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SPECTRUM OF THE ZERO-DIVISOR GRAPH ON THE RING OF INTEGERS MODULO n

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Abstract. For a commutative ring R with non-zero identity, let $Z^*(R)$ denote the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph with all non-zero zero-divisors as vertices and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. In this paper, the adjacency matrix and spectrum of $\Gamma(\mathbb{Z}_{p^k})$ are investigated. Also, the implicit computation of the spectrum of $\Gamma(\mathbb{Z}_n)$ is described.

Keywords: eigenvalues; zero-divisor graph; block matrix; adjacency matrix.

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1. INTRODUCTION

let G be a simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{uv})$, where a_{uv} is the number of edges joining vertices u and v, each loop counting as two edges. For a simple graph, $A(G)$ is real and symmetric with entries 0 and 1, where all diagonal entries are zeroes. That is , for a simple graph G, $A(G) = (a_{ij})$, where a_{ij}=1 or 0 according as $v_i \sim v_j$ in G or not.

The eigenvalues of a square matrix B are the roots of its characteristic polynomial $det(B - xI)$.

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The spectrum of a square matrix B, denoted by $\sigma(B)$, is the multi set of all the eigenvalues of B. If $\lambda_1,\lambda_2,...,\lambda_r$, are the distinct eigenvalues of B with respective multiplicities $m_1,m_2,...,m_r$, then we shall denote the spectrum of B, by ,

$$\sigma(B) = \left\{ \begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{array} \right\}.$$

The characteristic polynomial of a graph G is the characteristic polynomial of its adjacency matrix; denoted by $\Phi(G;x) = det(A-xI)$. Spectrum of a graph, denoted by $\sigma(G)$ is the spectrum of the adjacency matrix $A(G)$. Since the adjacency matrix of a graph G is real and symmetric, algebraic multiplicity of an eigenvalue is same as its geometric multiplicity[12]. The author refers to [5, 12] for a good introduction to Spectral Graph Theory.

Let R be a commutative ring with nonzero identity. A nonzero element $x \in R$ is called a zero divisor if there exists a nonzero element $y \in R$ such that $xy = 0$. Let $Z^*(R) = Z(R) \setminus (0)$, be the set of non-zero zero-divisors of R. In [8], Beck associated to a commutative ring R its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of R (including 0) and two distinct vertices a and b are adjacent if and only if $ab = 0$. In [4], Anderson and Livingston redefined the concept of the zero divisor graph and introduced the subgraph $\Gamma(R)$ (of $G(R)$) as zero divisor graph whose vertices are the non-zero zero-divisors of R and the authors studied the interplay between the ring-theoretic properties of a commutative ring and the graph theoretic properties of its zero-divisor graph.

The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph with $V(\Gamma(R)) = Z^*(R)$ and two distinct vertices $x,y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the null graph if R is an integral domain.

The study of zero divisor graph on commutative rings has attracted the attention of many researchers. We refer to [1, 6, 13, 15, 18, 20, 24] for a survey of results regarding both algebraic and graph parameters of the zero-divisor graphs on certain commutative rings. P. Sharma et.al [16] studied the adjacency matrix associated with zero-divisor graphs of finite commutative rings. In [18], R. G. Tirop et. al analysed the adjacency matrices of the zero-divisor graphs of Galois rings. Pranjali et.al [15] described results regarding the adjacency matrix of the zero-divisor graph over finite ring of Gaussian integers. The study of the zero-divisor graph of the ring on integers modulo n can be found in [10, 11, 21].

This paper aims to describe the spectrum of the zero-divisor graph of the ring of integers modulo n. The direct computation of the spectrum of $\Gamma(\mathbb{Z}_n)$, for $n = p^3, p^4, p^2q$, where p and q are distinct primes; is described in sections 3 and 4. The structure of $\Gamma(\mathbb{Z}_{p^k})$ is explored in section:5. In section:6, we propose an implicit method of computing the spectrum of $\Gamma(\mathbb{Z}_n)$, for $n \neq p$; a prime and illustrate the comparison between the direct and implicit computations. The notations and basic definitions in graph theory are standard and are taken from the books of graph theory such as, e.g.[5] ,[9] and [12].

2. PRELIMINARIES

Let G be a graph. G is connected if there is a path between any two distinct vertices. For distinct vertices x and y of G, let $d(x,y)$ be the length of a shortest path from x to y. Clearly $d(x,x) = 0$ and $d(x,y) = \infty$, if there is no path connecting x and y. The diameter of G is defined as $diam(G) = Sup\{d(x,y)$: x and y are vertices of $G\}$. The neighborhood (or open neighborhood) of a vertex v of G, denoted by $N(v)$, is the set of vertices adjacent to v.

Clique of a graph is a set of mutually adjacent vertices. The maximum size of a clique of a graph G, called the clique number of G, is denoted by $\omega(G)$. A graph G is said to be complete if any two distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n. The complement of K_n is a null graph and is denoted by $\overline{K}_n$. For a graph G, a stable set is a set of vertices, no two of which are adjacent. A stable (or independent) set in a graph is maximum if the graph contains no larger stable set. The cardinality of a maximum stable set in a graph G is called the stability number, denoted by $\alpha(G)$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. ($gr(G) = \infty$ if G contains no cycles).

Sabidussi [19, p. 396] has defined the X-join of a set of graphs, $\{Y_x\}_{x\in X}$, indexed by $V(X)$, as the graph Z with $V(Z) = \{(x,y) : x \in X, y \in Y_x\}$. and $E(Z) = \{((x,y),(x',y')) : (x,x') \in E(X)$ or else $x = x'$ and $(y,y') \in E(Y_x)\}$. Let G be a finite graph with vertices labeled as $1,2,...,n$ and let $H_1,H_2,...,H_n$ be a set of n graphs. The generalised join of $H_1,H_2,...H_n$, denoted by $G[H_1,H_2,...,H_n]$ is obtained by replacing each vertex i of G by the graph H_i and inserting all or none of the possible edges between H_i and H_j depending on whether or not i and j are adjacent in G.

For a natural number n, $\phi(n)$ is the number of positive integers less than n and relatively prime to n. $M_n(F)$ denotes the vector space of all square matrices of size $n \times n$ with entries from a field F. A circulant matrix of size $n \times n$, with entries a and b, where $a, b \in \mathbb{R}$, denoted by $C_{(a,b,n)}$, is of the form

$$C_{(a,b,n)} = \begin{bmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}_{n \times n}$$

The complexity of computing the characteristic polynomial of a $n \times n$ block matrix is often reduced to some extent by means of the circulant matrices defined above in section:4.

3. Adjacency Matrix and Spectrum of $\Gamma(\mathbb{Z}_{p^3})$ and $\Gamma(\mathbb{Z}_{p^4})$

D. F. Anderson and P. S. Livingston[4] gave several fundamental results regarding $\Gamma(R)$ of a commutative ring R. For any commutative ring R, $\Gamma(R)$ is connected[4]. The zero divisor of $\mathbb{Z}_n$ is a simple, connected and undirected graph. For an idempotent element x, $x.x = 0$, but x is not adjacent with itself in a zero divisor graph.
In any finite commutative ring with unity, every non-zero element is either a unit or a zero-divisor.

Proposition 3.1. [14] The number of non-zero zero-divisors of $\mathbb{Z}_n$ is $n - \phi(n) - 1$

B. Surendranath Reddy et.al[23] have studied the spectrum of $\Gamma(\mathbb{Z}_{p^3})$ and $\Gamma(\mathbb{Z}_{p^2q})$, where the graph $\Gamma(\mathbb{Z}_n)$ is not simple. But following the traditional system of considering $\Gamma(\mathbb{Z}_n)$ as simple, connected graphs, the adjacency matrix of $\Gamma(\mathbb{Z}_n)$ has to be reformed. We incorporate this modification and compute the spectrum of these respective graphs in two different ways in the next two sections.

3.1. The adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$.

Theorem 3.1. The adjacency matrix of $\Gamma(\mathbb{Z}_n)$ for $n = p^3$, where p is a prime integer, , is

$$A(\Gamma(\mathbb{Z}_n))= \left[\begin{array}{c|c} O_{(p^2-p)} & J_{(p^2-p)\times(p-1)} \\ \hline J_{(p-1)\times(p^2-p)} & J-I_{(p-1)} \end{array}\right]$$

where J is a matrix of all ones and I is an identity matrix.

The size of this matrix is p^2-1.

Proof. Let $n = p^3$. By Proposition 3.1, the number of non-zero zero-divisors of $\Gamma(\mathbb{Z}_{p^3})$ is p^2-1. These p^2-1 non-zero zero-divisors are partitioned as follows.

$P_1 = \{k_1 p : k_1 = 1,2,...p^2-1,\text{where } p \nmid k_1 \}$.

$P_2 = \{k_2 p^2 : k_2 = 1,2,...p-1,\text{where } p \nmid k_2\}$.

Using elementary number theory, it can be easily seen that the cardinality of P_1 is $|P_1| = p^2-p$. Similarly,

$|P_2| = p-1$. We also observe that,

(1) $xy \neq 0, \forall x,y \in P_1$.
(2) $xy = 0, \forall x \in P_1$ and $\forall y \in P_2$.
(3) $xy = 0, \forall x,y \in P_2$.

These simple observations gives rise to the partitioned structure of the adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$. The non-zero zero divisors of n are rearranged such that the elements of P_1 appear first and then P_2. Since no two vertices in P_1 are adjacent, it is an independent set in $\Gamma(\mathbb{Z}_{p^3})$ and hence it corresponds to a block of zeroes in the adjacency matrix. Also since all vertices of P_1 are adjacent to every vertex of P_2, it corresponds to a block of all ones and so on. Note that the zero-divisor graph of a commutative ring is a simple undirected graph. Hence, even though $xx = 0, \forall x, \in P_2$, x is not adjacent with x. Thus the adjacency of vertices among P_2 corresponds to the block $J-I$. Thus the adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$ is a 2×2 block matrix consisting of blocks of zeros and ones in the following form,

$$A(\Gamma(\mathbb{Z}_{p^3})) = \left[\begin{array}{c|c} O_{(p^2-p)} & J_{(p^2-p)\times(p-1)} \\ \hline J_{(p-1)\times(p^2-p)} & J-I_{(p-1)} \end{array}\right] \tag{1}$$

The size of this matrix is $|P_1|+|P_2| = p^2-1$. □

3.2. Spectrum of $\Gamma(\mathbb{Z}_{p^3})$. The size of the adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$ is p^2-1. Since the direct computation of the characteristic polynomial is tedious for large n, we adopt some tools of matrix theory among which schur complement and coronal plays vital role.

Lemma 3.1. [7] Let M,N,P,Q be matrices and let M be invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$, then $detS = detM.\, det(Q-PM^{-1}N)$.
$(Q-PM^{-1}N)$ is called the Schur complement of M in S.

Definition 3.1. [3] Let $\mathbf{1}_n$ denote an all-one vector. The coronal of a matrix A, denoted by $\Gamma_A(x)$, is defined as the sum of the entries of the matrix $(xI-A)^{-1}$.
That is, $\Gamma_A(x) = (\mathbf{1}_n)^T.(xI-A)^{-1}.\mathbf{1}_n$

Lemma 3.2. [22] Let G be a r-regular graph on n vertices, with adjacency matrix A. Then

$$\Gamma_A(x) = \frac{n}{x-r}.$$

Proposition 3.2. [25] Let A be an $n\times n$ matrix and $J_{n\times n}$ denote an all one matrix. Then,

$$det(xI_n - A - \alpha J_{n\times n}) = (1-\alpha\Gamma_A(x)).det(xI_n - A),$$

where α is a real number.

Let $G = \Gamma(\mathbb{Z}_{p^3})$ and $M = A(\Gamma(\mathbb{Z}_{p^3}))$. The vertex set of G is partitioned into P_1 and P_2, where P_1 induces the null subgraph $\overline{K}_{p^2-p}$ and P_2 induces a complete subgraph K_{p-1} both of which are regular of degree 0 and $p-2$ respectively.

Theorem 3.2. Let $G = \Gamma(\mathbb{Z}_{p^3})$ and let λ be an eigenvalue of G. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities p^2-p-1 and $p-2$ respectively. If $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies $\phi(\lambda) = \lambda^2 - (p-2)\lambda - p(p-1)^2 = 0$.

Proof. Let the adjacency matrix of $G = \Gamma(\mathbb{Z}_{p^3})$ be denoted by M. From equation (1),

$$M = \left[\begin{array}{c|c} O_{(p^2-p)} & J_{(p^2-p)\times(p-1)} \\ \hline J_{(p-1)\times(p^2-p)} & J - I_{(p-1)} \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Clearly A_1 and A_2 are the adjacency matrices of the induced subgraphs $\overline{K}_{p^2-p}$ and K_{p-1} of G

respectively. By Lemma 3.2, $\Gamma_{A_1}(x) = \dfrac{p^2-p}{x}$ and $\Gamma_{A_4}(x) = \dfrac{p-1}{x-p+2}$.

The eigenvalues of G are given by $det(xI-M)=0$.

By Lemma 3.1 ,

$$det(xI-M) = det(xI-A_1).det\left[(xI-A_4)-{A_3}^T.(xI-A_1)^{-1}.A_2\right], \tag{2}$$

where

$$det(xI-A_1) = x^{p^2-p}$$

and

$$det(xI-A_4) = (x-p+2).(x+1)^{p-2}.$$

Also, $det\left[(xI-A_4)-{A_3}^T.(xI-A_1)^{-1}.A_2\right] = det\left[(xI-A_4)-\Gamma_{A_1}(x).J_{p-1\times p-1}\right]$

$$= det\left[(xI-A_4)-(\frac{p^2-p}{x}).J_{p-1\times p-1}\right]$$

$$= \left(1-(\frac{p^2-p}{x})\Gamma_{A_4}(x)\right).det(xI-A_4)$$

$$= \left(1-(\frac{p^2-p}{x})(\frac{p-1}{x-p+2})\right).(x-p+2).(x+1)^{p-2},$$

by Lemma 3.2 and Proposition3.2.

Applying these steps in equation(2), the characteristic polynomial of G is given by,

$$\Phi(G;x) = x^{p^2-p-1}.(x+1)^{p-2}.\left(x^2-(p-2)x-p(p-1)^2\right)$$

Thus the spectrum of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ is

$$\begin{pmatrix} 0 & -1 & \dfrac{(p-2)+\sqrt{4p^3-7p^2+4}}{2} & \dfrac{(p-2)-\sqrt{4p^3-7p^2+4}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{pmatrix}$$ □

3.3. Adjacency matrix and eigenvalues of $\Gamma(\mathbb{Z}_{p^4})$. Partitioning the zero-divisors of $\mathbb{Z}_{p^4}$ into multiples of p, p^2, p^3 and labeling the vertices of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$ properly, we see that,

$$A(\Gamma(\mathbb{Z}_{p^4})) = \left[\begin{array}{c|cc} O_{p^2(p-1)} & O_{p^2(p-1)\times p(p-1)} & J_{p^2(p-1)\times(p-1)} \\ O_{p(p-1)\times p^2(p-1)} & J-I_{p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p^2(p-1)} & J_{(p-1)\times p(p-1)} & J-I_{(p-1)} \end{array}\right]$$

Computing the characteristic polynomial as in section 3.1, we arrive at the following corollary.

Corollary 3.1. Let $G = \Gamma(\mathbb{Z}_{p^4})$ and let λ be an eigenvalue of G. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $p^3 - p^2 - 1$ and $p^2 - 3$ respectively. If $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies $\phi(\lambda) = \lambda^3 - (p^2-3)\lambda^2 - (p^4 - 2p^3 + 2p^2 - 2)\lambda + p^2(p-1)^2(p^2-p-1) = 0$.

4. Adjacency Matrix and Eigenvalues of $\Gamma(\mathbb{Z}_{p^2q})$

P. M. Magi et.al[14] have studied the structure of $\Gamma(\mathbb{Z}_{p^2q^2})$ by partitioning the vertex set into 7 classes and computed the characteristic polynomial by employing Gaussian elimination and other matrix operations. In this section Let $n = p^2q$ where p and q are prime integers with $p < q$ and let $G = \Gamma(\mathbb{Z}_{p^2q})$. By proposition 3.1, n has $p(p+q-1)-1$ non-zero zero-divisors. As in the previous section, the non-zero zero-divisors of n are partitioned into four disjoint sets as given below.

$E_1 = \{k_1p : k_1 = 1,2,...pq-1, \text{where } p \nmid k_1 \text{ and } q \nmid k_1\}$.
$E_2 = \{k_2q : k_2 = 1,2,...p^2-1, \text{where } p \nmid k_2\}$.
$E_3 = \{k_3pq : k_3 = 1,2,...p-1\}$.
$E_4 = \{k_4p^2 : k_4 = 1,2,...q-1\}$.
Clearly, $|E_1| = (p-1)(q-1)$, $|E_2| = p(p-1)$, $|E_3| = (p-1)$, $|E_4| = (q-1)$.
The adjacency matrix of G is given by ,

(3)

$$A(\Gamma(\mathbb{Z}_{p^2q})) = \left[\begin{array}{cc|cc} O_{(p-1)(q-1)} & O_{(p-1)(q-1)\times p(p-1)} & J_{(p-1)(q-1)\times(p-1)} & O_{(p-1)(q-1)\times(q-1)} \\ O_{p(p-1)\times(p-1)(q-1)} & O_{p(p-1)} & O_{p(p-1)\times(p-1)} & J_{p(p-1)\times(q-1)} \\ \hline J_{(p-1)\times(p-1)(q-1)} & O_{(p-1)\times p(p-1)} & J-I_{(p-1)} & J_{(p-1)\times(q-1)} \\ O_{(q-1)\times(p-1)(q-1)} & J_{(q-1)\times p(p-1)} & J_{(q-1)\times(p-1)} & O_{(q-1)} \end{array}\right]$$

Remark 4.1. The stability number $\alpha(G) = |E_1| + |E_2| = (p-1)(p+q-1)$.

Remark 4.2. The clique number $\omega(G) = |E_3| = p-1$.

Remark 4.3. Since E_3 induces a complete subgraph of order $p-1$, the girth $gr(G) = 4$ if $p = 2$ and $gr(G) = 3$, if $p \geq 3$.

In this section we expose the direct computation of characteristic polynomial of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2q})$ using block diagonal matrix and certain circulant matrix.

Definition 4.1. [17] A matrix $A \in M_n(F)$, where F is a field of numbers(real or complex); of the form

$A=\begin{bmatrix} A_{11} & O & \dots & O \\ O & A_{22} & \dots & O \\ \vdots & & \ddots & \vdots \\ O & \dots & & A_{nn} \end{bmatrix}$ in which $A_{ii} \in M_{n_i}(F), i = 1,2,...k$, $\Sigma_{i=1}^{k} n_i = n$, and all blocks above and below the block diagonal are the zero blocks, is called a block diagonal matrix. Thus A=$A_{11} \oplus A_{22} ... \oplus A_{kk} = \bigoplus_{i=1}^{k} A_{ii}$, is the direct sum of matrices $A_{11}, A_{22}, ...A_{kk}$.

Lemma 4.1. [17] $det(\bigoplus_{i=1}^{k} A_{ii}) = \prod_{i=1}^{k} det(A_{ii})$.

In particular, if $A_{11} \in M_n(F)$ and $A_{22} \in M_m(F)$, then $det \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = det(A_{11}) \cdot det(A_{22})$.

Lemma 4.2. [17] If $A_{11} \in M_n(F)$ and $A_{22} \in M_m(F)$ are non singular,

then $\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}$

Proposition 4.1. [14] Let $C_{(a,b,n)}$= $\begin{bmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}_{n \times n}$ be a circulant matrix of size $n \times n$; with entries a and b.

Then $detC_{(a,b,n)}$; denoted by δ, is given by $\delta = (a+(n-1)b)(a-b)^{n-1}$.

Proposition 4.2. [14] If $C_{(a,b,n)}$ is nonsingular, then its inverse is given by

$$C^{-1}_{(a,b,n)} = \frac{1}{\delta}\begin{bmatrix} \delta_{n-1} & \Delta_{n-1} & \dots & \Delta_{n-1} \\ \Delta_{n-1} & \delta_{n-1} & \dots & \Delta_{n-1} \\ \vdots & & \ddots & \vdots \\ \Delta_{n-1} & \dots & & \delta_{n-1} \end{bmatrix} = \frac{1}{\delta}C(\delta_{n-1},\Delta_{n-1},n),$$

where $\delta_{n-1} = (a+(n-2)b)\,(a-b)^{n-2}$ and $\Delta_{n-1} = -b\cdot(a-b)^{n-2}$.

Theorem 4.1. Let $G = \Gamma(\mathbb{Z}_{p^2q})$ and let λ be an eigenvalue of G. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $(p-1)(p+q-1)+(q-4)$ and $p-2$ respectively. If $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies,
$\phi(\lambda) = \lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2 = 0.$

Proof. Let $M = A(\Gamma(\mathbb{Z}_{p^2q}))$. The eigenvalues of G are given by

$$det(M-\lambda I) = 0 \tag{4}$$

$$M-\lambda I = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -\lambda \end{bmatrix}_{(p-1)(p+q-1)},$$

$$B = \begin{bmatrix} 1 & \cdots\cdots & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots\cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots\cdots & 0 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots\cdots & 0 & 1 & \cdots & 1 \end{bmatrix}_{(p-1)(p+q-1)\times(p+q-2)},$$

and $C = \begin{bmatrix} -\lambda & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ 1 & -\lambda & & 1 & \vdots & & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 1 & \cdots & \cdots & -\lambda & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & -\lambda & 0 & \cdots & 0 \\ \vdots & & & \vdots & 0 & -\lambda & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & -\lambda \end{bmatrix}_{p+q-2}$

If $\lambda \neq 0$, then A is invertible and by Proposition 4.1 and Proposition 4.2

$$detA = (-\lambda)^{(p-1)(p+q-1)} \tag{5}$$

and $A^{-1} = \frac{-1}{\lambda} I$.

Also $B^T A^{-1} B = \frac{-1}{\lambda} \left[\begin{array}{c|c} (p-1)(q-1)J_{p-1} & O_{(p-1)\times(q-1)} \\ \hline O_{(q-1)\times(p-1)} & p(p-1)J_{q-1} \end{array} \right]$,

where J denotes a matrix of all ones.

Now, the schur complement of A in C is given by

$$C - B^T A^{-1} B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{6}$$

where $A_{11} = C_{\left(-\lambda + \frac{(p-1)(q-1)}{\lambda},\ 1 + \frac{(p-1)(q-1)}{\lambda},\ p-1\right)}$,

$A_{12} = J_{(p-1)\times(q-1)}$,

$A_{21} = J_{(q-1)\times(p-1)}$ and

$A_{22} = C_{\left(-\lambda + \frac{p(p-1)}{\lambda},\ \frac{p(p-1)}{\lambda},\ q-1\right)}$. Thus by lemma 3.1,

$$det(M - \lambda I) = detA.\ det(C - B^T A^{-1} B) \tag{7}$$

Applying Lemma 3.1 once again ,

$$det(C - B^T A^{-1} B) = detA_{11}.\ det(A_{22} - A_{21} A_{11}^{-1} A_{12}) \tag{8}$$

By Proposition 4.1 it can be seen that,

$$detA_{11} = \frac{h(\lambda)}{\lambda} (-\lambda - 1)^{p-2} \tag{9}$$

where $h(\lambda) = -\lambda^2 + (p-2)\lambda + (p-1)^2(q-1)$.

Also by Proposition 4.2,

$$A_{11}^{-1} = \frac{1}{h(\lambda)(\lambda+1)} C_{\left(\lambda^2-(p-3)\lambda-(p-1)(q-1)(p-2)),\ \lambda+(p-1)(q-1),\ p-1\right)} \tag{10}$$

Also,

$$A_{22} - A_{21}A_{11}^{-1}A_{12} = C_{\left(-\lambda+\frac{p(p-1)}{\lambda}-\frac{\lambda(p-1)}{h(\lambda)},\ \frac{p(p-1)}{\lambda}-\frac{\lambda(p-1)}{h(\lambda)}),\ q-1\right)} \tag{11}$$

(12)

$$det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = (-1)^{q-1}\lambda^{q-2}\frac{\lambda^2 h(\lambda) - h(\lambda)p(p-1)(q-1) + \lambda^2(p-1)(q-1)}{\lambda h(\lambda)}$$

Applying these in equation(7), the characteristic equation of $\Gamma(\mathbb{Z}_{p^2q})$ is obtained as

$$\lambda^{(p-1)(p+q-1)+q-4}(\lambda+1)^{p-2}\phi(\lambda) = 0 \tag{13}$$

where $\phi(\lambda) = \lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2$

Hence $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $(p-1)(p+q-1)+(q-4)$ and $p-2$ respectively. Also if $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies,

$\phi(\lambda) = \lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2 = 0$ □

5. ADJACENCY MATRIX OF $\Gamma(\mathbb{Z}_{p^k})$, $k \geq 3$

By a proper divisor of n, we mean a positive divisor d such that d/n , $1 < d < n$. Let $s(n)$ denote the number of proper divisors of n. Then, $s(n) = \sigma_0(n) - 2$, where $\sigma_k(n)$ is the sum of k powers of all divisors of n, including n and 1.

If $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes,

$$s(n) = \prod_{i=1}^{r}(n_i+1) \ -2.$$

Let $S(d) = \{k \in \mathbb{Z}_n : gcd(k,n) = d\}$. See[21]. Then $\{S(d_1), S(d_2), ..., S(d_{s(n)})\}$ is an equitable partition for the vertex set of $\Gamma(\mathbb{Z}_n)$ such that $S(d_i) \cap S(d_j) = \phi, i \neq j$, and any two vertices in $S(d_i)$ have the same number of neighbours in $S(d_j)$ for all divisors d_i, d_j of n.

Proposition 5.1. [11,prop2.1] $|S(d_i)| = \phi(\frac{n}{d_i})$, for every $i = 1,2,...s(n)$.

Also the subgraphs induced by $S(d_i)$ is either K_{d_i} or $\overline{K}_{d_i}$. For example, in $\Gamma(\mathbb{Z}_{p^3})$, $S(p)$ induces $\overline{K}_{p(p-1)}$ and $S(p^2)$ induces K_{p-1}. In $\Gamma(\mathbb{Z}_{p^2q})$, $S(p), S(q), S(p^2)$ induce $\overline{K}_{(p-1)(q-1)}$, $\overline{K}_{p(p-1)}, \overline{K}_{q-1}$ respectively while $S(pq)$ induces K_{p-1}; which is visible from the diagonal blocks $0, J-I$ in the adjacency matrices of respective graphs. See equation(1) and equation(3).

In this section, we analyse the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k}), k \geq 3$. Since $\mathbb{Z}_p$ is an integral domain for any prime p, $\Gamma(\mathbb{Z}_p)$ is a null graph. Hence to avoid triviality, we assume $k \geq 2$. Also for $k = 2$, $\Gamma(\mathbb{Z}_{p^2})$ is a complete graph on $p-1$ vertices; whose spectrum is known. Hence we assume $k \geq 3$. Also we note that the proper divisors of p^k are $p, p^2, ..., p^{k-1}$ and the number of non-zero zero-divisors of p^k is $p^{k-1}-1$, by Proposition 3.1.

The structure of $\Gamma(\mathbb{Z}_{p^4})$, $\Gamma(\mathbb{Z}_{p^5})$, $\Gamma(\mathbb{Z}_{p^6})$, $\Gamma(\mathbb{Z}_{p^7})$, which motivated us to some interesting results, are given below.(See Figure:1, Figure:2, Figure:3, Figure:4)

$$\begin{array}{c} \\ S(p) \\ S(p^2) \\ S(p^3) \end{array} \begin{array}{c} \begin{array}{ccc} S(p) & S(p^2) & S(p^3) \end{array} \\ \left[\begin{array}{c|cc} O & O & J \\ \hline O & J-I & J \\ J & J & J-I \end{array}\right] \end{array}$$

FIGURE 1. $\Gamma(\mathbb{Z}_{p^4})$

$$\begin{array}{c} \\ S(p) \\ S(p^2) \\ S(p^3) \\ S(p^4) \end{array} \begin{array}{c} \begin{array}{cccc} S(p) & S(p^2) & S(p^3) & S(p^4) \end{array} \\ \left[\begin{array}{cc|cc} O & O & O & J \\ O & O & J & J \\ \hline O & J & J-I & J \\ J & J & J & J-I \end{array}\right] \end{array}$$

FIGURE 2. $\Gamma(\mathbb{Z}_{p^5})$

Remark 5.1. As illustrated in section 3.1, the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$ contains blocks of all zero matrices, blocks of all one matrices and identity-matrix blocks.

$$
\begin{array}{c} \\ S(p) \\ S(p^2) \\ S(p^3) \\ S(p^4) \\ S(p^5) \end{array}
\begin{array}{c}
\begin{array}{ccccc} S(p) & S(p^2) & S(p^3) & S(p^4) & S(p^5) \end{array} \\
\left[\begin{array}{cc|ccc}
O & O & O & O & J \\
O & O & O & J & J \\
\hline
O & O & J-I & J & J \\
O & J & J & J-I & J \\
J & J & J & J & J-I
\end{array}\right]
\end{array}
$$

FIGURE 3. $\Gamma(\mathbb{Z}_{p^6})$

$$
\begin{array}{c} \\ S(p) \\ S(p^2) \\ S(p^3) \\ S(p^4) \\ S(p^5) \\ S(p^6) \end{array}
\begin{array}{c}
\begin{array}{cccccc} S(p) & S(p^2) & S(p^3) & S(p^4) & S(p^5) & S(p^6) \end{array} \\
\left[\begin{array}{ccc|ccc}
O & O & O & O & O & J \\
O & O & O & O & J & J \\
O & O & O & J & J & J \\
\hline
O & O & J & J-I & J & J \\
O & J & J & J & J-I & J \\
J & J & J & J & J & J-I
\end{array}\right]
\end{array}
$$

FIGURE 4. $\Gamma(\mathbb{Z}_{p^7})$

If all vertices of $S(p^i)$ are adjacent to every vertex of $S(p^j)$, we write $S(p^i) \sim S(p^j)$. Clearly, $S(p^i) \sim S(p^j)$ iff $i + j \geq k$. Also, $S(p^i) \sim S(p^i)$, indicates that every vertex of $S(p^i)$ is adjacent to every other vertex of $S(p^i)$ and clearly the equivalent condition of adjacency of vertices among $S(p^i)$ is that; $S(p^i) \sim S(p^i)$ iff $i \geq \lceil \frac{k}{2} \rceil$. Thus, the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$ is obtained as in Figure: 5 and Figure: 6.

5.1. Some graph parameters of $\Gamma(\mathbb{Z}_{p^k})$. The above analysis of the structure of $\Gamma(\mathbb{Z}_{p^k})$ leads to some results regarding the stability number, clique number and girth of $\Gamma(\mathbb{Z}_{p^k})$. As the matrix narrates the adjacency between $S(p^i)$ and $S(p^j)$, for $i, j = 1, 2, ..., k-1$ and the adjacency among the vertices of each $S(p^i)$, $i = 1, 2, ..., k-1$, it is clear that, any principal sub matrix of zero blocks corresponds to an independent set in $\Gamma(\mathbb{Z}_{p^k})$.

Theorem 5.1. Let $G = \Gamma(\mathbb{Z}_{p^k})$, $k \geq 3$. Then, $\alpha(G) = p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor}$

Proof. From the structure of the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$, it is clear the , the maximum size of a principal sub matrix of zero blocks is $|S(p)| + |S(p^2)| + ... + |S(p^{\lceil \frac{k}{2} \rceil - 1})|$. Thus by

$$
\begin{array}{c|cccc|ccccc|}
 & S(p) & S(p^2) & \cdots & S(p^{\lceil \frac{k}{2} \rceil -1}) & S(p^{\lceil \frac{k}{2} \rceil}) & \cdots & \cdots & & S(p^{k-1}) \\
\hline
S(p) & O & \cdots & \cdots & O & O & \cdots & & O & J \\
S(p^2) & \vdots & \ddots & & \vdots & \vdots & & O & J & J \\
\vdots & \vdots & & \ddots & \vdots & \vdots & & & \vdots & \vdots \\
S(p^{\lceil \frac{k}{2} \rceil -1}) & O & \cdots & \cdots & O & O & J & \cdots & \cdots & J \\
\hline
S(p^{\lceil \frac{k}{2} \rceil}) & & \cdots & \cdots & O & J-I & J & \cdots & \cdots & J \\
\vdots & \vdots & \cdots & O & J & J & J-I & J & \cdots & J \\
 & \vdots & & & \vdots & \vdots & & \ddots & & \vdots \\
\vdots & O & J & \cdots & J & \vdots & & & \ddots & J \\
S(p^{k-1}) & J & J & \cdots & J & J & J & \cdots & \cdots & J-I
\end{array}
$$

FIGURE 5. $\Gamma(\mathbb{Z}_{p^k})$; when k is even

$$
\begin{array}{c|cccc|ccccc|}
 & S(p) & S(p^2) & \cdots & S(p^{\lceil \frac{k}{2} \rceil -1}) & S(p^{\lceil \frac{k}{2} \rceil}) & \cdots & & & S(p^{k-1}) \\
\hline
S(p) & O & \cdots & \cdots & O & O & O & \cdots & O & J \\
S(p^2) & \vdots & \ddots & & \vdots & O & \vdots & \cdots & J & J \\
\vdots & \vdots & & \ddots & \vdots & \vdots & J & \cdots & \vdots & \vdots \\
S(p^{\lceil \frac{k}{2} \rceil -1}) & O & \cdots & \cdots & O & J & J & \cdots & \cdots & J \\
\hline
S(p^{\lceil \frac{k}{2} \rceil}) & O & O & \cdots & J & J-I & J & \cdots & \cdots & J \\
\vdots & O & \cdots & J & J & J & J-I & J & \cdots & J \\
 & \vdots & & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
 & O & J & \cdots & J & \vdots & & & \ddots & \vdots \\
S(p^{k-1}) & J & J & \cdots & J & J & J & \cdots & \cdots & J-I
\end{array}
$$

FIGURE 6. $\Gamma(\mathbb{Z}_{p^k})$; when k is odd

Proposition.5.1,

$$
\begin{aligned}
\alpha(G) = & \quad \phi(\tfrac{p^k}{p}) + \phi(\tfrac{p^k}{p^2}) + \ldots + \phi(\tfrac{p^k}{p^{\lceil \frac{k}{2} \rceil -1}}) \\
= & \quad \phi(p^{k-1}) + \phi(p^{k-2}) + \ldots + \phi(p^{k-\lceil \frac{k}{2} \rceil +1}) \\
= & \quad p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor}.
\end{aligned}
$$

□

Theorem 5.2. Let $G = \Gamma(\mathbb{Z}_{p^k})$, $k \geq 3$.Then,

$$\omega(G) = \begin{cases} p^{\frac{k}{2}} - 1; & \text{if k is even,} \\ p^{\lfloor \frac{k}{2} \rfloor}; & \text{k is odd.} \end{cases}$$

Proof. A clique of a graph G is a subset of $V(G)$ which induces a complete subgraph in G. Thus the maximum size of a clique in $\Gamma(\mathbb{Z}_{p^k})$ is $|S(p^{\lceil \frac{k}{2} \rceil})| + |S(p^{\lceil \frac{k}{2} \rceil + 1})| + ... + |S(p^{k-1})|$; if k is even and one more than this number if k is odd.

$$\begin{aligned} |S(p^{\lceil \frac{k}{2} \rceil})| + |S(p^{\lceil \frac{k}{2} \rceil + 1})| + ... + |S(p^{k-1})| &= \phi(\frac{p^k}{p^{\lceil \frac{k}{2} \rceil}}) + ... + \phi(\frac{p^k}{p^{k-1}}) \\ &= \phi(p) + ... + \phi(p^{\lfloor \frac{k}{2} \rfloor}) \\ &= p^{\lfloor \frac{k}{2} \rfloor} - 1. \end{aligned}$$

Thus, $\omega(G) = \begin{cases} p^{\frac{k}{2}} - 1; & \text{if k is even,} \\ p^{\lfloor \frac{k}{2} \rfloor}; & \text{if k is odd.} \end{cases}$ □

Theorem 5.3. Let $G = \Gamma(\mathbb{Z}_{p^k})$, $k \geq 3$ for any prime p. Then, $gr(G) = 3$ except that $gr(\Gamma(\mathbb{Z}_8)) = \infty$

Proof. Consider $k \geq 3$.

If k is even, from the above theorem, we see that $\omega(G) \geq 3$, for any prime p.

If k is odd, $\omega(G) \geq 3$, for any prime $p \geq 3$. Thus the length of the shortest cycle is 3 in these cases. Also for $p = 2$ and $k = 3$, we see that the zero divisor graph contains no cycle. □

5.2. The eigenvalues $\lambda = 0$ and $\lambda = -1$ of $\Gamma(\mathbb{Z}_{p^k})$.

Matrix theory is a mode of conveying very important information regarding both structural and algebraic parameters of a graph. Here, from the point of view of Linear Algebra, the multiplicities of the eigenvalues 0 and -1 are calculated.

Theorem 5.4. Let $G = \Gamma(\mathbb{Z}_{p^k}), k \geq 3$. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1$ and $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$ respectively.

Proof. The adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$ contains repeated rows. Hence the determinant is zero. This indicates that $\lambda = 0$ is an eigenvalue. Since the adjacency matrix of any simple graph is real and symmetric, it follows that the algebraic multiplicity of $\lambda = 0$ is the nullity of the adjacency matrix, which is exactly the number of dependent rows in the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$.See Figure:5 and Figure:6. If $M = A(\Gamma(\mathbb{Z}_{p^k}))$, then in each of the first $\lceil \frac{k}{2} \rceil - 1$ blocks of M , all but one, are dependent rows. Thus,

$$\begin{aligned} nullity(M) = &\ |S(p)| + |S(p^2)| + ... + |S(p^{\lceil \frac{k}{2} \rceil - 1})| - (\lceil \tfrac{k}{2} \rceil - 1) \\ = &\ \phi(\frac{p^k}{p}) + \phi(\frac{p^k}{p^2}) + ... + \phi(\frac{p^k}{p^{\lceil \frac{k}{2} \rceil - 1}}) - (\lceil \tfrac{k}{2} \rceil - 1) \\ = &\ p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \tfrac{k}{2} \rceil + 1 \end{aligned}$$

Thus multiplicity of $\lambda = 0$ is $p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1$.

Also, we can see that $det(M+I) = 0$. Hence $\lambda = -1$ is an eigenvalue of M and multiplicity of $\lambda = -1$ is the nullity of $M+I$. In each of the last $\lfloor \frac{k}{2} \rfloor$ blocks of $M+I$, all but one, are dependent rows. Thus nullity of $M+I$ is given by,

$$\begin{aligned} nullity(M+I) = &\ |S(p^{\lceil \frac{k}{2} \rceil})| + |S(p^{\lceil \frac{k}{2} \rceil + 1})| + ... + |S(p^{k-1})| - \lfloor \tfrac{k}{2} \rfloor \\ = &\ p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \tfrac{k}{2} \rfloor - 1. \end{aligned}$$

Thus multiplicity of the eigenvalue $\lambda = -1$ is $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$. □

The other eigenvalues of $\Gamma(\mathbb{Z}_{p^k})$ are computed in section:6.

6. Eigenvalues of $\Gamma(\mathbb{Z}_n), n \neq p$, for Any Prime p.

Sriparna Chattopadhyay et.al have studied the structure of $\Gamma(\mathbb{Z}_n)$ and found that it is a generalised join of certain regular graphs. [21]

Let $n = p_1^{n_1} p_2^{n_2} ... p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes and $d_1, d_2, ..., d_{s(n)}$ be the proper divisors of n. let $\Gamma(S(d_i)), i = 1, 2, ..., s(n)$; denote the subgraph of $\Gamma(\mathbb{Z}_n)$, induced by $S(d_i)$; which are either $K_{\phi(\frac{n}{d_i})}$ or its complement $\overline{K}_{\phi(\frac{n}{d_i})}$. It is obvious that $\Gamma(S(d_i))$ is regular for each $i = 1, 2, ..., s(n)$.

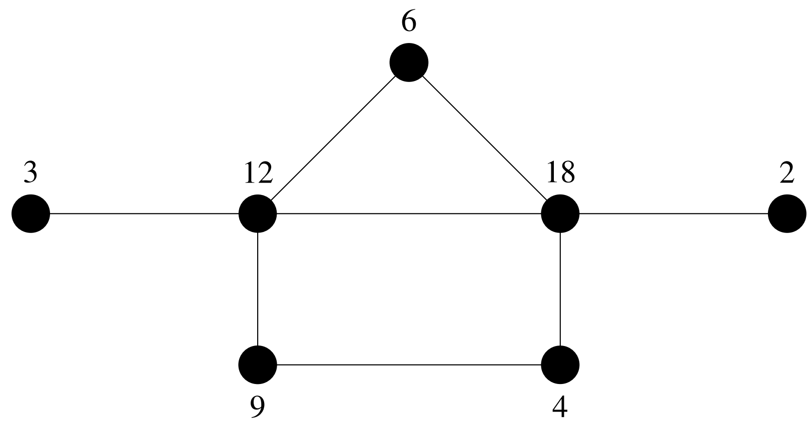

FIGURE 7. Υ_{36}

Proposition 6.1. [21] Let Υ_n denote the simple graph associated with $\Gamma(\mathbb{Z}_n)$ with vertices labeled as $d_1, d_2, ..., d_{s(n)}$ Then, the vertices d_i and d_j are adjacent in Υ_n if and only if n/d_id_j.

Example 6.1. For example, consider $\Gamma(\mathbb{Z}_{36})$. The number of proper divisors of 36 is $s(36) = 7$. They are precisely $2,3,4,6,9,12,18$. The non-zero divisors of $\Gamma(\mathbb{Z}_{36})$ is partitioned into 7 classes as follows. $S(2) = \{2,10,14,22,26,34\}$, $S(3) = \{3,15,21,33\}$, $S(4) = \{4,8,16,20,28,32\}$, $S(6) = \{6,30\}$, $S(9) = \{9,27\}$, $S(12) = \{12,24\}$, $S(18) = \{18\}$.
The graphs $\Gamma(\mathbb{Z}_{36})$ and Υ_{36} are given in Figure:7 and Figure:8, where the dotted lines indicate the join of graphs. Note that $\Gamma(S(6))$, $\Gamma(S(12))$ and $\Gamma(S(18)))$ are complete subgraphs, while the others are null graphs.

The following theorems are very crucial in this section.

Theorem 6.1. [21] $\Gamma(\mathbb{Z}_n) = \Upsilon_n \left[\Gamma(S(d_1)), \Gamma(S(d_2)), ..., \Gamma(S(d_{s(n)}))\right]$

The above theorem explains that the zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ is a generalised join of its subgraphs $\Gamma(S(d_i))$, for $i = 1,2...s(n)$.

Definition 6.1. [2] Let $V_1, V_2, ...V_m$ be an equitable partition of a graph G, with $|N(v) \cap V_j| = t_{ij}, 1 \leq i,j \leq m$, for all $v \in V_i$, then, $T = [t_{ij}]$ is called the matrix associated with the partition.

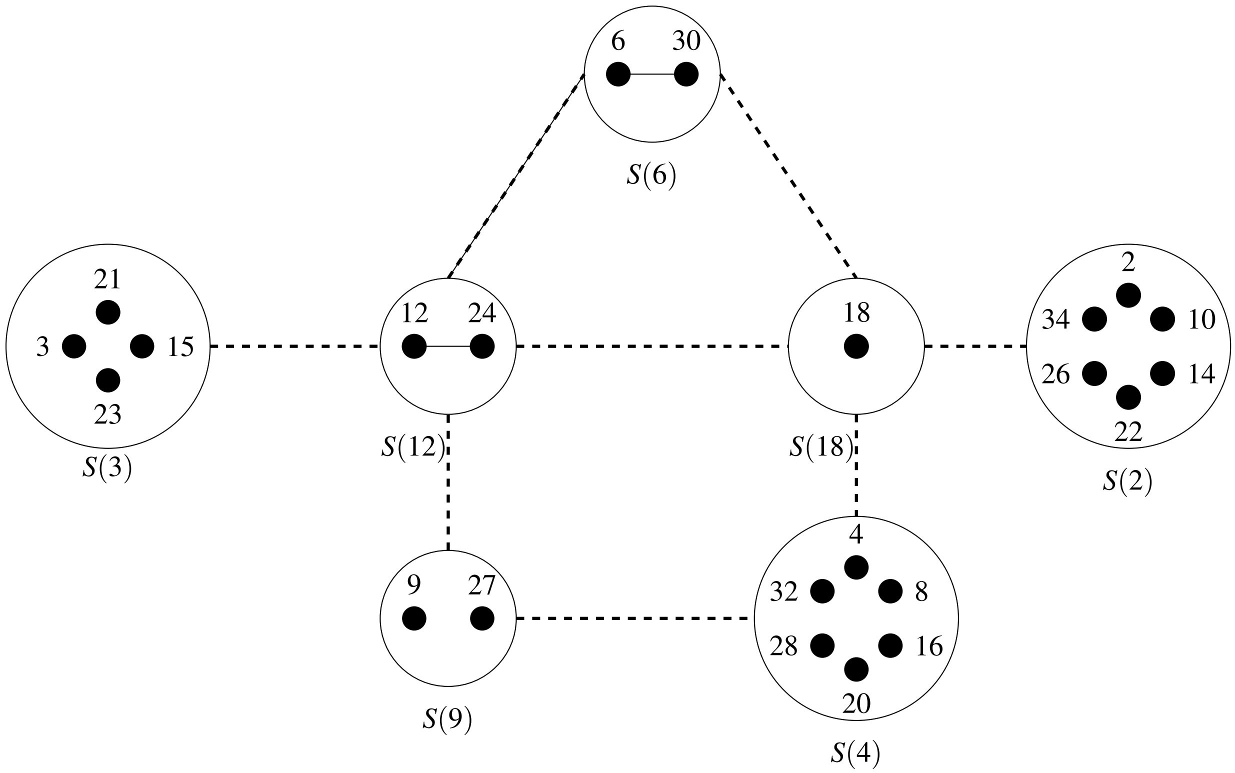

FIGURE 8. $\Gamma(\mathbb{Z}_{36})$

A.J. Schwenk [2] has described the spectrum of the generalised join of regular graphs.

Theorem 6.2. [2] Let G be a graph on p vertices. If $H_i, 1 \leq i \leq p$ are all r_i-regular graphs, then $V_1 \cup V_2 \cup ... \cup V_p$ is an equitable partition of $G[H_1, H_2, ...H_p]$. Let T denote the matrix associated with this partition, then the characteristic polynomial of the generalised composition is

$$\Phi(G[H_1, H_2, ...H_p]; x) = \Phi(T; x). \prod_{i=1}^{p} \frac{\Phi(H_i; x)}{(x - r_i)}.$$

The above theorem leads to a very exciting way of computing the eigenvalues of $\Gamma(\mathbb{Z}_n)$. First, we determine T, the matrix associated with the partition $S(d_1) \cup S(d_2) \cup ... \cup S(d_{s(n)})$. of the graph $\Gamma(\mathbb{Z}_n)$. Let $|N(v) \cap S(d_j)| = t_{ij}, 1 \leq i, j \leq s(n)$, for all $v \in S(d_i)$. Note that $S(d_i) \sim S(d_j)$ if and only if $n / d_i d_j$ and $S(d_i)$ induces a complete subgraph in G, if and only if n / d_i^2 and a null graph if and only if $n \nmid d_i^2$. Thus $T = [t_{ij}]_{s(n) \times s(n)}$ is defined as follows.

$$t_{ij} = \begin{cases} \phi(\frac{n}{d_j}); & \text{if } n \,/\, d_i d_j; \;\; i \neq j \\ \phi(\frac{n}{d_i}) - 1; & \text{if } n \,/\, d_i^2; \;\; i = j \\ 0; & \text{otherwise} \end{cases}$$

(14)

This description completely determines T and subsequently $\Phi(T;x)$. Now we explore the characteristic polynomial of $\Gamma(\mathbb{Z}_n)$ in a very convenient manner.

Theorem 6.3. Let $G = \Gamma(\mathbb{Z}_n), n \neq p$, for any prime p. Then the characteristic polynomial of G is $\Phi(G;x) = \Phi(T;x).\prod_{n/d_i^2}(x+1)^{\phi(\frac{n}{d_i})-1}.\prod_{n \nmid d_i^2} x^{\phi(\frac{n}{d_i})-1}$, where $T = [t_{ij}]$,

$$t_{ij} = \begin{cases} \phi(\frac{n}{d_j}); & \text{if } n \,/\, d_i d_j; \;\; i \neq j \\ \phi(\frac{n}{d_i}) - 1; & \text{if } n \,/\, d_i^2; \;\; i = j \\ 0; & \text{otherwise} \end{cases}$$

Proof. Let $n = p_1^{n_1} p_2^{n_2} ... p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes and assume that n is not a prime (to avoid triviality). Let $d_1, d_2, ...d_{s(n)}$ be the proper divisors of n. Let m_i denote the cardinality of $S(d_i), i = 1,2,...,s(n)$. Thus $m_i = \phi(\frac{n}{d_i}), i = 1,2,...,s(n)$. It is already seen that $\Gamma(S(d_i))$; the subgraph induced by $S(d_i)$ is either K_{m_i} or $\overline{K}_{m_i}$ which are regular of order $m_i - 1$ or 0 respectively; accordingly as n divides d_i^2 or not.

Thus, $\Phi(\Gamma(S(d_i));x) = \Phi(K_{m_i};x) = (x+1)^{m_i - 1}.(x - m_i + 1)$; if $n \,/\, d_i^2$ and

$\Phi(\Gamma(S(d_i));x) = \Phi(\overline{K}_{m_i};x) = x^{m_i}$; if $n \nmid d_i^2$.

Thus the conclusion follows from Theorem 6.1 and Theorem 6.2 .

□

Example 6.2. Consider $n = p^2 q$, where p and q are distinct primes $p < q$. The proper divisors of $p^2 q$ are $d_1 = p$, $d_2 = q$, $d_3 = p^2$ and $d_4 = pq$. Note that $S(d) = \{x \in \mathbb{Z}_n : gcd(x,n) = d\}$.

$S(d_1) = \{k_1 p : k_1 = 1,2,...pq-1; p \nmid k_1, q \nmid k_1.\}$

$S(d_2) = \{k_2 q : k_2 = 1,2,...p^2-1; p \nmid k_2.\}$

$S(d_3) = \{k_3 p^2 : k_3 = 1,2,...q-1.\}$

$S(d_4) = \{k_4 pq : k_4 = 1,2,...p-1.\}$

Clearly, $|S(d_1)| = (p-1)(q-1)$, $|S(d_2)| = p(q-1)$, $|S(d_3)| = q-1$, and $|S(d_4)| = p-1$. These sets form an equitable partition for the vertex set of $\Gamma(\mathbb{Z}_n)$ as we have seen in section 3.1. Also, the matrix of partition,

$$T = \begin{bmatrix} 0 & 0 & 0 & p-1 \\ 0 & 0 & q-1 & 0 \\ 0 & p(p-1) & 0 & p-1 \\ (p-1)(q-1) & 0 & q-1 & p-2 \end{bmatrix}.$$

The characteristic polynomial of this matrix is given by, $\Phi(T;x) = det(T-xI)$. Thus,
$\Phi(T;x) = x^4-(p-2)x^3-2p(p-1)(q-1)x^2+p(p-1)(p-2)(q-1)x+p(p-1)^3(q-1)^2$.
Let $G_i = \Gamma(S(d_i)), i = 1,2,3,4.$. Note that G_1, G_2, G_3 are null graphs of order $(p-1)(q-1)$, $p(p-1)$ and $q-1$ respectively and G_4 is a complete graph of order $p-1$ which is regular of degree $p-2$. Thus,

$\Phi(G_1;x) = x^{(p-1)(q-1)}$

$\Phi(G_2;x) = x^{p(p-1)}$

$\Phi(G_3;x) = x^{q-1}$

$\Phi(G_4;x) = (x+1)^{p-2}(x-p+2)$.

Hence the characteristic polynomial of $\Gamma(\mathbb{Z}_{p^2q})$ is,

$$\Phi(\Gamma(\mathbb{Z}_{p^2q};x)) = (x+1)^{p-2}.x^{(p-1)(p+q-1)+(q-4)}.\Phi(T;x);$$

where $\Phi(T;x) = x^4-(p-2)x^3-2p(p-1)(q-1)x^2+p(p-1)(p-2)(q-1)x+p(p-1)^3(q-1)^2$.

Remark 6.1. In the above example, the order of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2q})$ is $p^2+pq-p-1$, by Proposition 3.1. Theorem 6.3 reduces the inconvenience of handling a huge matrix of order $p^2+pq-p-1$ in finding the eigenvalues, by means of a 4×4 matrix of partition and thereby serves the purpose of bypassing the tedious traffic of direct computation using matrix operations.

Corollary 6.1. The characteristic polynomial of $\Gamma(\mathbb{Z}_{p^k}), k \geq 2$ is given by

$$\Phi(\Gamma(\mathbb{Z}_{p^k});x) = \Phi(T;x). \prod_{i<\lceil \frac{k}{2} \rceil} x^{(p-1)p^{(k-i-1)}}. \prod_{i\geq\lceil \frac{k}{2} \rceil} (x+1)^{(p-1)p^{(k-i-1)}},$$

where $T = [t_{ij}]_{(k-1)\times(k-1)}$,

$$t_{ij} = \begin{cases} (p-1)p^{k-j-1}; & \text{if } i+j \geq k;\ \ i \neq j \\ (p-1)p^{k-j-1}-1; & \text{if } i+j \geq k;\ \ i = j \\ 0; & \text{otherwise} \end{cases}$$

Proof. $S(d_i)$ induces a complete subgraph in $\Gamma(\mathbb{Z}_n)$ if and only if n/d_i^2; and a null graph otherwise. hence if $n = p^k; k \geq 2$, $S(p^i)$ induces a complete subgraph of order $p^{k-i-1}(p-1)$ if $i \geq \lceil \frac{k}{2} \rceil$ or a null graph otherwise. □

Example 6.3. For $G = \Gamma(\mathbb{Z}_{p^4})$, it can be seen from section 3.1 that the matrix of partition of the vertex set of G is given by,

$$T = \begin{bmatrix} 0 & 0 & p-1 \\ 0 & p(p-1)-1 & p-1 \\ p^2(p-1) & p(p-1) & p-2 \end{bmatrix}.$$

Thus $\Phi(T;x)$ is obtained as,
$\Phi(T;x) = x^3-(p^2-3)x^2-(p^4-2p^3+2p^2-2)x+p^2(p-1)^2(p^2-p-1)$. Applying Corollary 6.1, we see that the characteristic polynomial of $\Gamma(\mathbb{Z}_{p^4})$ is,

$$\Phi(\Gamma(\mathbb{Z}_{p^4});x) = (x+1)^{p^2-3}.x^{p^3-p^2-1}.\Phi(T;x),$$

where $\Phi(T;x) = x^3-(p^2-3)x^2-(p^4-2p^3+2p^2-2)x+p^2(p-1)^2(p^2-p-1)$

Thus Theorem 6.3 and Corollary 6.1 are the generalisation of results in section:3 and section:4.

Conclusion

This paper is an attempt to explore the spectrum of the zero-divisor graphs on the ring of integers modulo n. In this paper, the characteristic polynomial of the adjacency matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is investigated so that the spectrum of this graph can be found for any natural number using numerical methods. Zero-divisor graphs are used to model networks of communication , network flow and clique problems.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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