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A NOTE ON WEEK RADICAL CLASSES

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Abstract. In this article we introduce and investigate the concept of weak radical class of semirings.

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1. Introduction

In this article, we have introduced the weaker version of a radical class of semirings and established that the existing definition of the radical class of semirings is equivalent to the weaker version for additively cancellative and semisubtractive semirings. We have introduced and investigated the concept of weak radical classes for additively cancellative and semisubtractive semirings. In general addition of two k-ideals (subtractive ideals) is not a k-ideal, hence we have tried to improve some results for restricted class of semirings as given in [1].

Definition 1.1. A nonempty set R is said to form a semiring with respect to two binary operations, addition $(+)$ and multiplication $(\cdot)$ defined on it, if the following conditions are satisfied.
(1) $(R,+)$ is a commutative semigroup with zero,

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(2) $(R,.)$ is a semigroup,

(3) for any elements a, b, $c \in R$, the left distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$ and the right distributive law $(b+c) \cdot a = b \cdot a + c \cdot a$ both hold and

(4) $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$.

Definition 1.2. An ideal I of a semiring R is said to be a k- ideal or subtractive ideal if $a \in I$ and $a+b \in I$ for $b \in R$, then $b \in I$.

Definition 1.3. A semiring R is said to be semisubtractive if for $a \neq b \in R$, there exists $c \in R$ such that $a+c = b$ or $d \in R$ such that $a = d+b$.

Definition 1.4. Let $\alpha : R \to S$ be a homomorphism of semirings. Then α is said to be a steady homomorphism if for any $a, b \in R, a \equiv_{\alpha} b$ if and only if $a \equiv_{Ker\alpha} b$ ($a \equiv_{Ker\alpha} b$ if and only if $a+k = b+k'$ for some $k, k' \in Ker\alpha$ and $a \equiv_{\alpha} b$ if and only if $\alpha(a) = \alpha(b)$).

Definition 1.5. A semiring R is said to be a s-homomorphic image of S if there exists a steady homomorphism from S onto R.

Theorem 1.6. [5] Let S be a semiring, T a semiring with an absorbing zero 0_T , and $\phi : S \to T$ a surjective homomorphism. Then $K = \phi^{-1}(0_T)$ is a k-ideal of S (also called the kernel of ϕ) and $\phi([s]_K) = \phi(s)$ for all $s \in S$ defines a semi-isomorphism $\phi : S/K \to T$ which satisfies $\phi \ o \ k_K^{\#} = \phi$, where $k_K^{\#}$ denotes the natural homomorphism of S onto $S/K = S/k_K$.

Theorem 1.7. [5] For a semiring S with an absorbing zero 0 let A be a subsemiring which contains 0 and B an ideal of S. Then $\phi([a]_{A\cap\bar{B}}) = [a]_B$ for all $a \in A \subseteq A+B$ defines a semi-isomorphism

$$\phi : A/A \cap \bar{B} \to A+B/B.$$

Theorem 1.8. [5] Let A, B be ideals of a semiring S with the additional condition $A \subseteq B$. Then $\bar{\phi}([s]_B) = [[s]_A]_{\bar{B}/A}$ for all $s \in S$ defines an isomorphism

$$\bar{\phi} : S/B \to (S/A)/(\bar{B}/A).$$

Proposition 1.9. [2] A homomorphism f from any cancellative and semisubtractive semiring S to any cancellative semiring T is always a steady homomorphism.

Remark 1.10. [2] If S is cancellative and semisubtractive, then S/I is cancellative and semisubtractive for any k-ideal I of S. From Proposition 1.5 [1], the semi-isomorphism theorems for semirings (semimodules) are the isomorphism theorems for cancellative and semisubtractive semirings (semimodules).

2. Radical Classes

Definition 2.1. A class $\mathfrak{R}$ of semirings is a radical class whenever the following three conditions are satisfied:
(a) $\mathfrak{R}$ is homomorphically closed; i.e. if S is a homomorphic image of a $\mathfrak{R}$-semiring R, then S is also a $\mathfrak{R}$-semiring.
(b) Every semiring R contains a $\mathfrak{R}$-ideal $\mathfrak{R}(R)$ which in turn contains every other $\mathfrak{R}$-ideal of R.
(c) The factor semiring $R/\mathfrak{R}(R)$ does not contain any nonzero $\mathfrak{R}$-ideal; i.e. $\mathfrak{R}(R/\mathfrak{R}(R)) = 0$.

An ideal I of a semiring R is called an $\mathfrak{R}$-ideal if I is an $\mathfrak{R}$-semiring. A semiring which does not contain any non-zero $\mathfrak{R}$-ideals will be called $\mathfrak{R}$- semi-simple.

Definition 2.2. The maximal $\mathfrak{R}$-ideal S of any semiring R is called the $\mathfrak{R}$-radical of R and it is denoted by $\mathfrak{R}(R)$.

Note that $\mathfrak{R}(R)$ is a k-ideal.
By the union (not set theoretic union) or sum of two subsemirings I and J of a semiring R we mean the set of all $i + j$ where i is in I and j in J. More generally, if $I_1, I_2, I_3, ...$ is any (not-necessarily finite or even countable) class of subsemirings of R, then by $\cup I_k$ or $I_1 + I_2 + I_3 + ...$ we mean the set of all sums $i_1 + i_2 + i_3 + ...$ where i_k is in I_k and where only a finite number of the i_k are non-zero. It is easy to verify that the union of any set of ideals (left, right or two sided) is again an ideal (left, right or two sided).

Theorem 2.3. A class $\mathfrak{R}$ of additively cancellative and semisubtractive semirings is a radical class if and only if
(a) $\mathfrak{R}$ is homomorphically closed, that is, a homomorphic image of a $\mathfrak{R}$-semiring is an $\mathfrak{R}$-semiring.

(d) Every non zero homomorphic image of a semiring R contains a non-zero $\mathfrak{R}$-ideal, then $R \in \mathfrak{R}$.

Proof. Assume that (a), (b) and (c) hold. Then (a) implies (a). We shall prove that (b) and (c) imply (d). Let R be a semiring such that $R \notin \mathfrak{R}$. Then $\mathfrak{R}(R) = S$ and $S \neq R$. But then $R/S \neq 0$ and R/S is a homomorphic image of R which does not have a non zero $\mathfrak{R}$-ideal since $\mathfrak{R}(R/\mathfrak{R}(R)) = 0$. This proves (d). Thus (b) and (c) imply (d). Conversely assume that (a) and (d) hold. Claim that (a), (b) and (c) hold. $(a) \Rightarrow (a)$. Let J be the union of I_i, where I_i is an ideal of R for each i and $I_i \in \mathfrak{R}$. Claim that $J \in \mathfrak{R}$. If $J = 0$, then $J \in \mathfrak{R}$. If $J \neq 0$, let $J/K \neq 0$ be any factor semiring of J, where K ia a $k-$ ideal in J. Since $K \subset J$, there must exists in R an $\mathfrak{R}$-ideal L such that $L \nsubseteq K$. By theorem 1.7 and remark 1.10 $L+K/\ K \ \cong \ L/L\cap\bar{K}$. However $L+K/K \neq 0$ and is an ideal in J/K and $L/L\cap\bar{K}$ is a homomorphic image of $\mathfrak{R}$-ideal L and by (a) $L/L\cap\bar{K} \in \mathfrak{R}$. Therefore, every non-zero homomorphic image of J contains a non-zero $\mathfrak{R}$-ideal by (d), hence $J \in \mathfrak{R}$.

Finally we must establish (c). Take any semiring R. We know that R has an $\mathfrak{R}$-radical S since (b) is already established. Suppose that $R/\mathfrak{R}(R) = R/S$ is not semisimple as $(\mathfrak{R}(R/\mathfrak{R}(R)) \neq 0$ by (d). Let M/S be its non-zero $\mathfrak{R}$-radical (i.e. $(\mathfrak{R}(R/S) = M/S)$. Then M is an ideal of R and M contains S. Let M/N be any non-zero factor semiring of the semiring M, where N is a $k-$ ideal. If $N \supseteq S$, then M/N is a homomorphic image of the $\mathfrak{R}$-semiring M/S and by (a), M/N is an $\mathfrak{R}$-semiring. If $N \nsupseteq S$, then $N \cap S = S$ and again by theorem 1.7 and remark 1.10 $(N+S)/N \cong S/S\cap\bar{N}$. The left hand side of this isomorphism is a non-zero ideal of M/N and the right-hand side is a homomorphic image of the $\mathfrak{R}$-semiring S and therefore by (a) it is an $\mathfrak{R}$-semiring. Thus every non-zero homomorphic image of M contains a non-zero $\mathfrak{R}$-ideal, and by (d), M is an $\mathfrak{R}$-semiring. Then M must be in S, a contradiction. This establishes (c). □

Definition 2.4. A class $\bar{\mathfrak{R}}$ of semirings is said to be a semiradical class whenever the following three conditions are satisfied:

(a') $\bar{\mathfrak{R}}$ is s-homomorphically closed; i.e. if S is an s-homomorphic image of a $\bar{\mathfrak{R}}$-semiring R, then S is also a $\bar{\mathfrak{R}}$-semiring.

(b') Every semiring R contains a $\bar{\mathfrak{R}}$-subtractive ideal $\bar{\mathfrak{R}}$(R) which in turn contains every other $\bar{\mathfrak{R}}$-subtractive ideal of R.

(c') The factor semiring $R/\bar{\mathfrak{R}}(R)$ does not contain any non-zero $\bar{\mathfrak{R}}$-subtractive ideals; i.e. $\bar{\mathfrak{R}}(R/\bar{\mathfrak{R}}(R)) = 0$.

Proposition 2.5. If $\bar{\mathfrak{R}}$ is a semiradical, then R is an $\bar{\mathfrak{R}}$-semiradical semiring if and only if R cannot be mapped steady-homomorphically onto a non-zero $\bar{\mathfrak{R}}$-semisimple semiring.

Proof. If R is an $\bar{\mathfrak{R}}$-semiradical semiring then condition (a') implies that every non zero s-homomorphic image of R is also a $\bar{\mathfrak{R}}$-semiradical semiring and therefore it cannot be $\bar{\mathfrak{R}}$-semisimple. Conversely, if R is not a $\bar{\mathfrak{R}}$-semiradical semiring, then by (b') and (c') it can be mapped s-homomorphically onto the non-zero $\bar{\mathfrak{R}}$-semisimple semiring $R/\bar{\mathfrak{R}}(R)$, where $\bar{\mathfrak{R}}(R)$ is the $\bar{\mathfrak{R}}$-semiradical of R. □

3. WEAK RADICAL CLASSES

Definition 3.1. A class σ of semirings is a weak radical class whenever the following two conditions are satisfied:

(a1) If R is an σ-semiring, then every non-zero steady homomorphic image of R has a non-zero σ-subtractive ideal;

(d1) If every-non-zero steady homomorphic image of a semiring R has a non-zero σ-subtractive ideal then R is an σ-semiring.

Remark 3.2. Observe that by axiom (d1) the semiring 0 is always a σ-weak radical semiring. Furthermore, there are two trivial weak radical classes, the semiring 0 and the class of all semirings. Note that when σ is a class of rings, this definition defaults to the standard definition of a radical class.

Proposition 3.3. Every semiradical class of semirings is a weak radical class.

Proof. If $\mathfrak{R}$ is a semiradical class, then the class $\mathfrak{R}$ satisfies conditions (a'), (b'), and (c'). Condition (a') implies condition (a1) trivialy. If R is not a $\mathfrak{R}$-semiring, then by (b') we have $\mathfrak{R}(R) \neq R$, and so $R/\mathfrak{R}(R)$ is a non-zero homomorphic image of R which does not contain non-zero $\mathfrak{R}$-subtractive ideals by (c'). Hence the condition (d1). □

We develop weak radical classes in the same way that radical classes of rings are developed in [3].

Theorem 3.4. Let $\mathscr{M}$ be any class of semiring which satisfies the following condition:
(e) Every non-zero subtractive ideal of a semiring of $\mathscr{M}$ can be mapped s-homomorphically onto some non-zero semiring of $\mathscr{M}$.
Then the class $\sigma_{\mathscr{M}} = \{$R : R cannot be mapped s-homomorphically onto any non-zero semiring of $\mathscr{M}\}$ is a weak radical class.

Proof. Proof is similar to ([1], Theorem 3).

We prove the following fundamental theorem for any subclass of additively cancellative and semisubtractive semirings of an universal class £ of semirings. □

Theorem 3.5. A class $\mathscr{M} \subset$ £ is a class of all $\mathfrak{R}$-semisimple semirings with respect to some semiradical class $\mathfrak{R}$ over the class £ if and only if $\mathscr{M}$ satisfies the following conditions:
(e) Every non-zero subtractive ideal of a semiring of $\mathscr{M}$ can be mapped s-homomorphically onto some non-zero semiring of $\mathscr{M}$.
(f) If every non-zero subtractive ideal of a semiring R can be mapped s-homomorphically onto some non-zero semiring of $\mathscr{M}$, then the semiring R must be in $\mathscr{M}$.

Proof. Proof follows by Theorem 2.3 and Proposition 2.5. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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