagonal Metric Space, J. Adv. Stud. Topol. 3 (2012), 12-18.

[5] L.-G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468–1476.

[6] A. Ran, M. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2003), 1435-1443.

[7] R. Rashwan, S.M. Saleh, Some fixed point theorems in Cone Rectangular Metric Spaces, Math. Aeterna, 2 (2012), 573-587.