Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 5, 1697-1711
https://doi.org/10.28919/jmcs/4742
ISSN: 1927-5307

k-TOTAL MEAN CORDIAL GRAPHS

R. PONRAJ[1,*], S. SUBBULAKSHMI[2,†], S. SOMASUNDARAM[2]

[1]Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, Tamilnadu, India

[2]Department of Mathematics, Manonmaniam Sundarnar University, Abishekapatti, Tirunelveli-627012, Tamilnadu, India

Abstract. Let G be a (p,q) graph. Let $f : V(G) \to \{0,1,2,3,\ldots,k-1\}$ be a function where $k \in N$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean cordial labeling of G if $\left|t_{mf}(i) - t_{mf}(j)\right| \leq 1$, $i, j \in \{0,1,2,\ldots,k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0,1,2,\ldots,k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.

Keywords: path; cycle; complete graph; star; bistar; comb; crown.

2010 AMS Subject Classification: 05C78.

1. Introduction

Graphs in this paper are finite, simple and undirected. Graph labeling was first initiated by in the name of graceful labeling by Rosa [5]. Subsequently harmonious labeling introduced by Graham and Solane [3] and cordial labeling by Cahit [1]. In this paper, we introduce k-total mean cordial graphs and studied the k-total mean cordial behaviour of some graphs and

*Corresponding author

E-mail address: ponrajmaths@gmail.com

†Research Scholar, Reg. No: 19124012092011

Received May 31, 2020

investigate 4-total mean cordial labeling behaviour of cycle, complete graph, star, bistar, comb and crown. Terms are not defined here follow from Harary[4] and Gallian[2].

2. PRELIMINARIES

Definition 2.1. Let G be a (p,q) graph. Let $f: V(G) \rightarrow \{0,1,2,3,\ldots,k-1\}$ be a function where $k \in N$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean cordial labeling of G if $\left|t_{mf}(i) - t_{mf}(j)\right| \leq 1$, $i, j \in \{0,1,2,\ldots,k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0,1,2,\ldots,k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.

Remark. 2-total mean cordial labeling is a total product cordial labeling.

Remark. 3-total mean cordial labeling is a total mean cordial labeling.

3. MAIN RESULTS

Theorem 3.1. Every graph is a subgraph of a connected k-total mean cordial graph.

Proof. Let G be a (p,q) graph. Consider k-copies of the complete graph K_p and $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_p^{(i)}$ be the vertices of the i^{th} copy of K_p $(1 \leq i \leq p)$.
The super graph G^* of G is obtained from k-copies of K_p by joining the vertices $v_1^{(i)}$ and $v_1^{(i+1)}$ $(1 \leq i \leq p-1)$.
Now assign the label 0 to the all the vertices of the first copy of K_p. Next assign the label 1 to all the vertices of the second copy of K_p. Proceeding like this assign the label $k-1$ to the all the vertices of k^{th} copy of K_p. That is assign the label r to all the vertices of $(r+1)^{th}$ copy of K_p $(0 \leq r \leq k-1)$.
Clearly $t_{mf}(0) = \frac{p(p+1)}{2}$, $t_{mf}(1) = t_{mf}(2) = \ldots = t_{mf}(k-1) = \frac{p(p+1)}{2} + 1$.
Therefore G^* is a connected k-total mean cordial graph. □

Theorem 3.2. Any path is k-total mean cordial.

Proof. Let P_n be the path $u_1u_2u_3\ldots u_n$ and $n = kt + r$, $0 \leq r < n$.

Consider the vertices $u_1, u_2, u_3, \dots, u_t$. Assign the label $k-1$ to the vertices $u_1, u_2, \dots, u_t$. Next assign the label $k-2$ to the vertices $u_{t+1}, u_{t+2}, \dots, u_{2t}$. We now assign the label $k-3$ to the vertices $u_{2t+1}, u_{2t+2}, \dots, u_{3t}$. Proceeding like this assign the label 1 to the vertices $u_{(k-2)t+1}, u_{(k-2)t+2}, \dots, u_{(k-1)t}$ and 0 to the vertices $u_{(k-1)t+1}, u_{(k-1)t+2}, \dots, u_{(k)t}$.
Now we consider the vertices $u_{kt+1}, u_{kt+2}, \dots, u_{kt+r}$. Assign the even integer $0, 2, 4, \dots$ to the vertices $u_{kt+1}, u_{kt+2}, \dots$with the condition that even number are $\leq k-1$. If all the even numbers $(\leq k-1)$ are exhausted then assign the odd integers $k, k-2, k-4, \dots$ if k is odd or $k-1, k-3, k-5, \dots$ if k is even consecutively to the remaining non-labelled vertices. It is easy to verify that this vertex labeling is a k-total mean cordial labeling. □

Theorem 3.3. The cycle C_n is 4-total mean cordial for all n.

Proof. Let C_n be the cycle $u_1 u_2 u_3 \dots u_n u_1$.

Case 1. $n \equiv 0 \pmod 4$.
Assign the label 0 to the $\frac{n-4}{4}$ vertices $u_1, u_2, \dots, u_{\frac{n-4}{4}}$. We now assign the label 1 to the $\frac{n-4}{4}$ vertices $u_{\frac{n}{4}}, u_{\frac{n+4}{4}}, \dots, u_{\frac{n-2}{2}}$. Next assign the label 2 to the $\frac{n-4}{4}$ vertices $u_{\frac{n}{2}}, u_{\frac{n+2}{2}}, \dots, u_{\frac{3n-12}{4}}$. Now assign the label 3 to the $\frac{n-4}{4}$ vertices $u_{\frac{3n-8}{4}}, u_{\frac{3n-4}{4}}, \dots, u_{n-4}$. Finally assign the labels $3, 0, 0, 2$ to the non-labelled vertices $u_{n-3}, u_{n-2}, u_{n-1}, u_n$.

Case 2. $n \equiv 1 \pmod 4$.
Assign the label 0 to the $\frac{n-1}{4}$ vertices $u_1, u_2, \dots, u_{\frac{n-1}{4}}$. Next assign the label 1 to the $\frac{n-1}{4}$ vertices $u_{\frac{n+3}{4}}, u_{\frac{n+7}{4}}, \dots, u_{\frac{n-1}{2}}$. We now assign the label 2 to the $\frac{n-1}{4}$ vertices $u_{\frac{n+1}{2}}, u_{\frac{n+3}{2}}, \dots, u_{\frac{3n-3}{4}}$. Next assign the label 3 to the $\frac{n-1}{4}$ vertices $u_{\frac{3n+1}{4}}, u_{\frac{3n+5}{4}}, \dots, u_{n-1}$. Finally assign the label 0 to the vertex u_n.

Case 3. $n \equiv 2 \pmod 4$.
Assign the label 0 to the $\frac{n-2}{4}$ vertices $u_1, u_2, \dots, u_{\frac{n-2}{4}}$. Next assign the label 1 to the $\frac{n-2}{4}$ vertices $u_{\frac{n+2}{4}}, u_{\frac{n+6}{4}}, \dots, u_{\frac{n-2}{2}}$. We now assign the label 2 to the $\frac{n-2}{4}$ vertices $u_{\frac{n}{2}}, u_{\frac{n+2}{2}}, \dots, u_{\frac{3n-6}{4}}$. Now assign the label 3 to the $\frac{n-2}{4}$ vertices $u_{\frac{3n-2}{4}}, u_{\frac{3n+2}{4}}, \dots, u_{n-2}$. Finally assign the labels 2,0 to the

non-labelled vertices u_{n-1},u_n.

Case 4. $n \equiv 3 \pmod 4$.

Assign the label 0 to the $\frac{n-3}{4}$ vertices $u_1, u_2, \ldots, u_{\frac{n-3}{4}}$. Next assign the label 1 to the $\frac{n-3}{4}$ vertices $u_{\frac{n+1}{4}}, u_{\frac{n+5}{4}}, \ldots, u_{\frac{n-3}{2}}$. We now assign the label 2 to the $\frac{n-3}{4}$ vertices $u_{\frac{n-1}{2}}, u_{\frac{n+1}{2}}, \ldots, u_{\frac{3n-9}{4}}$. Now assign the label 3 to the $\frac{n-3}{4}$ vertices $u_{\frac{3n-5}{4}}, u_{\frac{3n-1}{4}}, \ldots, u_{n-3}$. Finally assign the labels 2,0,0 to the non-labelled vertices u_{n-2},u_{n-1},u_n.

This vertex labeling f is a 4-total mean cordial labeling follows from the Table 1.

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n \equiv 0 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$
$n \equiv 1 \pmod 4$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$
$n \equiv 3 \pmod 4$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$

TABLE 1

□

The following lemmas will be used for investigation of Complete graph.

Lemma 1. n^2+n+1 is not a perfect square for all n.

Proof. Suppose $n^2+n+1 = m^2$

$\Rightarrow 4n^2+4n+4 = 4m^2$

$\Rightarrow (2n+1)^2+3 = (2m)^2$

$\Rightarrow (2m)^2-(2n+1)^2 = 3$

$\Rightarrow (2m+2n+1)(2m-2n-1) = 3$

$\Rightarrow 2m+2n+1 = 3 \longrightarrow (1)$

and $2m-2n-1 = 1 \longrightarrow (2)$

From (1) and (2), $\Rightarrow m = 1$ and $n = 0$. □

Lemma 2. n^2+n+3 is not a square for all $n \neq 2$.

Proof. As in Lemma 1, we get the relations $\Rightarrow 2m+2n+1=3 \longrightarrow (1)$

and $2m-2n-1=1 \longrightarrow (2)$

From (1) and (2) $\Rightarrow m=3$ and $n=2$

It follows that n^2+n+3 is square for $n=2$ only. □

Lemma 3. n^2+n+5 is not a square for all $n \neq 4$.

Proof. As in the same tecnique in Lemma 1, we get the relations $\Rightarrow 2m+2n+1=19$

and $2m-2n-1=1$

This implies $m=5$ and $n=4$. □

Lemma 4. n^2+n+7 is not a square if $n \notin \{1,6\}$.

Proof. As in Lemma 3, we get $2m+2n+1=27$

and $2m-2n-1=1$

(or)

$2m+2n+1=27$

and $2m-2n-1=1$

$\Rightarrow m=7$ and $n=6$ (or) $m=3$ and $n=1$

$\Rightarrow n=1$ (or) $n=6$. □

Lemma 5. If $n>1$, n^2+n-1 is not a square.

Proof. As in Lemma 3, we get the relations $2m+2n+1=5$

and $2n-2m+1=1$

$\Rightarrow m=1$ and $n=1$. □

Lemma 6. If $n \neq 3$, n^2+n-3 is not a square.

Proof. We get the relations $2m+2n+1=13$

and $2n-2m+1=1$

Consequently we have $m=3$ and $n=3$. □

Lemma 7. If $n \notin \{2,5\}$, n^2+n-5 is not a square.

Proof. Here we have $2m+2n+1=7$

and $2n-2m+1=1$

(or)

$2m+2n+1=21$

and $2n-2m+1=1$

$\Rightarrow n=2 \ (or) \ n=5$. □

Theorem 3.4. The complete graph K_n is 4-total mean cordial if and only if $n \leq 4$

Proof. Suppose f is a 4-total mean cordial label of K_n.

Clearly $|V(K_n)|+|E(K_n)| = \frac{n(n+1)}{2}$.

Suppose s vertices are labelled with 0.

$\Rightarrow t_{mf}(0) = s+\frac{s(s-1)}{2}$

$= \frac{s(s+1)}{2} \longrightarrow (1)$

Case 1. $n \equiv 0,7 \pmod 8$.

In this case $t_{mf}(0) = \frac{n(n+1)}{8} \longrightarrow (2)$

from (1) and (2), $\Rightarrow \frac{n(n+1)}{8} = \frac{s(s+1)}{2}$

$\Rightarrow \frac{n(n+1)}{4} = s(s+1)$

$\Rightarrow 4s^2+4s-n^2-n=0$

$\Rightarrow s = \frac{-4\pm\sqrt{16+16\left(n^2+n\right)}}{8}$

$\Rightarrow s = \frac{-4\pm4\sqrt{n^2+n+1}}{8}$

$\Rightarrow s = \frac{-1\pm\sqrt{n^2+n+1}}{2}$, a contradiction to Lemma 1

Case 2. $n \equiv 2,5 \pmod 8$. $n \neq 2$ and $n \neq 5$.

In this case $t_{mf}(0) = \frac{n^2+n+1}{8}$ (or)

$t_{mf}(0) = \frac{n^2+n-6}{8}$

Subcase 1. $t_{mf}(0) = \frac{n^2+n+2}{8}$

$\Rightarrow \frac{s(s+1)}{2} = \frac{n^2+n+2}{8}$

$\Rightarrow 4s^2 + 4s - \left(n^2+n+2\right) = 0$

$\Rightarrow s = \frac{-4 \pm \sqrt{16+16\left(n^2+n\right)}}{8}$

$= \frac{-1 \pm \sqrt{n^2+n+3}}{2}$, a contradiction to Lemma 2.

Subcase 2. $t_{mf}(0) = \frac{n^2+n-6}{8}$

In this case, $s = \frac{-1 \pm \sqrt{n^2+n-5}}{2}$, a contradiction to Lemma 7.

Case 3. $n \equiv 3,4 \pmod 8$. $n \neq 3$ and $n \neq 4$.

In this case $t_{mf}(0) = \frac{n^2+n-4}{8}$ (or)

$t_{mf}(0) = \frac{n^2+n+4}{8}$

Subcase 1. $t_{mf}(0) = \frac{n^2+n-4}{8}$

Clearly $s = \frac{-1 \pm \sqrt{n^2+n-3}}{2}$, a contradiction to Lemma 6.

Subcase 2. $t_{mf}(0) = \frac{n^2+n+4}{8}$

Here, $s = \frac{-1 \pm \sqrt{n^2+n+5}}{2}$, a contradiction to Lemma 3.

Case 4. $n \equiv 1,6 \pmod 8$. $n \neq 1$ and $n \neq 6$.

In this case $t_{mf}(0) = \frac{n^2+n-2}{8}$ (or)

$t_{mf}(0) = \frac{n^2+n+6}{8}$

Subcase 1. $t_{mf}(0) = \frac{n^2+n-2}{8}$

Clearly $s = \frac{-1\pm\sqrt{n^2+n-1}}{2}$, a contradiction to Lemma 5.

Subcase 2. $t_{mf}(0) = \frac{n^2+n+6}{8}$

In this case, $s = \frac{-1\pm\sqrt{n^2+n+7}}{2}$, a contradiction to Lemma 4.

Case 5. $n \in \{1,2,3,4\}$.

A 4-total mean cordial labeling is given in Table 2

n	u_1	u_2	u_3	u_4
1	0			
2	0	2		
3	0	2	3	
4	0	0	2	3

TABLE 2

Case 6. $n = 5$.

Suppose $t_{mf}(0) = 3$

$\Rightarrow \frac{s(s+1)}{2} = 3$

$\Rightarrow s = 2\,(or) - 3$.

$s = -3$ is not possible.

When $s = 2$, Assume $f(u_1) = f(u_2) = 0$.

Then atleast two vertices receive the label 3.

Assume $f(u_3) = f(u_4) = 3$.

If $f(u_5) = 3$, then $t_{mf}(2) \geq 5$, a contradiction

If $f(u_5) = 1$, then $t_{mf}(2) \geq 6$, a contradiction

Case 7. $n = 6$.

Suppose $t_{mf}(0) = 6$

$\Rightarrow \frac{s(s+1)}{2} = 6$

$\Rightarrow s = 3\,(or) - 4.$

Clearly $s = -4$ is not possible.

When $s = 3$, Assume $f(u_1) = f(u_2) = f(u_3) = 0$.

If more than one vertex receive the label 3, then $t_{mf}(2) \geq 6$, a contradiction

Assume $f(u_4) = 3$, this implies $t_{mf}(3) = 3$, a contradiction. □

Theorem 3.5. The star $K_{1,n}$ is a 4-total mean cordial for all values of n.

Proof. Let u be the centre vertex of the star $K_{1,n}$. Let u_i $(1 \leq i \leq n)$ be the pendant vertices adjacent to u.

Assign the label 1 to the vertex u.

Case 1. n is even.

Consider the vertices $u_1, u_2, \ldots, u_n$. Assign the label 0 to the $\frac{n}{2}$ vertices $u_1, u_2, \ldots, u_{\frac{n}{2}}$. Next assign the label 3 to the $\frac{n}{2}$ vertices $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, \ldots, u_n$.

Case 2. n is odd.

Assign the label 0 to the $\frac{n-1}{2}$ vertices $u_1, u_2, \ldots, u_{\frac{n-1}{2}}$. Next assign the label 3 to the $\frac{n+1}{2}$ vertices $u_{\frac{n+1}{2}}, u_{\frac{n+3}{2}}, \ldots, u_n$.

This vertex labeling f is a 4-total mean cordial labeling follows from the Table 3 □

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
n is even	$\frac{n}{2}$	$\frac{n+2}{2}$	$\frac{n}{2}$	$\frac{n}{2}$
n is odd	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$

TABLE 3

Theorem 3.6. The bistar $B_{n,n}$ is 4-total mean cordial for all n.

Proof. Let u, v be the centre vertices of the bistar $B_{n,n}$. Let u_i $(1 \le i \le n)$ be the pendant vertices adjacent to u and v_i $(1 \le i \le n)$ be the pendent vertices adjacent to v. $E(B_{n,n}) = \{uv\} \cup \{uu_i, vv_i : 1 \le i \le n\}$.

Case 1. n is even.

Let $n = 2t$, $t \in N$.

Assign the labels 0, 2 respectively to the central vertices u,v.

Consider the vertices $u_1, u_2, \ldots, u_n$. Assign the label 0 to the t vertices $u_1, u_2, \ldots, u_t$. Next assign the label 1 to the t vertices $u_{t+1}, u_{t+2}, \ldots, u_{2t}$. We now move to the vertices $v_1, v_2, \ldots, v_n$. Assign the label 2 to the t vertices $v_1, v_2, \ldots, v_t$. Next assign the label 3 to the t vertices $v_{t+1}, v_{t+2}, \ldots, v_{2t}$.

Case 2. n is odd.

Let $n = 2t + 1$, $t \in N$.

Assign the labels 1, 2 to the central vertices u,v respectively.

Assign the label 0 to the $2t+1$ vertices $u_1, u_2, \ldots, u_{2t+1}$. We now assign the label 2 to the t vertices $v_1, v_2, \ldots, v_t$. Next assign the label 3 to the $t+1$ vertices $v_{t+1}, v_{t+2}, \ldots, v_{2t+1}$.

This vertex labeling f is a 4-total mean cordial labeling follows from the Table 4

□

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2t$	$2t+1$	$2t+1$	$2t+1$	$2t$
$n = 2t+1$	$2t+1$	$2t+2$	$2t+2$	$2t+2$

TABLE 4

Theorem 3.7. The comb $P_n \odot K_1$ is 4-total mean cordial for all values of n.

Proof. Let P_n be the path $u_1 u_2 \ldots u_n$.

Let $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \le i \le n\}$ and $E(P_n \odot K_1) = E(P_n) \cup \{u_i v_i : 1 \le i \le n\}$.

Case 1. n is odd.

Consider the vertices $u_1, u_2, \ldots, u_n$. Assign the label 0 to the $\frac{n+1}{2}$ vertices $u_1, u_2, \ldots, u_{\frac{n+1}{2}}$. Next assign the label 1 to the next $\frac{n-1}{2}$ vertices $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \ldots, u_n$. We now move to the vertices $v_1, v_2, \ldots, v_n$. Assign the label 3 to the n vertices $v_1, v_2, \ldots, v_n$.

Case 2. n is even.

Assign the label 0 to the $\frac{n}{2}$ vertices $u_1, u_2, \ldots, u_{\frac{n}{2}}$. Next assign the label 1 to the $\frac{n}{2}$ vertices $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, \ldots, u_n$. Finally assign the label 3 to the n vertices $v_1, v_2, \ldots, v_n$.

This vertex labeling f is 4-total mean cordial labeling follows from the Tabel 5

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
n is odd	n	$n-1$	n	n
n is even	$n-1$	n	n	n

TABLE 5

□

Theorem 3.8. The crown $C_n \odot K_1$ is 4-total mean cordial for all n.

Proof. Let C_n be the cycle $u_1u_2\ldots u_nu_1$. Let $V(C_n \odot K_1) = V(C_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_1) = E(C_n) \cup \{u_iv_i : 1 \leq i \leq n\}$.

Case 1. n is odd.

Clearly assign the vertex labeling as in Case 1 of Theorem3.7 is also a 4-total mean cordial labeling.

Case 2. n is even.

Consider the vertices $u_1, u_2, \ldots, u_n$. Assign the label 0 to the $\frac{n}{2}$ vertices $u_1, u_2, \ldots, u_{\frac{n}{2}}$. Next assign the label 3 to the vertex $u_{\frac{n+2}{2}}$. We now assign the label 1 to the non-labelled vertices

$u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \ldots, u_n$. We now move to the vertices $v_1, v_2, \ldots, v_n$. Assign the label 2 to the vertex v_1. Next assign the label 3 to the vertices $v_2, v_3, \ldots, v_{n-1}$. Finally assign the label 0 to the vertex v_n.

This vertex labeling f is 4-total mean cordial labeling follows from the Tabel 6

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
n is odd	n	n	n	n
n is even	n	n	n	n

TABLE 6

□

Theorem 3.9. The Book with triangular pages, $K_2 + mK_1$ is 4-total mean cordial if and only if $m \equiv 0, 1, 2, 4, 5, 6 \pmod 8$.

Proof. Let $V(K_2 + mK_1) = \{u, v, u_j : 1 \leq j \leq m\}$ and
$E(K_2 + mK_1) = \{uv, uu_j, vu_j : 1 \leq j \leq m\}$.
Note that the order and size of $K_2 + mK_1$ are $m + 2$ and $2m + 1$ respectively. Assign the labels 0, 2 respectively to the vertices u, v.

Case 1. $m \equiv 0 \pmod 8$.
Let $m = 8r$, $r \in N$.

Now we consider the vertices $u_1, u_2, \ldots, u_r$. Assign the label 0 to the $3r$ vertices $u_1, u_2, \ldots, u_{3r}$. Next assign the label 1 to the r vertices $u_{3r+1}, u_{3r+2}, \ldots, u_{4r}$. We now assign the label 2 to the r vertices $u_{4r+1}, u_{4r+2}, \ldots, u_{5r}$ and finally assign the label 3 to the remaining $3r$ vertices $u_{5r+1}, u_{5r+2}, \ldots, u_{8r}$.

Case 2. $m \equiv 1 \pmod 8$.

Let $m = 8r+1$, $r \geq 0$.

As in Case 1 assign the label to the vertices $u_i\,(1 \leq i \leq 8r)$. Finally assign the label 3 to the vertex u_{8r+1}.

Case 3. $m \equiv 2 \pmod 8$.

Let $m = 8r+2$, $r \geq 0$.

Label the vertices $u_i\,(1 \leq i \leq 8r+1)$ as in Case 2. Next assign the label 0 to the vertex u_{8r+2}.

Case 4. $m \equiv 4 \pmod 8$.

Let $m = 8r+4$, $r \geq 0$.

In this case assign the label for the vertices $u_i\,(1 \leq i \leq 8r+2)$ as in Case 3. We now assign the labels $1,3$ to the vertices u_{8r+3}, u_{8r+4}.

Case 5. $m \equiv 5 \pmod 8$.

Let $m = 8r+5$, $r \geq 0$.

Assign the label for the vertices $u_i\,(1 \leq i \leq 8r+4)$ as in Case 4. Now assign the label 0 to the vertex u_{8r+5}.

Case 6. $m \equiv 6 \pmod 8$.

Let $m = 8r+6$, $r \geq 0$.

As in Case 5 assign the label to the vertices $u_i\,(1 \leq i \leq 8r+5)$. Finally assign the label 3 to the vertex u_{8r+6}.

Thus this vertex labeling f is 4-total mean cordial labeling follows from the Tabel 7

Case 7. $m \equiv 3 \pmod 8$.

Let $m = 8r+3$, $r \geq 0$.

$\Rightarrow t_{mf}(0) = t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 6r+3.$

Nature of m	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$m=8r$	$6r+1$	$6r+1$	$6r+1$	$6r$
$m=8r+1$	$6r+1$	$6r+1$	$6r+2$	$6r+2$
$m=8r+2$	$6r+3$	$6r+2$	$6r+2$	$6r+2$
$m=8r+4$	$6r+3$	$6r+4$	$6r+4$	$6r+4$
$m=8r+5$	$6r+5$	$6r+5$	$6r+4$	$6r+4$
$m=8r+6$	$6r+5$	$6r+5$	$6r+5$	$6r+6$

TABLE 7

Subcase (i). $f(u)=f(v)=0$

In this case, 3 is the label of the vertices only.

Assume $f(u_i)=3$, $1\le i\le 6r+3$.

This implies $t_{mf}(2)\ge 6r+3+6r+3=12r+6$, a contradiction.

Subcase (ii). $f(u)=0$, $f(v)=2$

As in Subcase (i), $t_{mf}(2)\ge 12r+6$, a contradiction.

Subcase (iii).$f(u)=0$, $f(v)=1$

Here also, $t_{mf}(2)\ge 12r+6$, a contradiction.

Subcase (iv).$f(u)=0$, $f(v)=3$

Clearly $3r+1$ vertices receive the label 3. Without loss of generality assume $f(u)=3$, $1\le i\le 3r+1$. Similarly $3r+1$ vertices receive the label 3 and assume $f(u_i)=0$, $3r+2\le i\le 6r+2$.

For the label 1, $\left[\frac{6r+3}{2}\right]$ vertices receives 1.

This implies $t_{mf}(2)\ge 6r+3$, a contradiction.

Case 8. $m\equiv 7 \pmod 8$.

Let $m=8r+7$, $r\ge 0$.

$\Rightarrow\Rightarrow t_{mf}(0) = t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 6r+6.$

Similar to Case 7, a contradiction. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] I. Cahit, Cordial Graphs: A weaker version of Graceful and Harmonious graphs, Ars Combin. 23 (1987), 201-207.

[2] J.A. Gallian, A Dynamic survey of graph labeling, Electron. J. Comb. 19 (2016), #DS6.

[3] R.L. Graham, N.J.A. Solane, On additive bases and Harmonious graphs, SIAM J. Algebraic Discrete Methods, 1 (1980), 382-404.

[4] F. Harary, Graph theory, Addision wesley, New Delhi, 1969.

[5] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs, Internat. Sympos., ICC Rome 1966, Paris, Dunod (1967), 349-355.