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MULTISWITCHING HYBRID SYNCHRONIZATION USING DISTURBANCE OBSERVER ADAPTIVE SLIDING MODE CONTROL ON FRACTIONAL ORDER CHAOTIC SYSTEM

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Abstract. In this manuscript we have analysed identical financial systems of fractional order with external disturbances taken into consideration and introduced the disturbance observer to estimate the disturbance. We have applied adaptive sliding mode technique to synchronize these identical systems. Numerical simulations have been performed in MATLAB to validate the effectiveness of the method proposed. The obtained results show the usefulness and suitability of the method used to achieve the synchronization. A comparison has been made between the obtained and published results and application in secure communication has also been displayed.

Keywords: chaos synchronization; secure communication; adaptive sliding mode synchronization; multi-switching synchronization; hybrid synchronization.

2010 AMS Subject Classification: 37D45, 37N10, 37E99, 37F99.

1. INTRODUCTION

Chaos synchronization [5] [3]is a process having two or more chaotic systems(identical or non-identical) which follow the same path. The dynamics of one system is locked into the

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another and thereby causes their synchronization in the sense that the state of one asymptotically approaches to the other. Untill 1990 synchronization between chaotic systems [19] was considered impractical because of the divergence of trajectories. But it was because of the pioneering work of Pecora and Caroll the synchronization between chaotic systems came into existence and emerged an interesting area of research. Chaos being the inherent property of nonlinear systems and has various applications such as viscoelasticity, dielectric polarization, electromagnetic waves, diffusion, signal processing, mathematical biology and chaotic systems in many disciplines [12] [17] [10]. The nonlinear systems which shows such type of behaviour are known as chaotic systems. Various methods are used to know about the chaotic behaviour of a system, some of them are by plotting phase portrait, poincare section or by finding the lyapunov exponents. In order to know about their behaviour and making them stable or to control them, it has become the topic of interest of the current era. Various techniques has been designed to synchronize the chaotic systems, some of them are active control technique, adaptive control technique, sliding mode technique, time-delay feedback control etc. Various studies have been done comprehensively in the last two decades. Different methods have also been designed for synchronization of chaotic systems such as adaptive feedback control, optimal control, linear and nonlinear feedback synchronization, active control, sliding mode control, adaptive sliding mode technique, time delay feedback approach , tracking control, backstepping design method and so on. Due to the increased interest in chaos synchronization various kinds of synchronization schemes have been proposed such as lag synchronization, complete synchronization, phase and anti-phase synchronization, anti-synchronization [24], hybrid synchronization [25] [18], dual synchronization [8], double synchronization [11], projective synchronization [27] [28], compound synchronization [9] [21] [6] [22] [26], hybrid funchtion projective synchronization,generalised synchronization etc.

Many attempts have been made to synchronize similar systems with different techniques. Moreover the non-identical systems have also been synchronized by many researchers. In this manuscript we have tried to synchronize the two identical financial system of fractional order by taking external unknown disturbance into consideration.To estimate the disturbance, we have introduced the disturbance observer and we have considered the external disturbance as bounded.

We have also introduced multiswitching due to which the number of choices of switching increases and enhances the analysis of our study.In multi-switching, the slave system states are synchronized with the desired state of the master system in a multi-switching manner.

In the process of hybrid synchronization, coexistence of complete and anti-synchronization occurs. This co-existence of synchronization is also referred as mixed synchronization.

Motivated by the above studies, in this paper we present the adaptive sliding mode hybrid multiswitching synchronization. We believe that it is the first kind of study addressing the problem of fractional order multi-switching hybrid synchronization of chaotic systems using disturbance observer. During our studies,we have synchronized the two identical financial system of fractional order with external unknown bounded disturbances, compared the obtained results with published literature and applied the results in field of secure communication.

2. PRELIMINARIES

Definition: As various definitions have been available for fractional order derivative [23] [2] [4] [20], we have considered Caputo's definition:

$$ {}_a^cD_t^q x(t) = \frac{d^q x(t)}{dt^q} = \begin{cases} \frac{1}{\Gamma(n-q)\int_a^t \frac{x^{(n)}(\tau)d\tau}{(t-\tau)^{q+1-n}}} & \text{for} \quad n-1 < q < n \\ \frac{d^{(n)}x(t)}{dt^n} \end{cases} $$

where $0 < q \in R$ and $\Gamma(.)$ is the Gamma function.

3. SYSTEM DESCRIPTION

3.1. The fractional order Financial system model is given by

$$ \begin{aligned} \frac{d^\alpha u_1}{dt^\alpha} &= u_3 + (u_2 - f)u_1 \\ \frac{d^\alpha u_2}{dt^\alpha} &= 1 - gu_2 - u_1^2 \\ \frac{d^\alpha u_3}{dt^\alpha} &= -u_1 - hu_3 \end{aligned} \tag{1} $$

where u_1, u_2, u_3 are state variables representing interst rate, demand for investment and the price index respectively. Parameters f, g, h are non-negative constants. Here f represents the saving amount, g represents cost per investment, h represents the elasticity of demand of the market.parameters are chosen as $f = 3, g = 0.1, h = 1$ and initial conditions as $(u_1(0), u_2(0), u_3(0)) = (2.0, 3.0, 2.0)$.

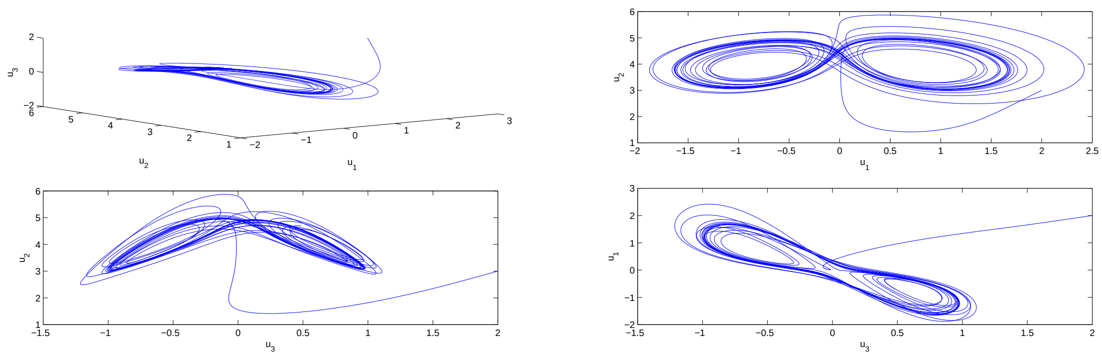

Fig.1: Phase portraits of fractional order financial model for fractional order $\alpha = 0.85$ with initial condition $(2,3,2)$.

4. Scheme for Synchronization

In order to conduct the adaptive sliding mode hybrid synchronization using disturbance observer we consider eq.1 as master system and the corresponding identical slave system with external disturbances is given by

$$\begin{aligned} \frac{d^\alpha v_1}{dt^\alpha} &= v_3 + (v_2 - f)v_1 + \psi_1 + S_1 \\ \frac{d^\alpha v_2}{dt^\alpha} &= 1 - g v_2 - v_1^2 + \psi_2 + S_2 \\ \frac{d^\alpha v_3}{dt^\alpha} &= -v_1 - h v_3 + \psi_3 + S_3 \end{aligned} \tag{2}$$

where v_1, v_2, v_3 are state variables representing interst rate, demand for investment and the price index respectively.ψ_1, ψ_2, ψ_3 are external disturbances and S_1, S_2, S_3 are controllers.

System (1) and (2) are said to be in multiswitching hybrid synchronization if there exist suitable parameters $\delta_j = (\delta_1, \delta_2, \delta_3)$and suitable controller (S_1, S_2, S_3), such that

$$\lim_{t\to\infty} \| e_{i,j}(t) \| = \lim_{t\to\infty} \| v_i(t) - \delta_j u_j(t) \| = 0 \tag{3}$$

As we have introduced $\psi_i, i = 1,2,3$(disturbance) in the slave system (2) and these disturbances are not known so it would not work in developing the scheme to synchronize the master and slave system.In order to resolve this problem,we have designed a nonlinear disturbance observer of fractional order to have an estimate of the unknown disturbance.Before that we have introduced some lemmas and assumptions.

Lemma 1:-Consider a continuous derivable function $\eta(t) \in R$.Then ,for any time $t \geq t_0$ we have $\frac{D^{\alpha}\eta^2(t)}{2} \leq \eta(t)D^{\alpha}\eta(t)$,where $0 < \alpha < 1$.

Lemma 2:-For the following fractional order system

$D^{\alpha}q(t) \leq -H_0 q(t) + H_1$

there exists a constant $t_1 > 0$, such that for all $t \in (t_1, \infty)$,we have

$||q(t)|| \leq \frac{2H_1}{H_0}$,

where q(t) is the state variable and H_0 and H_1 are two positive constants.

Assumption:-The Caputo fractional order derivative of the external disturbance $\psi_i(t)$ with $i = 1,2,.....n$ is bdd, i.e $\mid D^{\alpha}\psi_i(t) \mid \leq \upsilon_i$,where $\upsilon_i > 0$ are unknown positive constants.

5. Fractional Order Disturbance Observer Scheme

Here we design a scheme for a non-linear FODO to measure the external unknown disturbance being in the slave system.Essentially the scheming of disturbance observer is to increase the disturbance reduction and robustness of system performance and to measure up the system disturbance.We have proposed a subsidiary variable for scheming the nonlinear disturbance observer(FODO) of fractional order, using the same technique as used in integer order system as follows:-

$$\chi_i = \psi_i - \omega_i v_i(t) \tag{4}$$

where ω_i $(i = 1,2,3)$ are non-zero positive constants to be determined.

The fractional order Caputo's derivative of $\chi_i(t)$ can be written as

$$D^\alpha \chi_i = D^\alpha \psi_i - \omega_i D^\alpha v_i(t) \tag{5}$$

Substitute (2) in (5)

$$\begin{aligned} D^\alpha \chi_1 &= D^\alpha \psi_1 - \omega_1(v_3 + (v_2 - f)v_1 + \chi_1 + \omega_1 v_1) - \omega_1 S_1 \\ D^\alpha \chi_2 &= D^\alpha \psi_2 - \omega_2(1 - g v_2 - v_1^2 + \chi_2 + \omega_2 v_2) - \omega_2 S_2 \\ D^\alpha \chi_3 &= D^\alpha \psi_3 - \omega_3(-v_1 - h v_3 + \chi_3 + \omega_3 v_3) - \omega_3 S_3 \end{aligned} \tag{6}$$

The estimate of $\chi_i(t)$ are given as

$$\begin{aligned} D^\alpha \widehat{\chi_1(t)} &= -\omega_1(v_3 + (v_2 - f)v_1 + \omega_1 v_1) - \omega_1 \hat{\chi}_1 - \omega_1 S_1 \\ D^\alpha \widehat{\chi_2(t)} &= -\omega_2(1 - g v_2 - v_1^2 + \omega_2 v_2) - \omega_2 \hat{\chi}_2 - \omega_2 S_2 \\ D^\alpha \widehat{\chi_3(t)} &= -\omega_3(-v_1 - h v_3 + \omega_3 v_3) - \omega_3 \hat{\chi}_3 - \omega_3 S_3 \end{aligned} \tag{7}$$

where $\hat{\chi}_i$ denote the estimate of χ_i.
Using equation (4),the estimated disturbance $\widehat{\psi_i(t)}$ can be written as

$$\hat{\psi}_i = \hat{\chi}_i + \omega_i v_i \tag{8}$$

Error of the disturbance estimation can be stated as

$$\tilde{\psi}_i = \psi_i - \hat{\psi}_i \tag{9}$$

Using equation (4)

$$\tilde{\chi}_i = \chi_i - \hat{\chi}_i = \psi_i - \hat{\psi}_i = \tilde{\psi}_i \tag{10}$$

where $i = 1,2,3$,then the fractional order Caputo's derivatives of $\tilde{\chi}_i$ $(i = 1,2,3)$ can be depicted as

$$D^\alpha \tilde{\chi}_i = -\omega_i \tilde{\chi}_i + D^\alpha \psi_i \tag{11}$$

In order to study the convergence of disturbance estimate error, we construct a Lyapunov function $V_{\psi_i}(t)(i=1,2,3)$ as

$$V_{\psi_i}(t) = \frac{1}{2}\tilde{\psi}_i^2 = \frac{1}{2}\tilde{\chi}_i^2, i = 1,2,3 \tag{12}$$

on employing Lemma 1,the fractional order derivative of V_{ψ_i} can be depicted as

$$D^\alpha V_{\psi_i}(t) \leq \tilde{\chi}_i D^\alpha \tilde{\chi}_i \tag{13}$$

Using the equation (11) and (13),we get

$$D^\alpha V_{\psi_i}(t) \leq \tilde{\chi}_i(-\omega_i\tilde{\chi}_i + D^\alpha \psi_i) \tag{14}$$

Using the assumption (1) in (14),we get

$$\begin{aligned} D^\alpha V_{\psi_i}(t) &\leq -\omega_i\tilde{\chi}_i{}^2 + \frac{\tilde{\chi}_i^2}{2} + \frac{\rho_i^2}{2} \\ &= -(\omega_i - \frac{1}{2})\tilde{\chi}_i{}^2 + \frac{\rho_i^2}{2} \\ &= -n_0 V_{\psi_i}(t) + n_1 \end{aligned} \tag{15}$$

where $n_0 = 2\omega_i - 1$ and $n_1 = \frac{\rho_i^2}{2}$

The control gain ω_i of non linear FODO should be selected to make $\omega_i > 0.5$, and to assure the bound of the estimated error.

Applying lemma (2) and (15),we get

$$| V_{\psi_i}(t) | \leq \frac{\rho_i^2}{2(\omega_i - 0.5)} \tag{16}$$

which means

$$| \tilde{\psi}_i | \leq \sqrt{\frac{\rho_i^2}{(\omega_i - 0.5)}} \tag{17}$$

From the equation (17) it is clear that $\tilde{\psi}_i(t)$ is bdd above. Therefore for the external disturbances $\psi_i(t)$(i=1,2,3),the error obtained from the disturbance approximation $\tilde{\psi}_i$ satisfies $| \tilde{\psi}_i | \leq \varsigma_i$, where ς_i is the unknown positive real constant. As in real practice the upper bound of $| \tilde{\psi}_i |$ is not as simple to find and so the estimated value $\hat{\varsigma}_i$ of ς_i (i=1,2,3) have been introduced .

From the above analysis we observe that the estimated error of the disturbance of the financial system is bounded.

6. Adaptive Sliding Mode Multiswitching Hybrid Synchronization of Fractional Order Financial Systems

In order to have bounded hybrid synchronization between the two systems,we have designed the nonlinear FODO based adaptive sliding mode control scheme.First we define the error states for synchronization between the response system and the drive system as there are various possibilities of switches some of them are:-

$$\text{Switch 1:-}\begin{cases} e_{21} = v_2 - u_1 \\ e_{32} = v_3 + u_2 \\ e_{13} = v_1 - u_3 \end{cases}$$

$$\text{Switch 2:-}\begin{cases} e_{12} = v_1 - u_2 \\ e_{23} = v_2 + u_3 \\ e_{31} = v_3 - u_1 \end{cases}$$

$$\text{Switch 3:-}\begin{cases} e_{22} = v_2 - u_2 \\ e_{33} = v_3 + u_3 \\ e_{11} = v_1 - u_1 \end{cases}$$

$$\text{Switch 4:-}\begin{cases} e_{23} = v_2 - u_3 \\ e_{21} = v_3 + u_1 \\ e_{12} = v_1 - u_2 \end{cases}$$

Now we will discuss Switch 1 and others will follow the same

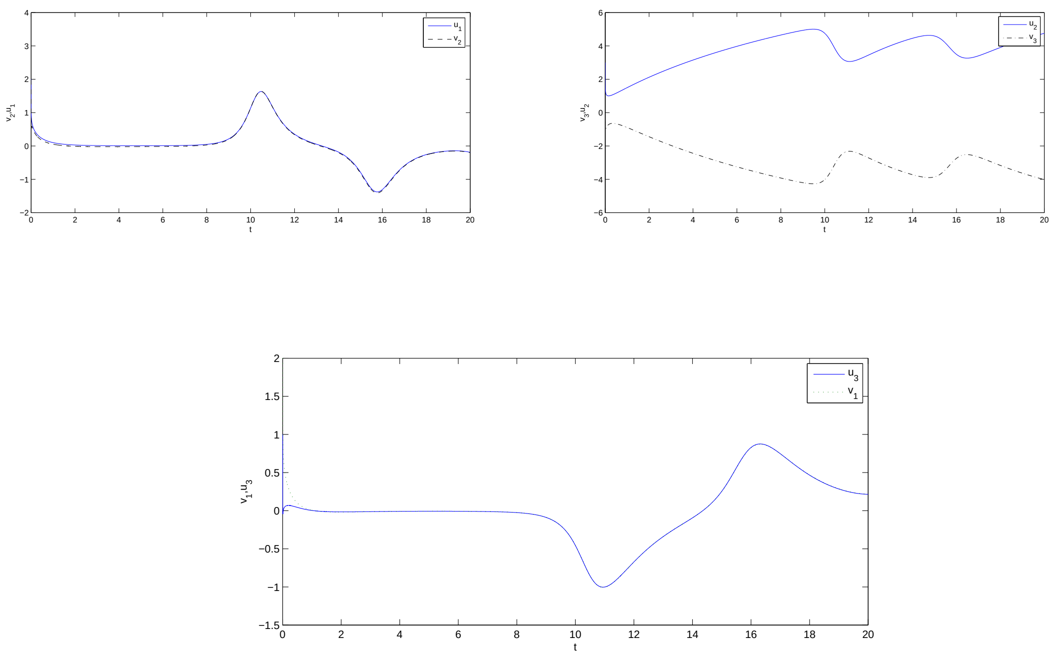

Fig.2: Different synchronized state trajectories of switch 1.

The error dynamics of Switch 1 can be written as:-

$$
\begin{aligned}
D^{\alpha}e_{21} &= D^{\alpha}v_2 - D^{\alpha}u_1 \\
D^{\alpha}e_{32} &= D^{\alpha}v_3 + D^{\alpha}u_2 \\
D^{\alpha}e_{13} &= D^{\alpha}v_1 - D^{\alpha}u_3
\end{aligned}
\tag{18}
$$

Using drive system (1) and response system (2), the error dynamics can be written as

$$
\begin{aligned}
D^{\alpha}e_{21} &= 1 - ge_{21} + (f-g)u_1 - u_3 - v_1^2 - u_2u_1 + \psi_2 + S_2 \\
D^{\alpha}e_{32} &= -e_{13} - he_{32} + (h-g)u_2 - u_3 - u_1^2 + \psi_3 + S_3 \\
D^{\alpha}e_{13} &= e_{32} - fe_{13} + v_2v_1 - u_2 + (h-f)u_3 + u_1 + \psi_1 + S_1
\end{aligned}
\tag{19}
$$

To investigate the stability of error systems (19),we employ a sliding mode surface described by:

$$\begin{aligned} s_1(t) &= e_{21}(t) \\ s_2(t) &= e_{32}(t) \\ s_3(t) &= e_{13}(t) \end{aligned} \tag{20}$$

Taking the fractional derivative of (20) we get

$$\begin{aligned} D^\alpha s_1(t) &= D^\alpha e_{21}(t) \\ D^\alpha s_2(t) &= D^\alpha e_{32}(t) \\ D^\alpha s_3(t) &= D^\alpha e_{13}(t) \end{aligned} \tag{21}$$

The control input has been designed as:-

$$\begin{aligned} S_1 &= -e_{32} + fe_{13} - v_2 v_1 + u_2 - (h-f)u_3 - u_1 - \hat{\psi}_1 - \mu_3 s_3 - \hat{\varsigma}_3 signs_3 \\ S_2 &= -1 + ge_{21} - (f-g)u_1 + u_3 + v_1^2 + u_2 u_1 - \hat{\psi}_2 - \mu_1 s_1 - \hat{\varsigma}_1 signs_1 \\ S_3 &= e_{13} + he_{32} - (h-g)u_2 + u_3 + u_1^2 - \hat{\psi}_3 - \mu_2 s_2 - \hat{\varsigma}_2 signs_2 \end{aligned} \tag{22}$$

where sign(.) is the signum function and $\mu_i > 0$ are constants.The updated estimated value $\hat{\varsigma}_i$ is given by:-

$D^\alpha \hat{\varsigma}_i = m_i(\mid s_i(t) \mid - \hat{\varsigma}_i)$,

where $m_i(> 0)$(i=1,2,3) are the constants designed.

Using the values of (22) in (19)we get:-

$$\begin{aligned} D^\alpha e_{21} &= -\mu_1 s_1 - \hat{\varsigma}_1 signs_1 + \psi_2 - \hat{\psi}_2 \\ D^\alpha e_{32} &= -\mu_2 s_2 - \hat{\varsigma}_2 signs_2 + \psi_3 - \hat{\psi}_3 \\ D^\alpha e_{13} &= -\mu_3 s_3 - \hat{\varsigma}_3 signs_3 + \psi_1 - \hat{\psi}_1 \end{aligned} \tag{23}$$

$$D^{\alpha}e_{21} = -\mu_1 s_1 - \hat{\varsigma}_1 signs_1 + \tilde{\psi}_2$$

$$D^{\alpha}e_{32} = -\mu_2 s_2 - \hat{\varsigma}_2 signs_2 + \tilde{\psi}_3 \tag{24}$$

$$D^{\alpha}e_{13} = -\mu_3 s_3 - \hat{\varsigma}_3 signs_3 + \tilde{\psi}_1$$

Choosing (22) as controllers for (19),then the sliding surface $s_i(t)$ is stable and bounded,i.e

$$\mid s_i(t) \mid \leq P \tag{25}$$

where $P > 0$ is unknown,

Using (20) and (25),we get

$$\mid e_{ij}(t) \mid \leq P, \tag{26}$$

From (25),we get that $s_i(t)$,the sliding surface is bounded and hence the error $e_{ij}(t)$ is bounded. The nonlinear FODO-based adaptive sliding mode multiswitching hybrid synchronization scheme for fractional order financial system with external disturbances can be summarised as following theorem.

Theorem 1: For multiswitching hybrid synchronization error system (19) with $0 < \alpha < 1$, on considering the sliding mode surface according to (20) and approximately the external bounded disturbance by using non-linear FODO (9) and (10). Then multiswitching hybrid synchronization error e(t) is stable and bounded under the adaptive sliding control scheme (22). Proof:To prove the multiswitching hybrid synchronization error convergence e(t), we consider the Lyapunov function **V(t)** as

$$\mathbf{V(t)} = \sum_{i=1}^{3}\frac{1}{2}s_i^2(t) + \sum_{i=1}^{3}\frac{1}{2}\tilde{\psi}^2(t) + \sum_{i=1}^{3}\frac{1}{2}\Big(\frac{1}{\sqrt{m_i}}(\hat{\varsigma}_i - \varsigma_i)\Big)^2 \tag{27}$$

Differentiating and using Property 2 in eq. (27), we obtain

$$D^{\alpha}\mathbf{V(t)} = \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}s_i^2(t) + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\psi}^2(t) + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\Big(\frac{1}{\sqrt{m_i}}(\hat{\varsigma}_i - \varsigma_i)\Big)^2 \tag{28}$$

Substitute $\hat{\varsigma}_i - \varsigma_i = \tilde{\varsigma}_i$ and using Lemma 2 in in eq.(28), we get

$$D^{\alpha}\mathbf{V(t)} \leq \sum_{i=1}^{3} s_i(t)D^{\alpha}s_i(t) + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\psi}^2(t) + \sum_{i=1}^{3}\frac{1}{\sqrt{m_i}}\tilde{\varsigma}_i D^{\alpha}\tilde{\varsigma}_i \tag{29}$$

Using (21) and substitute the value of (24) in (29), we have

$$\begin{aligned} D^{\alpha}\mathbf{V(t)} &\leq \sum_{i=1}^{3} s_i(t)(-\mu_i s_i - \hat{\varsigma}_i signs_i + \tilde{\psi}_i) \\ &+ \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\psi(t)}^2(t) + \sum_{i=1}^{3}\frac{1}{\sqrt{m_i}}\tilde{\varsigma}_i D^{\alpha}\tilde{\varsigma}_i \end{aligned} \tag{30}$$

Using Property 1 and $\tilde{\varsigma}_i = \hat{\varsigma}_i - \varsigma_i$, we get

$$D^{\alpha}\tilde{\varsigma}_i = D^{\alpha}\hat{\varsigma}_i \tag{31}$$

From update laws and (30), we get

$$\begin{aligned} \sum_{i=1}^{3}\frac{1}{\sqrt{m_i}}\tilde{\varsigma}_i D^{\alpha}\tilde{\varsigma}_i &= \sum_{i=1}^{3}\tilde{\varsigma}_i(\mid s_i(t)\mid - \hat{\varsigma}_i) \\ &= \sum_{i=1}^{3}\tilde{\varsigma}_i \mid s_i(t)\mid - \sum_{i=1}^{3}\tilde{\varsigma}_i\hat{\varsigma}_i \\ &\leq \sum_{i=1}^{3}\tilde{\varsigma}_i \mid s_i(t)\mid - \sum_{i=1}^{3}\frac{1}{2}\tilde{\varsigma}_i^2 + \sum_{i=1}^{3}\frac{1}{2}\varsigma_i^2 \end{aligned} \tag{32}$$

After substituting (32) in (30), we get

$$\begin{aligned} D^{\alpha}\mathbf{V(t)} &\leq \sum_{i=1}^{3} s_i(t)(-\omega_i s_i - \hat{\varsigma}_i signs_i + \tilde{\theta}_i) + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\omega}^2(t) \\ &+ \sum_{i=1}^{3}\tilde{\phi}_i \mid s_i(t)\mid - \sum_{i=1}^{3}\frac{1}{2}\tilde{\phi}_i^2 + \sum_{i=1}^{3}\frac{1}{2}\varsigma_i^2 \end{aligned} \tag{33}$$

Eq.(33) can be written as

$$\begin{aligned} D^{\alpha}\mathbf{V(t)} &\leq -m_i s_i^2 + \sum_{i=1}^{3}\hat{\varsigma}_i \mid s_i\mid + \sum_{i=1}^{3}\tilde{\psi}_i \mid s_i\mid + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\psi}^2(t) \\ &+ \sum_{i=1}^{3}\tilde{\varsigma}_i \mid s_i(t)\mid - \sum_{i=1}^{3}\frac{1}{2}\tilde{\varsigma}_i^2 + \sum_{i=1}^{3}\frac{1}{2}\varsigma_i^2 \end{aligned} \tag{34}$$

Using

$$\sum_{i=1}^{3}\tilde{\varsigma}_i \mid s_i(t)\mid - \sum_{i=1}^{3}\hat{\varsigma}_i \mid s_i\mid = -\sum_{i=1}^{3}\varsigma_i \mid s_i\mid \tag{35}$$

From (34-35) and $|\tilde{\psi}_i(t)| \leq \varsigma_i(t)$, we get

$$D^{\alpha}\mathbf{V(t)} \leq -m_i s_i^2 + \sum_{i=1}^{3}\frac{1}{2}D^{\alpha}\tilde{\omega}^2(t) - \sum_{i=1}^{3}\frac{1}{2}\tilde{\varsigma}_i^2 + \sum_{i=1}^{3}\frac{1}{2}\varsigma_i^2 \tag{36}$$

From eq. (16) and (36), we have

$$\begin{aligned} D^{\alpha}\mathbf{V(t)} &\leq -m_i s_i^2 - \sum_{i=1}^{3}\frac{1}{2}\tilde{\varsigma}_i^2 + \sum_{i=1}^{3}\frac{1}{2}\varsigma_i^2 + \sum_{i=1}^{3} -(\omega_i - \frac{1}{2})\tilde{\chi}_i{}^2 + \sum_{i=1}^{3}\frac{\rho_i^2}{2} \\ &\leq -n_2 V(t) + n_3 \end{aligned} \tag{37}$$

where $n_2 = min(2m_i, 1, 2\omega_i - 1)$ and $n_3 = \sum_{i=1}^{3}\frac{1}{2}\rho_i^2 + \sum_{i=1}^{3}\frac{1}{2}\varsigma_i^2$. The synchronization error is bounded on choosing the values $m_i > 0$ and $\omega_i > 0.5$. Using the Lemma 2 in (37), we get

$$\begin{aligned} |V(t)| &\leq \frac{2n_3}{n_2} \\ &= \frac{\sum_{i=1}^{3}\rho_i^2 + \sum_{i=1}^{3}\varsigma_i^2}{g_2} \end{aligned} \tag{38}$$

Eq.(38) implies that

$$\| s(t) \| \leq \sqrt{\frac{2(\sum_{i=1}^{3}\rho_i^2 + \sum_{i=1}^{3}\varsigma_i^2)}{2_2}} \tag{39}$$

Therefore, the inequality (38) and (39) implies that the synchronization errors e(t) and s(t) will be bounded as $t \to \infty$. Hence, the error dynamical system (19) is bounded and stable and we achieve bounded multiswitching hybrid synchronization between master and slave systems. This completes the proof.

7. NUMERICAL SIMULATIONS AND DISCUSSIONS

Simulations have been carried out using MATLAB.The system parameters for master system and slave system are taken as $(f = 3, g = 0.1, h = 1)$ and the initial conditions for drive and response system respectively have been taken as $(u_1(0) = 2, u_2(0) = 3, u_3(0) = 2)$ and $(v_1(0) = 1, v_2(0) = 4, v_3(0) = -1)$.

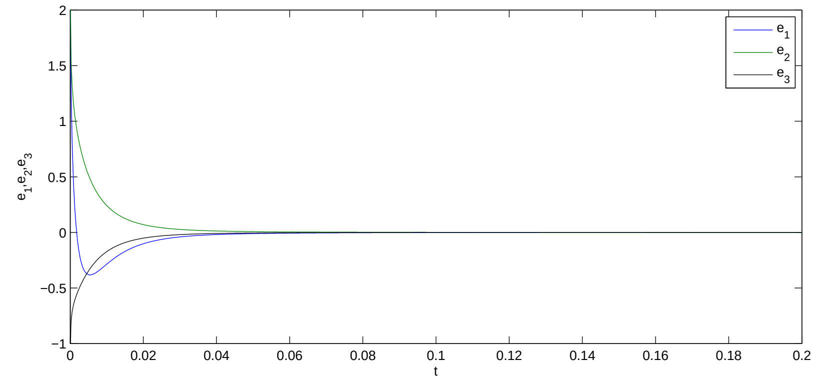

Fig.3: Synchronization error approaches zero at t=0.06(approximately) for $\alpha = 0.85$.

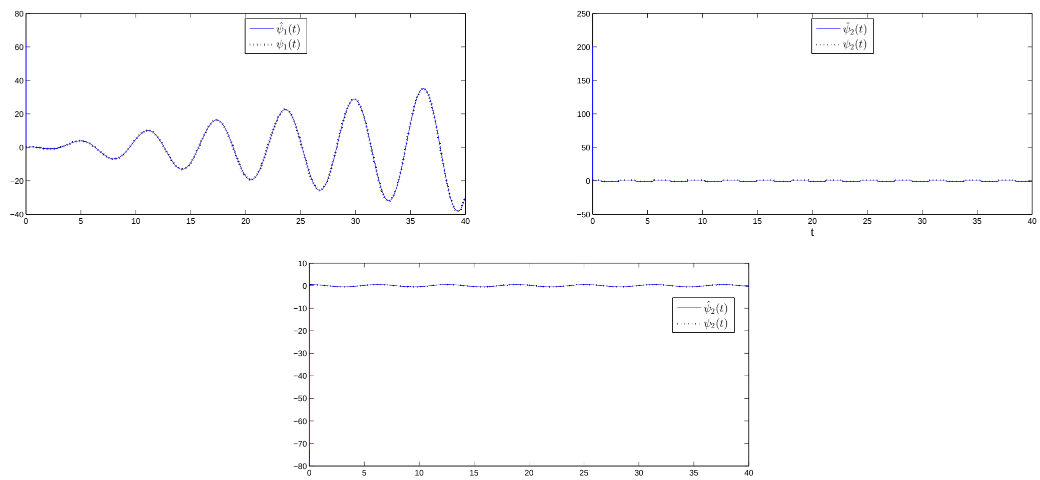

Fig.4: Synchronized Disturbance observers

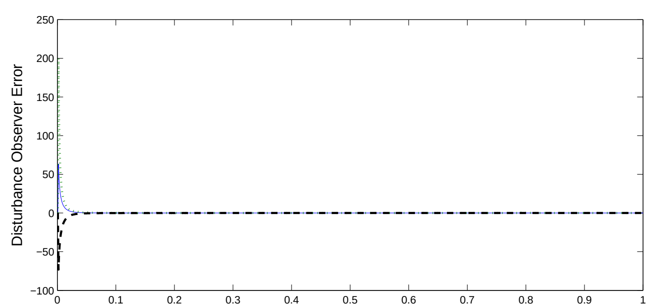

Fig.5: Disturbance observers Error

The initial condition for switch 1 is $(e_{21}(0) = 2, e_{32}(0) = 2, e_{13}(0) = -1)$, $\hat{\varsigma}(0) = (0.2, 0.2, 0.2)$ and $\hat{\chi}(0) = (0.2, 0.2, 0.2)$ .We select the described parameter as $(\omega_1, \omega_2, \omega_3) = (110, 110, 110)$, $(\mu_1, \mu_2, \mu_3) = (90, 90, 90)$ and $(\phi_1, \phi_2, \phi_3) = (0.1, 0.1, 0.1)$. We consider the disturbances as

$\psi_1 = (1-t)sint$, $\psi_2 = sign(cos2t)$ and $\psi_3 = \frac{1}{2}cost$. For numerical simulations we consider the scaling factors as $\delta_1 = 1$, $\delta_2 = -1$ and $\delta_3 = 1$. Fig.1 shows the phase portrait of fractional order financial model. Fig.2 shows the trajectories of variables of switch-1. Fig.3 shows the synchronization error converges to zero. Fig.4 shows the trajectories of estimated disturbance and actual disturbance and Fig.5 shows the disturbance error.

8. Comparison with Published Literature

Synchronization techniques are being developed every now and then with the aim of increasing their robustness and reducing their synchronization time. A few popular synchronization techniques are the adaptive sliding mode technique and disturbance observer based adaptive sliding mode technique. In [16] synchronization is performed considering uncertainties and disturbances using adaptive sliding mode technique at approx. 1.5 units of time and in [7] at 3 units hybrid projective synchronization is performed . In [1] hybrid projective compound synchronization is attained at 0.2 units using disturbance observer based adaptive sliding mode technique. In [15], [14] and [13]disturbance observer based adaptive sliding mode technique on Newton Leipnik system ,financial models and Genesio-Tesi system is achieved at 0.1,0.2 and 0.07 units respectively.

In this paper multi-switching hybrid synchronization using disturbance observer adaptive sliding mode control on fractional financial system is achieved at 0.05 units indicating the efficacy of the deigned controllers.

9. Illustration in Secure Communication

Many chaos synchronization techniques using various control methods are being developed recently. Chaotic systems shows high sensitivity to initial conditions and parameter values, hence they prove to be highly suitable in secure communication, image encryption, control processes etc. The achieved synchronization in this paper is illustrated in field of secure communication with help of an example. Let the message to be sent secretly be $p(t) = sin(t).cos(t)$. We hide it among the chaotic signals from the chaotic system and transmit the encrypted message

$p_1(t)$. After performing the desired synchronization using designed controllers at the receiving end the secret message is decoded as $p_2(t)$. The results are displayed in Fig. 6.

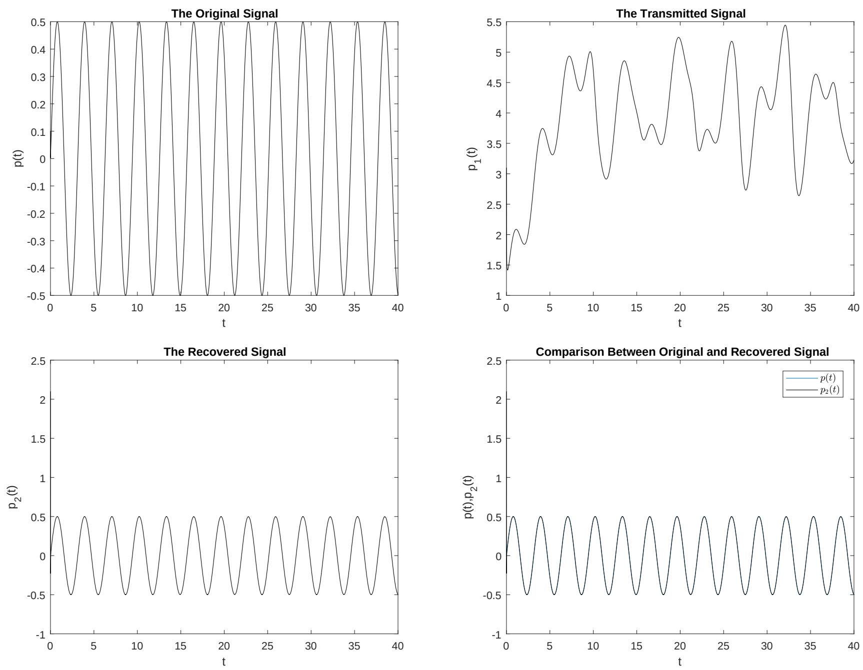

Fig.6: Illustration in Secure Communication

10. Conclusion

During the whole analysis we have been successful in estimating the external disturbances. In order to realize the hybrid multiswitching synchronization we have used the nonlinear sliding mode scheme. Also the error has been displayed to be stable and bounded using the adaptive sliding control scheme. The multi-switching synchronization scheme presented in this paper gives more switching options for constructing the error states and hence makes it more strong. Comparison of the obtained results show the efficacy of designed controllers. This method has been illustrated to be applicable in secure communication with help of an example.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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