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THE ADOMIAN DECOMPOSITION METHOD SOLUTION OF THE INVISCID BURGERS EQUATION

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Abstract. We study the inviscid Burgers equation which models nonlinear wave propagation. We derive the inviscid Burgers equation from the Navier-Stokes equation and solve it using the Adomian decomposition method. By means of numerical examples we show that the Adomian decomposition method produces results that compare favourably with the exact solution obtained using the method of characteristics.

Keywords: Burgers equation; inviscid Burgers equation; Navier-Stokes equation; characteristic solution.

2010 AMS Subject Classification: 49M27, 35L60, 35Q30, 35Q35.

1. INTRODUCTION

Burgers equation is one of the most important partial differential equations (PDEs) in the theory of nonlinear conservation laws and occurs in many areas of applied mathematics, e.g., fluid mechanics, nonlinear acoustics, gas dynamics, cosmology, quantum mechanics and traffic flow. Burgers equation, which is an approximation of the one-dimensional nonlinear propagation of weak shock waves in a fluid [1], has been studied extensively in the literature. Named after Johannes Martinus Burgers (1895-1981), Burgers equation was derived in a physical context by

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Bateman in 1915. Whitham [2] argues that Burgers equation is the simplest PDE combining both nonlinear propagation and diffusive effects. El Malek and El-Mansi [3] estimated the solution of Burgers equation initially obtained by Cole [4].

On the whole line Burgers equation, which may be viscid or inviscid, is known to possess travelling shock wave solutions. Kuo and Lee [5] have shown that though Burgers equation is a nonlinear PDE, it can be solved exactly. Using the method of characteristics Bendaas [6] gave the exact solution to the inviscid Burgers equation $u_t + uu_x = 0,\ \forall x \in \mathbb{R},\ t > 0,\ u(\xi, 0) = f(\xi)$ in parametric form as $u(x,t) = f(\xi),\ x = \xi + f(\xi)t$. This is equivalent to the implicit solution $u(x,t) = f(x - u(x,t)t)$ or, recursively, $u_{n+1} = f(x - u_n t),\ n \geq 0$, for any $f(x)$. The inviscid Burgers equation is a prototype for equations for which the solutions can develop discontinuities, i.e., shock waves [7]. A recent study describes the Burgers equation as a mathematical model for the one-dimensional groundwater recharge by spreading [8].

The value of studying the Burgers equation has been underscored by Bonkile et al. [1] as follows:

(1) Its exact solution is well known.
(2) The equation can be thought of as a hyperbolic equation with artificial diffusion for small kinematic viscosity ν or as a heat equation for very small fluid velocity u.
(3) It can be used in boundary layer calculations for viscous fluid flow.
(4) It constitutes a standard test problem for use in PDE solvers.
(5) It is suitable for analysis in various fields such as those earlier highlighted, necessitating a multidisciplinary approach in its study (see Figure 1).

More on Burgers equation can be found in the excellent review paper by Bonkile et al. [1] and references therein. A wide range of numerical methods have been used to solve Burgers equation, including finite differences [9, 10, 11], finite elements [12], boundary elements [13], parameter-uniform implicit difference schemes [14], exponential finite differences [15], implicit exponential finite differences [16] and Crank-Nicolson exponential finite differences [17], all applied on the viscid Burgers equation. In this paper, we apply the Adomian decomposition method (ADM) to the solution of the inviscid Burgers equation and compare the results with the analytical solution obtained from the method of characteristics. The rest of the paper is

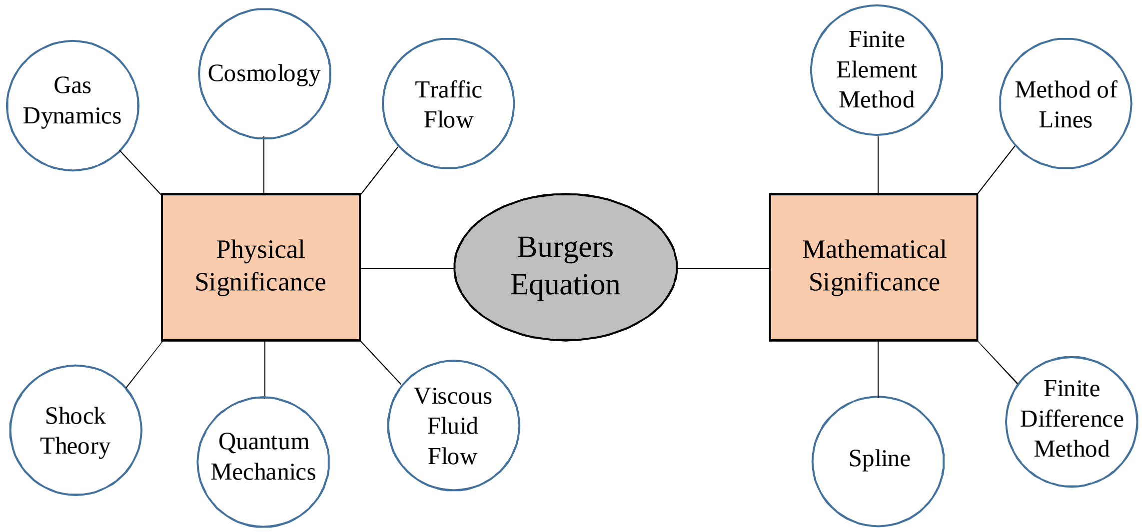


FIGURE 1. A multidisciplinary approach [Adapted from [1]]

organised as follows: Section 2 shows a derivation of the inviscid Burgers equation, Section 3 outlines the method of solution, Section 4 gives some numerical examples and Section 5 concludes the study.

2. Derivation of the Burgers Equation

The general form of the one-dimensional Burgers equation may be obtained as a simplifying case of the Navier-Stokes equations. For a Newtonian incompressible fluid, the Navier-Stokes equation is given by

$$\rho(u_t + u\cdot\nabla u) = -\nabla P + \mu\nabla^2 u + F, \tag{1}$$

where u is the fluid velocity vector field, P the fluid pressure, μ the viscosity, ρ the fluid density, $\nabla u = \frac{\partial u}{\partial x_1}\vec{i} + \frac{\partial u}{\partial x_2}\vec{j} + \frac{\partial u}{\partial x_3}\vec{k}$ and F the source term representing an external force. If we assume that the external force is zero and divide through by ρ, the Navier-Stokes equation takes the form

$$u_t + u\cdot\nabla u = -\frac{\nabla P}{\rho} + \nu\nabla^2 u, \tag{2}$$

where $\nu = \frac{\mu}{\rho} > 0$ is the diffusion coefficient or the kinematic viscosity of the fluid, since for an incompressible fluid ρ is a constant. If we now drop the pressure term, we obtain the *viscid*

Burgers equation

$$u_t + u \cdot \nabla u = \nu \nabla^2 u \tag{3}$$

Equation (3) is a parabolic PDE. When the kinematic viscosity $\nu = 0$, then the diffusion term on the RHS vanishes and (3) becomes the *inviscid* Burgers equation

$$u_t + u \cdot \nabla u = 0 \tag{4}$$

with initial condition $u(x,0) = f(x)$, which is a first-order one-dimensional nonlinear hyperbolic PDE. Equation (4) can be written as

$$u_t + uu_x = 0 \tag{5}$$

Equation (5) is the simplest model of turbulence and is a conservation equation.

3. Adomian Decomposition Method

The Adomian decomposition method (ADM) was formulated by George Adomian [18, 19] and is a technique used for solving both linear and nonlinear ordinary and partial differential equations. This method makes it possible to express analytic solutions in terms of a rapidly converging series. The method identifies and separates the linear and nonlinear parts of a differential equation. By inverting and applying the highest order differential operator that is contained in the linear part of the equation, it is possible to express the solution in terms of the rest of the equation affected by this inverse operator. The nonlinear part is expressed in terms of Adomian polynomials. The initial or boundary conditions and the terms that contain the independent variables, e.g., the source term which is a function of the independent variables only, are taken as the initial approximation. Thus, using a recurrence relation, we compute the terms of the series to obtain the approximate solution of the differential equation.

3.1. Operator form.

In the standard operator form, (5) can be rewritten as

$$L_t(u(x,t)) + Nu(x,t) = 0 \tag{6}$$

where $L_t = \frac{\partial}{\partial t}$ is a first order differential operator and $Nu(x,t) = uu_x$ is the nonlinear term which is assumed to be analytic. Making $L_t(u(x,t))$ the subject gives

$$L_t(u(x,t)) = -Nu(x,t) \tag{7}$$

Assuming that L_t is invertible, we apply the one-fold integral operator $L_t^{-1}(\cdot) = \int_0^t (\cdot)ds$ to both sides of (7) to obtain

$$L_t^{-1}\left[L_t(u(x,t))\right] = -L_t^{-1}\left[Nu(x,t)\right]. \tag{8}$$

3.2. Application of the ADM to the solution of the problem. Applying the given initial condition to (8) gives

$$u(x,t) = f(x) - L_t^{-1}[Nu(x,t)], \tag{9}$$

where $f(x)$ is the constant of integration with respect to t that satisfies $L_t f = 0$. In equations where the initial value $t = t_0$, the inverse operator L^{-1} can be conveniently defined. The ADM proposes a decomposition series solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{10}$$

with u_0 identified as $u(x,0)$ and the components $u_n(x,t)$ obtained from the recursive formula

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) - L_t^{-1}[Nu(x,t)]. \tag{11}$$

The nonlinear term $Nu(x,t) = \beta(u)$ is given by

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n),$$

where the A_n, depending on $u_0,\ u_1,\ u_2, \ldots,\ u_n$, are called *Adomian polynomials* obtained as shown in the following theorem.

Theorem 3.1. *Assume that the following hypotheses hold:*

(1) The series solution $u = \sum_{n=0}^{\infty} u_n$ of the problem given in equation (6) *is absolutely convergent.*

(2) The nonlinear term Nu can be expressed by means of a power series whose radius of convergence is infinite, i.e.,

$$Nu = \sum_{n=0}^{\infty} \beta^{(n)}(0)\frac{u^n}{n!},\ |u| < \infty. \tag{12}$$

Then the Adomian polynomials A_n are given by

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}L_\lambda \left[\beta\left(\sum_{i=0}^{\infty} u_i \lambda^i\right)\right]_{\lambda=0},\ n = 0,\ 1,\ 2, \ldots, \tag{13}$$

where λ is a grouping parameter of convenience and $L_\lambda = \frac{d^n}{d\lambda^n}$.

Proof. Assuming the above hypotheses, the series whose terms are the Adomian polynomials $\{A_n\}_{n=0}^{\infty}$ results from a generalization of the Taylor series

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) = \sum_{n=0}^{\infty} \beta^{(n)}(u_0)\frac{(u-u_0)^n}{n!}, \tag{14}$$

It should be noted that (14) is a rearrangement of the series (13) and that due to the given hypotheses this series is convergent. Now, consider the parametrization proposed by Adomian [18] given by

$$u_\lambda(x,t) = \sum_{n=0}^{\infty} u_n(x,t) g^n(\lambda), \tag{15}$$

where λ is a parameter in $\mathbb{R}$ and g is a complex-valued function such that $|g| < 1$. With this choice of g and using the above hypotheses, the series (15) is absolutely convergent. Substituting (15) in (14), we obtain

$$Nu_\lambda = \sum_{n=0}^{\infty} \beta^{(n)}(u_0)\frac{\left(\sum_{j=1}^{\infty} u_j(x,t) g^j(\lambda)\right)^n}{n!}, \tag{16}$$

Due to the absolute convergence of

$$\sum_{j=1}^{\infty} u_j(x,t) g^j(\lambda)$$

we can rearrange Nu_λ so as to obtain the series in the form $\sum_{n=0}^{\infty} A_n g^n(\lambda)$. Using (15) we can finally get the coefficients A_i of $g^i(\lambda)$ and deduce the Adomian polynomials, i.e.,

$$\begin{aligned}
Nu_\lambda &= \beta(u_0) + \beta'(u_0)\left(u_1 g(\lambda) + u_2 g^2(\lambda) + u_3 g^3(\lambda) + \cdots\right) \\
&\quad + \frac{\beta''(u_0)}{2!}\left(u_1 g(\lambda) + u_2 g^2(\lambda) + u_3 g^3(\lambda) + \cdots\right)^2 \\
&\quad + \frac{\beta'''(u_0)}{3!}\left(u_1 g(\lambda) + u_2 g^2(\lambda) + u_3 g^3(\lambda) + \cdots\right)^3 \\
&\quad + \frac{\beta^{(4)}(u_0)}{4!}\left(u_1 g(\lambda) + u_2 g^2(\lambda) + u_3 g^3(\lambda) + \cdots\right)^4 + \cdots \\
&= \beta(u_0) + \beta'(u_0) u_1 g(\lambda) + \left(\beta'(u_0) u_2 + \beta''(u_0)\frac{u_1^2}{2!}\right) g^2(\lambda) \\
&\quad + \left(\beta'(u_0) u_3 + \beta''(u_0) u_1 u_2 + \beta'''(u_0)\frac{u_1^3}{3!}\right) g^3(\lambda) + \cdots \\
&= \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n) g^n(\lambda)
\end{aligned}$$

Setting $g(\lambda) = \lambda$ and taking derivatives on both sides enables us to make the following identification

$$\begin{aligned}
A_0(u_0) &= \beta(u_0) \\
A_1(u_0, u_1) &= \beta'(u_0) u_1 \\
A_2(u_0, u_1, u_2) &= \beta'(u_0) u_2 + \frac{u_1^2}{2!}\beta''(u_0) \\
A_3(u_0, u_1, u_2, u_3) &= \beta'(u_0) u_3 + \beta''(u_0) u_1 u_2 + \frac{u_1^3}{3!}\beta'''(u_0) \\
A_4(u_0, \ldots, u_4) &= \beta'(u_0) u_4 + \beta''(u_0)\left(u_1 u_3 + \frac{u_2^2}{2!}\right) + \frac{u_1^2 u_2}{2!}\beta'''(u_0) + \frac{u_1^4}{4!}\beta^{(4)}(u_0) \\
&\vdots
\end{aligned}$$

Summing these up gives the result (equation (13)):

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\beta\left(\sum_{i=0}^{\infty} u_i \lambda^i\right)\right]_{\lambda=0}, \; n = 0,\ 1,\ 2, \ldots \tag{17}$$

□

For Burgers equation (9), the Adomian polynomials are found as

$$
\begin{aligned}
A_0(u_0) &= \beta(u_0) = u_0 u_{0_x}, \text{ where } u_0 = f(x) \\
A_1(u_0,u_1) &= \beta^{'}(u_0)u_1 = (u_0 u_{0_x})^{'} u_1 \\
A_2(u_0,u_1,u_2) &= \beta^{'}(u_0)u_2 + \frac{u_1^2}{2!}\beta^{''}(u_0) = (u_0 u_{0_x})^{'} u_2 + \frac{u_1^2}{2}(u_0 u_{0_x})^{''} \\
&\vdots
\end{aligned}
\tag{18}
$$

It should be noted that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on $u_0,\ u_1$ and u_2, and so on. Substituting (10) and (18) into (9) gives the recursive scheme

$$
\begin{aligned}
u_0 &= f(x) \\
u_{n+1} &= -L_t^{-1}[A_n],\ n \geq 0
\end{aligned}
\tag{19}
$$

The scheme (19) is equivalent to

$$
\begin{aligned}
u_0 &= f(x) \\
u_1 &= -L_t^{-1}[A_0] \\
u_2 &= -L_t^{-1}[A_1] \\
u_3 &= -L_t^{-1}[A_2] \\
&\vdots \\
u_{n+1} &= -L_t^{-1}[A_n],\ n \geq 0
\end{aligned}
\tag{20}
$$

In this way, the components $u_0,\ u_1,\ u_2,\ldots$ are identified and the series solution to the Burgers equation is completely determined. The exact solution may be determined using the approximation

$$u(x,t) = \lim_{n\to\infty} \Phi_n,$$

where $\Phi_n = \sum_{k=0}^{n-1} u_k$.

Remark 3.2. For the non-homogeneous inviscid Burgers equation

$$u_t + uu_x = F(x,t),$$

with initial condition $u(x,0) = f(x)$, the operator form is

$$L_t(u(x,t)) + Nu(x,t) = F(x,t),$$

so that, applying the initial conditions, the solution is

$$u(x,t) = f(x) - L_t^{-1}[Nu(x,t)] + L_t^{-1}[F(x,t)]$$

from the recursive relationship

$$\begin{aligned} u_0 &= f(x) + L_t^{-1}[F(x,t)] \\ u_{n+1} &= -L_t^{-1}[A_n],\ n \geq 0 \end{aligned} \tag{21}$$

Remark 3.3. Note that the recursive relationship (20) is constructed on the basis that the zeroth component $u_0(x,t)$ is defined by all the terms that arise from the initial conditions and from integrating the source term $F(x,t)$. The remaining components $u_n(x,t),\ n \geq 1$ are completely determined recursively such that each term is computed by using the immediately preceding term. Accordingly, considering the first few terms only, the recursive relation (20) gives $u_0(x,t),\ u_1(x,t),\ u_2(x,t),\ldots$

4. Numerical Examples

In this section we give numerical examples of homogeneous inviscid Burgers equations which we solve using the ADM. The examples show that the ADM solution gives a good approximation to the exact solution. All the computations associated with these examples were performed using a Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10GHz and 6.0GB internal memory. The figures were constructed using MATLAB R2016a. In each of the following examples, we solve the equation (5) with different initial conditions.

Problem 1. Consider the homogeneous inviscid Burgers equation (5) with initial condition $u(x,0)=x$. Using (21) we find the Adomian polynomials as

$$\begin{aligned}
A_0(u_0) &= u_0 u_{0_x} = x(x)_x = x \\
A_1(u_0,u_1) &= (u_0 u_{0_x})' u_1 = -xt \\
A_2(u_0,u_1,u_2) &= (u_0 u_{0_x})' u_2 + \frac{u_1^2}{2!}(u_0 u_{0_x})'' = \frac{1}{2}xt^2 \\
A_3(u_0,\ldots,u_3) &= (u_0 u_{0_x})' u_3 + (u_0 u_{0_x})'' u_1 u_2 + \frac{u_1^3}{3!}(u_0 u_{0_x})''' = -\frac{1}{6}xt^3 \\
&\vdots
\end{aligned}$$

and the individual terms of the decomposition as

$u_0 = x$

$u_1 = -L_t^{-1}[A_0] = -L_t^{-1}[x] = -\int_0^t x\,ds = -xt$

$u_2 = -L_t^{-1}[A_1] = -L_t^{-1}[-xt] = \int_0^t xs\,ds = \frac{1}{2}xt^2$

$u_3 = -L_t^{-1}[A_2] = -L_t^{-1}\left[\frac{1}{2}xt^2\right] = -\int_0^t \frac{1}{2}xs^2\,ds = -\frac{1}{6}xt^3$

and so on. So the ADM solution is the partial sum of the approximants

$$\begin{aligned}
u &= u_0 + u_1 + u_2 + u_3 + \cdots \\
&= x - xt + \frac{xt^2}{2} - \frac{xt^3}{6} + \cdots \\
&= x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)
\end{aligned}$$

i.e.,

$$u(x,t) = x\mathrm{e}^{-t} \tag{22}$$

The exact solution is

$$u = x - ut$$

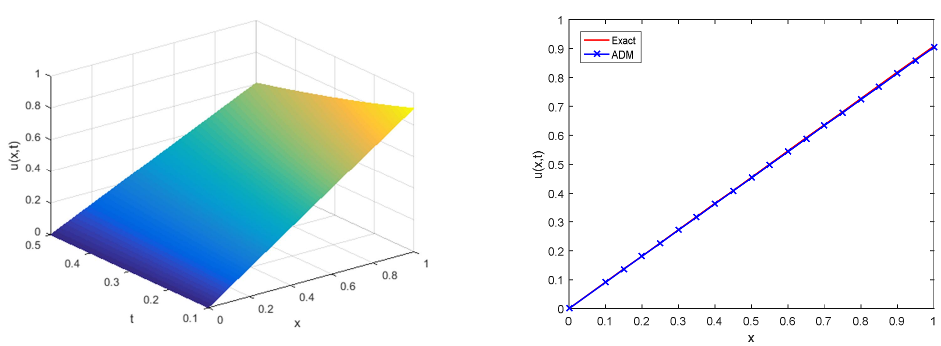
or

$$u(x,t) = \frac{x}{1+t} \tag{23}$$

Let $u_t = u_t(x,t)$, where $0.1 \leq t \leq 0.5$, be the exact solution when t is fixed, and let u_{ADM} be the approximate solution from the ADM. Then the ADM and exact results for $0 \leq x \leq 1$ and $0.1 \leq t \leq 0.5$ are shown in Table 1. Figure 2(a) shows the ADM results for $0 \leq x \leq 1$ and $0.1 \leq t \leq 0.5$, while the projection of the surface is shown in Figure 2(b) and compares the approximate and exact solutions for $0 \leq x \leq 1$ and a fixed $t = 0.1$.

TABLE 1. Approximate solution $u_{\mathrm{ADM}}(x,t)$ for Problem 1

x	$u_{0.1}$	u_{ADM}	$u_{0.2}$	u_{ADM}	$u_{0.3}$	u_{ADM}	$u_{0.4}$	u_{ADM}	$u_{0.5}$	u_{ADM}
0	0	0	0	0	0	0	0	0	0	0
0.2	0.1818	0.1810	0.1667	0.1637	0.1538	0.1482	0.1429	0.1341	0.1500	0.1213
0.4	0.3636	0.3619	0.3333	0.3275	0.3077	0.2963	0.2857	0.2681	0.2781	0.2426
0.6	0.5455	0.5429	0.5000	0.4912	0.4615	0.4445	0.4286	0.4022	0.4111	0.3639
0.8	0.7273	0.7239	0.6667	0.6550	0.6134	0.5927	0.5714	0.5363	0.5444	0.4852
1.0	0.9091	0.9048	0.8333	0.8187	0.7692	0.7408	0.7143	0.6703	0.6778	0.6065



(a) (b)

FIGURE 2. (a) Approximate solution for Problem 1 for $0 \leq x \leq 1$ and $0.1 \leq t \leq 0.5$ (b) Projection of surface for fixed $t = 0.1$

Problem 2. Consider the homogeneous inviscid Burgers equation (5) with initial condition $u(x,0) = -x$. The first few approximants are given below:

$u_0 = -x$

$u_1 = -L_t^{-1}[A_0] = -L_t^{-1}[x] = -xt$

$u_2 = -L_t^{-1}[A_1] = -L_t^{-1}[-xt] = \frac{1}{2}xt^2$

$u_3 = -L_t^{-1}[A_2] = -L_t^{-1}\left[\frac{1}{2}xt^2\right] = -\frac{1}{6}xt^3$

$u_4 = -L_t^{-1}[A_3] = -L_t^{-1}\left[-\frac{1}{6}xt^3\right] = \frac{1}{24}xt^4$

$u_5 = -L_t^{-1}[A_4] = -L_t^{-1}\left[\frac{1}{24}xt^4\right] = -\frac{1}{120}xt^5$

and so on. Thus, the solution by the ADM is

$$\begin{aligned} u &= u_0 + u_1 + u_2 + u_3 + \cdots \\ &= -x - xt + \frac{xt^2}{2} - \frac{xt^3}{6} + \frac{xt^4}{24} - \frac{xt^5}{120} + \cdots \\ &= x\left(-1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\right) \\ &= x\left(-2 + 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\right) \end{aligned}$$

i.e.,

$$u(x,t) = x(\mathrm{e}^{-t} - 2) \tag{24}$$

The exact solution is

$$u = -(x - ut)$$

or

$$u(x,t) = \frac{x}{t-1} \tag{25}$$

The results for $0 \le x \le 1$ and $0.1 \le t \le 0.5$ are shown in Table 2. Figure 3(a) shows the approximate solution for $0 \le x \le 1$ and $0.1 \le t \le 0.5$ while the projection of the surface for a fixed $t = 0.1$ is shown in Figure 3(b).

Problem 3. Consider a homogeneous Burgers equation (5) having initial condition $u(x,0) = 2x$. The series solution will consist of the partial sum of the approximants:

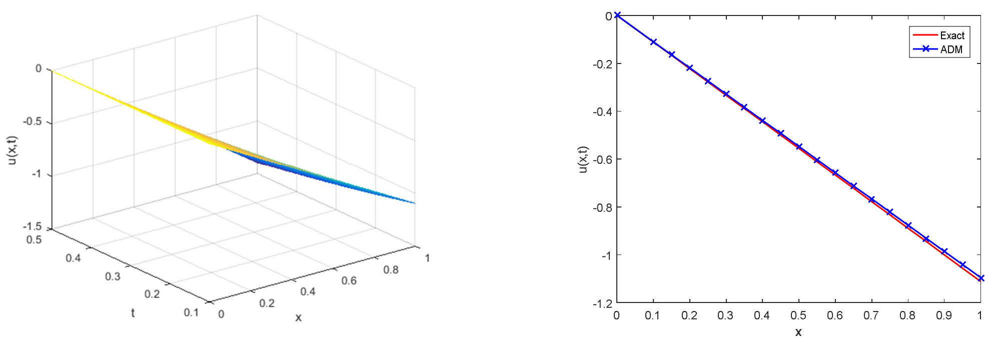
$u_0 = 2x$

$u_1 = -L_t^{-1}[A_0] = -L_t^{-1}[4x] = -4xt$

$u_2 = -L_t^{-1}[A_1] = -L_t^{-1}[-16xt] = 8xt^2$

TABLE 2. Approximate solution $u_{\mathrm{ADM}}(x,t)$ for Problem 2

x	$u_{0.1}$	u_{ADM}	$u_{0.2}$	u_{ADM}	$u_{0.3}$	u_{ADM}	$u_{0.4}$	u_{ADM}	$u_{0.5}$	u_{ADM}
0	0	0	0	0	0	0	0	0	0	0
0.2	-0.222	-0.219	-0.250	-0.236	-0.286	-0.253	-0.333	-0.266	-0.400	-0.279
0.4	-0.444	-0.438	-0.500	-0.473	-0.571	-0.504	-0.667	-0.532	-0.800	-0.557
0.6	-0.667	-0.657	-0.750	-0.709	-0.857	-0.756	-1.000	-0.798	-1.200	-0.836
0.8	-0.889	-0.876	-1.000	-0.945	-1.143	-1.007	-1.333	-1.064	-1.600	-1.115
1.0	-1.111	-1.095	-1.250	-1.181	-1.429	-1.259	-1.667	-1.330	-2.000	-1.393



(a) (b)

FIGURE 3. (a) Approximate solution for Problem 2 for $0 \leq x \leq 1$ and $0.1 \leq t \leq 0.5$ (b) Projection of surface for fixed $t = 0.1$

$$u_3 = -L_t^{-1}[A_2] = -L_t^{-1}\left[32xt^2\right] = -\tfrac{32}{3}xt^3$$
$$u_4 = -L_t^{-1}[A_3] = -L_t^{-1}\left[-\tfrac{128}{3}xt^3\right] = \tfrac{32}{3}xt^4$$
$$u_5 = -L_t^{-1}[A_4] = -L_t^{-1}\left[\tfrac{128}{3}xt^4\right] = -\tfrac{128}{15}xt^5$$

and so on. The ADM solution is therefore

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + u_3 + \cdots \\ &= 2x - 4xt + 8xt^2 - \frac{32}{3}xt^3 + \frac{32}{3}xt^4 - \frac{128}{15}xt^5 + \frac{512}{90}xt^6 \\ &\quad - \frac{1024}{315}xt^7 + \frac{512}{315}xt^8 - \frac{2048}{2835}xt^9 + \cdots \end{aligned}$$

The exact solution is

$$u = 2(x - ut)$$

or

$$u(x,t) = \frac{2x}{1+2t} \tag{26}$$

The ADM and exact results for $0 \le x \le 1$ and $0.1 \le t \le 0.5$ are shown in Table 3. Figure 4(a) gives the surface representing the approximate solution for Problem 3 for $0 \le x \le 1$ and $0.1 \le t \le 0.5$, while Figure 4(b) shows a projection of the surface for $0 \le x \le 1$ and a fixed $t = 0.1$.

TABLE 3. Approximate solution $u_{\mathrm{ADM}}(x,t)$ for Problem 3

x	$u_{0.1}$	u_{ADM}	$u_{0.2}$	u_{ADM}	$u_{0.3}$	u_{ADM}	$u_{0.4}$	u_{ADM}	$u_{0.5}$	u_{ADM}
0	0	0	0	0	0	0	0	0	0	0
0.2	0.3333	0.3341	0.2857	0.2899	0.2500	0.2602	0.2222	0.2403	0.2000	0.2270
0.4	0.6667	0.6681	0.5714	0.5797	0.5000	0.5205	0.4444	0.4807	0.4000	0.4540
0.6	1.0000	1.0022	0.8571	0.8696	0.7500	0.7807	0.6667	0.7210	0.6000	0.6811
0.8	1.3333	1.3363	1.1429	1.1591	1.0000	1.0410	0.8889	0.9613	0.8000	0.9081
1.0	1.6667	1.6703	1.4286	1.4493	1.2500	1.3012	1.1111	1.2017	1.0000	1.1351

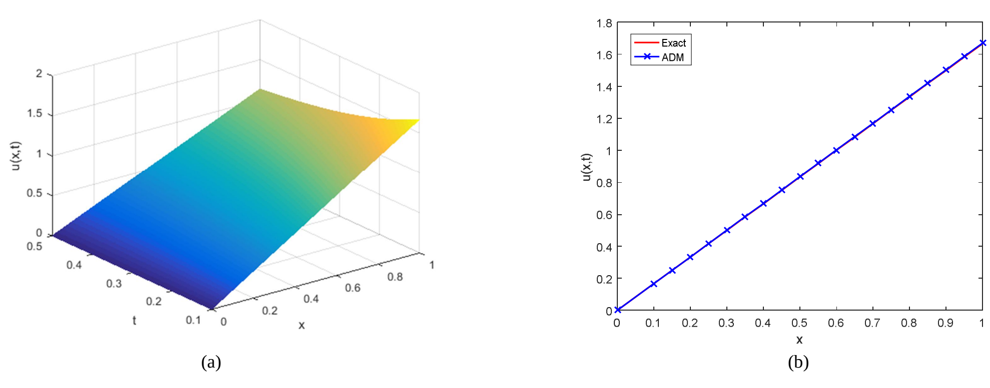


FIGURE 4. (a) Approximate solution for Problem 3 for $0 \le x \le 1$ and $0.1 \le t \le 0.5$ (b) Projection of surface for fixed $t = 0.1$

Let $e_{t,\text{Prob.xx}} = |u_t - u_{ADM}|$ be the absolute error for $t = 0.1$ and $t = 0.5$ for Problems 1, 2 and 3. A comparison of absolute errors using results in Tables 1, 2 and 3 shows that the ADM approximation is better for smaller values of t (see Table 4 and Figure 5).

TABLE 4. Absolute errors for $t = 0.1$ and $t = 0.5$ for Problems 1, 2 and 3

$e_{0.1,\text{Prob.1}}$	$e_{0.5,\text{Prob.1}}$	$e_{0.1,\text{Prob.2}}$	$e_{0.5,\text{Prob.2}}$	$e_{0.1,\text{Prob.3}}$	$e_{0.5,\text{Prob.3}}$
0	0	0	0	0	0
0.0008	0.0120	0.0032	0.1213	0.0007	0.0270
0.0017	0.0241	0.0064	0.2426	0.0015	0.0540
0.0026	0.0361	0.0096	0.3639	0.0022	0.0811
0.0034	0.0481	0.0128	0.4852	0.0029	0.1081
0.0043	0.0601	0.0159	0.6065	0.0037	0.1351

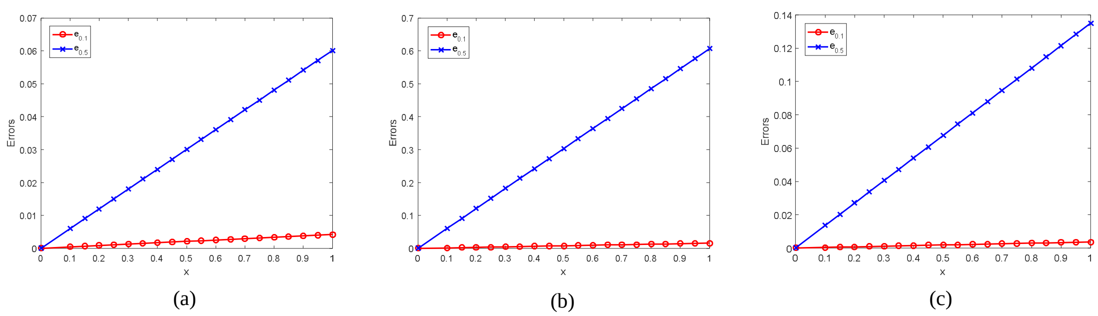


FIGURE 5. Absolute errors for $t = 0.1$ and $t = 0.5$ for (a) Problem 1 (b) Problem 2 (c) Problem 3

5. CONCLUSION

In this paper we have successfully used the Adomian decomposition method to find the solution of the inviscid Burgers equation which is a one-dimensional quasilinear PDE. Numerical results based on selected Burgers equations show that the ADM solution compares favourably with the exact solution. Possible extensions to this work include (1) use of one of the many modifications of the ADM, e.g., [20, 21, 22, 23]; (2) solution of the non-homogeneous inviscid

Burgers equation, since it has been shown that the homogeneous Burgers equation lacks the most important property attributed to turbulence, i.e., the solutions do not exhibit chaotic features such as sensitivity with respect to initial conditions [4]; (3) application of the ADM to the viscid Burgers equation and other nonlinear PDES like the Korteweg-de Vries (KdV) equation.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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