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FIXED POINT THEOREMS FOR DUALISTIC CONTRACTIONS OF RATIONAL TYPE IN PARTIALLY ORDERED DUALISTIC PARTIAL METRIC SPACES

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Abstract. The purpose of this paper is to establish some fixed point theorems for mappings involving rational expressions in the frame work of complete ordered dualistic partial metric spaces using a class of pairs of functions satisfying certain assumptions. These results unify, extend and generalize most of the existing relevant fixed point theorems from the literature. We give examples to explain our findings.

Keywords: fixed point; dualistic partial metric; dualistic contractions.

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1. INTRODUCTION

The Banach contraction principle is a classical and powerful tool in nonlinear analysis. Banach contraction principle has been generalized in various ways either by using contractive conditions or by imposing some additional conditions on the ambient spaces. Das and Gupta [7] were the pioneers in proving fixed point theorems using contractive conditions involving rational expressions. Following Das and Gupta [7], Cabrera *et al.* [5] proved a fixed point

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theorem in the context of partially ordered metric spaces. For more fixed point results, see ([9], [13], [24]). One of the most interesting and important space is partial metric space introduced by Matthews (see [12]) as a part of the study of denotational semantics of dataflow networks. In particular, he established the precise relationship between partial metric spaces and the so-called weightable quasi-metric spaces and proved a partial metric generalization of Banach contraction mapping theorem. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties ([3], [10], [17], [22], [25], [26]). Ran-Reuring's fixed point theorem [24] is a fixed point theorem in metric space with a partial order. Existence of fixed point in partially ordered metric spaces has been considered recently by many authors (see, [2], [5], [9], for example). In the same spirit, O'Neill [23] introduced the concept of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. Oltra and Valero [11] presented a Banach fixed point theorem on complete dualistic partial metric spaces. Valero also showed that the contractive condition in Banach fixed point theorem in complete dualistic partial metric spaces cannot be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Following Oltra and Valero [22], Nazam et al. [3] established some fixed point results in dualistic partial metric spaces for Greghty [8] type contraction and monotone mappings and discussed an application of fixed point theorem to show the existence of solution of integral equation. For the fixed point results on dualistic partial metric spaces, the readers may refer to [14] [15],[18],[20],[21].

Recently, Nazam et al. [16] studied behavior of a rational type contraction in context of ordered dualistic partial metric spaces and investigated sufficient conditions for the existence of a fixed point in this space. The main purpose of this paper is to present some fixed point theorems for mappings involving rational expressions in the frame work of complete ordered dualistic partial metric spaces using a class of pairs of functions satisfying certain assumptions. We shall show that our results generalize Theorem 2 and Theorem 3 of Nazam et al. [16] in many ways.

2. PRELIMINARIES

Throughout this paper the letters $\mathbb{R}_0^+, \mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, set of real numbers and set of natural numbers, respectively. We recall some basics definitions and results to make this paper self-sufficient.

Definition 2.1 (see [12]) A partial metric on a non-empty set $\mathfrak{D}$ is a function $\eta: \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}_0^+$ complying with following axioms, for all $\sigma, \varsigma, \nu \in \mathfrak{D}$

(η_1) $\sigma = \varsigma \Leftrightarrow \eta(\sigma,\varsigma) = \eta(\sigma,\sigma) = \eta(\varsigma,\varsigma)$;

(η_2) $\eta(\sigma,\sigma) \leq \eta(\sigma,\varsigma)$;

(η_3) $\eta(\sigma,\varsigma) = \eta(\varsigma,\sigma)$;

(η_4) $\eta(\sigma,\varsigma) \leq \eta(\sigma,\nu) + \eta(\nu,\varsigma) - \eta(\nu,\nu)$

The pair $(\mathfrak{D},\eta)$ is known as partial metric space.

O'Neill [23] introduced the concept of dualistic partial metric as a generalization of partial metric in order to expand the connections between partial metrics and semantics via valuation spaces. He did one significant change to the partial metric η by extending its range from $\mathbb{R}_0^+$ to $\mathbb{R}$. The partial metric η with extended range is called a dualistic partial metric, denoted by η^*.

Definition 2.2 (see [23]) A dualistic partial metric on a non-empty set $\mathfrak{D}$ is a function $\eta^*: \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}$ satisfying the following axioms, for all $\sigma, \varsigma, \nu \in \mathfrak{D}$

(η_1^*) $\sigma = \varsigma \Leftrightarrow \eta^*(\sigma,\varsigma) = \eta^*(\sigma,\sigma) = \eta^*(\varsigma,\varsigma)$;

(η_2^*) $\eta^*(\sigma,\sigma) \leq \eta^*(\sigma,\varsigma)$;

(η_3^*) $\eta^*(\sigma,\varsigma) = \eta^*(\varsigma,\sigma)$;

(η_4^*) $\eta^*(\sigma,\nu) + \eta^*(\varsigma,\varsigma) \leq \eta^*(\sigma,\varsigma) + \eta^*(\varsigma,\nu)$

The pair $(\mathfrak{D},\eta^*)$ is called a dualistic partial metric space.

Remark 2.3 Each partial metric is a dualistic partial metric but the converse is false. To prove this important fact, let $\mathfrak{D} = \mathbb{R}$ and define η^* on $\mathfrak{D}$ as $\eta^*(\sigma,\varsigma) = \max\{\sigma,\varsigma\}, \forall \sigma,\varsigma \in \mathfrak{D}$. Clearly, η^* satisfies $(\eta_1^*) - (\eta_4^*)$ and hence η^* is a dualistic partial metric on $\mathfrak{D}$. Refer that η^* is not a partial metric on $\mathfrak{D}$ because $\eta^*(\sigma,\varsigma) < 0 \notin \mathbb{R}_0^+$, $\forall\, \sigma < 0, \varsigma < 0$. Unlike other metrics, in dualistic partial metric $\eta^*(\sigma,\varsigma) = 0$ does not imply $\sigma = \varsigma$. Indeed, $\eta^*(-2,0) = 0$ and $-2 \neq 0$. The self-distance $\eta^*(\sigma,\sigma)$ is a feature utilized to describe the amount of information contained in σ.The restriction of η^* to $\mathbb{R}_0^+$ is a partial metric. This situation creates a problem in obtaining a fixed point of a self-mapping in dualistic partial metric space. For the solution of this problem, Nazam [16] introduced concept of *convergence comparison property* (CCP) and established some fixed point by using CCP along with axioms (η_1^*) and (η_2^*).

Definition 2.4 (see [16]) Let $(\mathfrak{D},\eta^*)$ be a dualistic partial metric space and $\mathcal{T}$ be a self-mapping on $\mathfrak{D}$. We say that T has a convergence comparison property (CCP) if for every sequence $\{\nu_n\}_{n\in\mathbb{N}}$ in $\mathfrak{D}$ such that $\nu_n \to \nu$, $\mathcal{T}$ satisfies

$$\eta^*(\nu,\nu) \leq \eta^*(\mathcal{T}\nu,\mathcal{T}\nu). \tag{2.1}$$

Example 2.5

(1) Define η^* as in Remark 2.3. Consider any sequence $\{\nu_n\}$ converging to ν in $(\mathfrak{D},\eta^*)$. Consider $\left\{\nu_n = \frac{1-n}{n}, n \geq 1\right\}_{n\in\mathbb{N}} \subset \mathfrak{D}$. We have $\lim_{n\to\infty} \nu_n = -1 \in \mathfrak{D}$. Define a self-map $\mathcal{T}$ on $\mathfrak{D}$ by $\mathcal{T}\nu = \exp \nu$. For such $\nu = -1$, note that

$$\eta^*(\nu,\nu) = -1 \leq \exp(-1) = \eta^*(\mathcal{T}\nu,\mathcal{T}\nu).$$

Hence $\mathcal{T}$ satisfies (CCP).

(2) Let $\mathfrak{D} = (-\infty, 0]$ and define the mapping η^* by $\eta^*(\sigma,\varsigma) = |\sigma - \varsigma|$ if $\sigma \neq \varsigma$ and $\eta^*(\sigma,\varsigma) = \sigma \vee \varsigma$ if $\sigma = \varsigma$, then, $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space. Define $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$ by $\mathcal{T}\sigma = -1$ if $\sigma \in (-\infty, -4]$ and $\mathcal{T}\sigma = 0$ if $\sigma \in (-4,0]$. Notice that $\mathcal{T}$ has (CCP). Indeed, if $\left\{\sigma_n = -\frac{(2n+1)}{n}\right\}_{n\in\mathbb{N}} \subset \mathfrak{D}$. Here $\lim_{n\to\infty} \sigma_n = -2 \in \mathfrak{D}$. For such $\sigma = -2$, we have

$$\eta^*(-2,-2) = (-2) \vee (-2) = -2 \leq 0 = \eta^*(0,0) = \eta^*\big(\mathcal{T}(-2),\mathcal{T}(-2)\big)$$

We present some examples to explain dualistic partial metric.

Example 2.6 (see [16], [23])

(a) If we take $\eta^*(\sigma,\varsigma) = d(\sigma,\varsigma) + b$, where d is a metric on a nonempty set $\mathfrak{D}$ and $b \in \mathbb{R}$ is arbitrary constant, then it is easy to check that η^* verifies axioms $(\eta_1^*) - (\eta_4^*)$ and hence $(\mathfrak{D},\eta^*)$ is a dualistic partial metric space.

(b) Let η be a partial metric defined on a non empty set $\mathfrak{D}$. The function $\eta^*: \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}$ defined by $\eta^*(\sigma,\varsigma) = \eta(\sigma,\varsigma) - \eta(\sigma,\sigma) - \eta(\varsigma,\varsigma)$ satisfies the axioms $(\eta_1^*) - (\eta_4^*)$ and so it defines a dualistic partial metric on $\mathfrak{D}$. Note that $\eta^*(\sigma,\varsigma)$ may have negative values.

(c) Define $\eta^*: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\eta^*(\sigma,\varsigma) = |\sigma - \varsigma|$ if $\sigma \neq \varsigma$ and $\eta^*(\sigma,\varsigma) = -\beta$ if $\sigma = \varsigma$ and $\beta > 0$. We can easily see that η^* is a dualistic partial metric on $\mathbb{R}$.

O'Neill [23] established that each dualistic partial metric η^* on $\mathfrak{D}$ generates a T_0 topology $\tau(\eta^*)$ on $\mathfrak{D}$ having a base, the family of η^*-balls $\{\mathcal{B}_{\eta^*}(\sigma,\epsilon) \mid \sigma \in \mathfrak{D}, \epsilon > 0\}$, where

$$\mathcal{B}_{\eta^*}(\sigma,\epsilon) = \{\varsigma \in \mathfrak{D} \mid \eta^*(\sigma,\varsigma) < \eta^*(\sigma,\sigma) + \epsilon\}.$$

If $(\mathfrak{D},\eta^*)$ is a dualistic partial metric space, then the function $d_{\eta^*}: \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}_0^+$, defined by

$$d_{\eta^*}(\sigma,\varsigma) = \eta^*(\sigma,\varsigma) - \eta^*(\sigma,\sigma) \tag{2.2}$$

defines a quasi metric on $\mathfrak{D}$ such that $\tau(\eta^*) = \tau(d_{\eta^*})$ and

$$d_{\eta^*}^s(\sigma,\varsigma) = \max\{d_{\eta^*}(\sigma,\varsigma), d_{\eta^*}(\varsigma,\sigma)\} \tag{2.3}$$

defines a metric on $\mathcal{A}$.

Definition 2.7 (see [22]) Let $(\mathfrak{D}, \eta^*)$ be a dualistic partial metric space.

1. A sequence $\{\sigma_n\} \subset \mathfrak{D}$ converges to a point $\sigma \in \mathfrak{D}$ if $\lim_{n\to\infty} \eta^*(\sigma_n, \sigma) = \eta^*(\sigma, \sigma)$.
2. A sequence $\{\sigma_n\} \subset \mathfrak{D}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \eta^*(\sigma_n, \sigma_m)$ exists and is finite.
3. A dualistic partial metric space $(\mathfrak{D}, \eta^*)$ is said to be complete if every Cauchy sequence $\{\sigma_n\} \subset \mathfrak{D}$ converges, with respect to $\tau(\eta^*)$, to a point $\sigma \in \mathfrak{D}$ such that $\eta^*(\sigma, \sigma) = \lim_{n,m\to\infty} \eta^*(\sigma_n, \sigma_m)$.

Remark 2.8 For a sequence, convergence with respect to metric space may not imply convergence with respect to dualistic partial metric space. Indeed, if we take $\beta = 1$ and $\{\sigma_n\} \subset \mathbb{R}$, where $\sigma_n = \frac{1-n}{n}$ as in Example 2.6 (c). Mention that $\lim_{n\to\infty} d(\sigma_n, -1) = -1$ and therefore, $\sigma_n \to -1$ with respect to d. On the other hand, we make a conclusion that $\sigma_n \nrightarrow -1$ with respect to η^* because $\lim_{n\to\infty} \eta^*(\sigma_n, -1) = \lim_{n\to\infty} \eta^* |\sigma_n - (-1)| = \lim_{n\to\infty} \left|\frac{1-n}{n} + 1\right| = 0$ and $\eta^*(-1, -1) = -1$.

Lemma 2.9 (see [22]) Let $(\mathfrak{D}, \eta^*)$ be a dualistic partial metric space.

(1) Every Cauchy sequence in $(\mathfrak{D}, d^s_{\eta^*})$ is also a Cauchy sequence in $(\mathfrak{D}, \eta^*)$.

(2) A dualistic partial metric $(\mathfrak{D}, \eta^*)$ is complete if and only if the induced metric space $(\mathfrak{D}, d^s_{\eta^*})$ is complete.

(3) A sequence $\{\sigma_n\} \subset \mathfrak{D}$ converges to a point $\sigma \in \mathfrak{D}$ with respect to $\tau\left(d^s_{\eta^*}\right)$ if and only if $\eta^*(\sigma, \sigma) = \lim_{n\to\infty} \eta^*(\sigma_n, \sigma) = \lim_{n\to\infty} \eta^*(\sigma_n, \sigma_m)$.

Definition 2.10 (see [16]) Let $(\mathfrak{D}, \preceq)$ be a partially ordered set and $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$, we say that $\mathcal{T}$ is monotone non-decreasing if $\sigma, \varsigma \in \mathfrak{D}, \sigma \preceq \varsigma \Longrightarrow \mathcal{T}\sigma \preceq \mathcal{T}\varsigma$.

Definition 2.11 (see [16]) Let $(\mathfrak{D}, \preceq)$ be a partially ordered set and $(\mathfrak{D}, \eta^*)$ be a dualistic partial metric space. A mapping $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$ is said to be a dualistic contraction of rational type if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that:

$$|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)| \leq \alpha \left|\frac{\eta^*(\varsigma, \mathcal{T}\varsigma)\left(1 + \eta^*(\sigma, \mathcal{T}\sigma)\right)}{1 + \eta^*(\sigma, \varsigma)}\right| + \beta |\eta^*(\sigma, \varsigma)| \tag{2.4}$$

$\forall \sigma, \varsigma \in \Delta = \left\{(\sigma, \varsigma) \in \mathfrak{D} \times \mathfrak{D} \,\middle|\, \sigma \preceq \varsigma \wedge \eta^*(\sigma, \varsigma) = -1\right\}$.

The rational contractive condition (2.4) has some differences with rational contractive condition (1) of Cabrera et al. [5]. Indeed, for a metric $d, d(\sigma, \sigma) = 0, \forall \sigma \in \mathfrak{D}$, which ensures that condition (1) of [5] holds for all σ such that $\mathcal{T}\sigma = \sigma$ and conversely. However, in general, $\eta^*(\sigma,\sigma) \neq 0$ for any $\sigma \in \mathfrak{D}$. For if $\mathcal{T}\sigma = \sigma$, then from (2.4) one can follow that $\eta^*(\sigma,\sigma) = 0$. So if $\mathcal{T}\sigma = \sigma$ such that $\eta^*(\sigma,\sigma) \neq 0$, then but $\sigma \preceq \sigma$ does not satisfy (2.4).

Nazam *et al.* [16] studied the following fixed point theorems on dualistic contraction of rational type.

Theorem 2.12 (see [16]) Let $(\mathfrak{D},\preceq)$ be a partially ordered set and $(\mathfrak{D},\eta^*)$ be a complete dualistic partial metric space. If $\mathcal{T}:\mathfrak{D} \to \mathfrak{D}$ is a non-decreasing, dualistic contraction of rational type satisfying the following conditions:

(1) there exists $\sigma_0 \in \mathfrak{D}$ such that $\sigma_0 \preceq \mathcal{T}\sigma_0$,

(2) if $\{\sigma_n\} \subset \mathfrak{D}$ is non-decreasing sequence such that $\sigma_n \to \nu$, then $\sigma_n \preceq \nu, \forall n \in \mathbb{N}$.

Then $\mathcal{T}$ has a fixed point.

Theorem 2.13 (see [16]) Let $\mathcal{T}:\mathfrak{D} \to \mathfrak{D}$ be defined on $(\mathfrak{D},\eta^*,\preceq)$ and satisfies conditions assumed in Theorem 2.12. If there exists an element $\omega \in \mathfrak{D}$ such that it is comparable with every fixed point of $\mathcal{T}$, then $\mathcal{T}$ has a unique fixed point in $\mathfrak{D}$.

One of the most important ingredients of a contractive condition is to study the kind of involved functions, like altering distance functions introduced by Khan et al. [11] as follows.

Definition 2.14 (see [11]) A function $\varphi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be altering distance function if

(a1) φ is monotone increasing and continuous,

(a2) $\varphi(\kappa) = 0 \Leftrightarrow \kappa = 0$.

Definition 2.15 (see [4]) The pair (φ,ϕ), where $\varphi,\phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a pair of generalized altering distance functions if

(b1). φ is continuous;

(b2). φ is non-decreasing;

(b3). $\lim_{n\to\infty} \phi(\kappa_n) = 0 \Rightarrow \lim_{n\to\infty} \kappa_n = 0$.

The condition (b3) was introduced by Moradi and Farajzadeh [13]. The above conditions do not determine the values of $\varphi(0)$ and $\phi(0)$.

Definition 2.16 (see [1]) We will denote by $\mathcal{F}$ the family of all pairs (φ,ϕ), where $\varphi,\phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are functions satisfying the following conditions.

(F1). φ is non-decreasing;

(F2). if $\exists\, \kappa_0 \in \mathbb{R}_0^+$ such that $\phi\,(\kappa_0) = 0$, then $\kappa_0 = 0$ and $\varphi^{-1}(0) = \{0\}$.

(F3). if $\{\alpha_n\},\{\beta_n\} \subset \mathbb{R}_0^+$ such that, $\{\alpha_n\} \to \lambda, \{\beta_n\} \to \lambda$ satisfying $\lambda < \{\beta_n\}$ and $\varphi(\beta_n) \leq (\varphi - \phi)(\alpha_n), \forall\, n \in \mathbb{N}$, then $\lambda = 0$.

Definition 2.17(see [27]) A pair of functions (φ,ϕ) is said to belong to the class $\mathfrak{F}$, if they satisfy the following conditions:

(c1). $\varphi, \phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$;

(c2). if $\kappa, \mu \in \mathbb{R}_0^+$, $\varphi(\kappa) \leq \phi(\mu)$ then $\kappa \leq \mu$;

(c3). if $\{\kappa_n\}, \{\mu_n\} \subset \mathbb{R}_0^+$, $\lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} \mu_n = \delta$ and $\varphi(\kappa_n) \leq \phi(\mu_n), \forall\, n \in \mathbb{N}$, then $\delta = 0$.

If (φ, ϕ) satisfies (F1) and (F2), then $(\varphi, \phi = \varphi - \phi)$ satisfies (c1) and (c2). Furthermore, if $(\varphi, \phi = \varphi - \phi)$ satisfies (c3), then (φ, ϕ) satisfies (F3).

Remark 2.18 (see [27]) If $(\varphi, \phi) \in \mathfrak{F}$ and $\varphi(\kappa) \leq \phi(\kappa)$, then $\kappa = 0$, since we can take $\kappa_n = \mu_n = \kappa, \forall\, n \in \mathbb{N}$ and by (c3), we deduce $\kappa = 0$.

Example 2.19 The conditions (c1)-(c3) of definition 2.17 are fulfilled for the functions $\varphi, \phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

(1) $\varphi(\kappa) = \ln\left(\frac{5\kappa+1}{12}\right)$ and $\phi(\kappa) = \ln\left(\frac{3\kappa+1}{12}\right), \forall \kappa \in \mathbb{R}_0^+$.

(2) $\varphi(\kappa) = \ln\left(\frac{2\kappa+1}{2}\right)$ and $\phi(\kappa) = \ln\left(\frac{\kappa+1}{2}\right), \forall \kappa \in \mathbb{R}_0^+$

Example 2.20 (see [27]) Let $\mathcal{S} = \left\{\ell: \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,\middle|\, \ell(\kappa_n) \to 1 \Rightarrow \kappa_n \to 0\right\}$. Consider the pairs of functions $\left(1_{\mathbb{R}_0^+}, \ell\left(1_{\mathbb{R}_0^+}\right)\right)$, where $\ell \in \mathcal{S}$ and $\ell\left(1_{\mathbb{R}_0^+}\right)$ is defined as $\left(\ell\left(1_{\mathbb{R}_0^+}\right)\right)(\kappa) = \ell(\kappa)\kappa, \kappa \in \mathbb{R}_0^+$. It is easy to check that $\left(1_{\mathbb{R}_0^+}, \ell\left(1_{\mathbb{R}_0^+}\right)\right) \in \mathfrak{F}$.

Example 2.21 (see [27]) Let $\varphi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and increasing function such that $\varphi(\kappa) = 0 \Leftrightarrow \kappa = 0$. Let $\phi: [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\phi(\kappa) = 0 \Leftrightarrow \kappa = 0$ and $\phi \leq \varphi$. We make a conclusion that $(\varphi, \varphi - \phi) \in \mathfrak{F}$.

An interesting particular case is when φ is the identity mapping, $\varphi = 1_{\mathbb{R}_0^+}$ and $\phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a non-decreasing function such that $\phi(\kappa) = 0 \Leftrightarrow \kappa = 0$ and $\phi(\kappa) \leq \kappa, \forall\, \kappa \in \mathbb{R}_0^+$.

Remark 2.22 (see [27]) Let $g: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an increasing function and $(\varphi, \phi) \in \mathfrak{F}$. Then $(g \circ \varphi, g \circ \phi) \in \mathfrak{F}$.

3. MAIN RESULTS

We state our main result as follows:

Theorem 3.1 Let $(\mathfrak{D}, \preceq)$ is a partially ordered set and suppose that $(\mathfrak{D}, \eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$ be a non-decreasing and satisfies (CCP) such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi(|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)|) \leq \max\left\{\phi(|\eta^*(\sigma, \varsigma)|), \phi\left(\left|\frac{\eta^*(\varsigma, \mathcal{T}\varsigma)\left(1+\eta^*(\sigma, \mathcal{T}\sigma)\right)}{1+\eta^*(\sigma, \varsigma)}\right|\right)\right\} \tag{3.1}$$

for all comparable elements $\sigma, \varsigma \in \Delta$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\} \subset \mathfrak{D}$ is non-decreasing sequence such that $\sigma_n \to \nu$, then $\sigma_n \preceq \nu, \forall n \in \mathbb{N}$. If $\exists\ \sigma_0 \in \mathfrak{D}$ such that $\sigma_0 \preceq \mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point.

Proof Let $\sigma_0 \in \mathfrak{D}$ be an initial element such that $\sigma_0 \preceq \mathcal{T}\sigma_0$ and let us define Picard iterative sequence $\{\sigma_n\}$ by $\mathcal{T}\sigma_{n-1} = \sigma_n, \forall n \in \mathbb{N}$. If there exists a positive integer n_0 such that $\sigma_{n_0} = \sigma_{n_0+1}$, then $\sigma_{n_0} = \sigma_{n_0+1} = \mathcal{T}\sigma_{n_0}$. So σ_{n_0} is a fixed point of $\mathcal{T}$. In this case, the proof is complete. On the other hand, if $\sigma_n \neq \sigma_{n+1}, \forall n \in \mathbb{N}$, then $\sigma_n \preceq \sigma_{n+1}$. Indeed by $\sigma_0 \preceq \mathcal{T}\sigma_0$, we obtain $\sigma_0 \preceq \sigma_1$. Since T is non-decreasing, $\sigma_0 \preceq \sigma_1$ implies $\mathcal{T}\sigma_0 \preceq \mathcal{T}\sigma_1$ and thus $\sigma_1 \preceq \sigma_2$. Continuing in this fashion, we get

$$\sigma_0 \preceq \sigma_1 \preceq \sigma_2 \preceq \sigma_3 \preceq \cdots \preceq \sigma_{n-1} \preceq \sigma_n \preceq \sigma_{n+1} \preceq \cdots \tag{3.2}$$

Since $\sigma_n \preceq \sigma_{n+1}$, applying contractive condition (3.1), we have

$$\begin{aligned}\varphi(|\eta^*(\sigma_{n+1}, \sigma_n)|) &= \varphi(|\eta^*(\mathcal{T}\sigma_n, \mathcal{T}\sigma_{n-1})|)\\ &\leq \max\left\{\phi(|\eta^*(\sigma_n, \sigma_{n-1})|), \phi\left(\left|\frac{\eta^*(\sigma_{n-1}, \mathcal{T}\sigma_{n-1})(1+\eta^*(\sigma_n, \mathcal{T}\sigma_n))}{1+\eta^*(\sigma_n, \sigma_{n-1})}\right|\right)\right\}\\ &= \max\left\{\phi(|\eta^*(\sigma_n, \sigma_{n-1})|), \phi\left(\left|\frac{\eta^*(\sigma_{n-1}, \sigma_n)(1+\eta^*(\sigma_n, \sigma_{n+1}))}{1+\eta^*(\sigma_n, \sigma_{n-1})}\right|\right)\right\} \end{aligned}\tag{3.3}$$

Now, we can distinguish two cases.

Case 1. Consider

$$\max\left\{\phi(|\eta^*(\sigma_n, \sigma_{n-1})|), \phi\left(\left|\frac{\eta^*(\sigma_{n-1}, \sigma_n)(1+\eta^*(\sigma_n, \sigma_{n+1}))}{1+\eta^*(\sigma_n, \sigma_{n-1})}\right|\right)\right\} = \phi(|\eta^*(\sigma_n, \sigma_{n-1})|) \tag{3.4}$$

Due to inequality (3.3), we have

$$\varphi(|\eta^*(\sigma_{n+1}, \sigma_n)|) \leq \phi(|\eta^*(\sigma_n, \sigma_{n-1})|) \tag{3.5}$$

Since $(\varphi, \phi) \in \mathfrak{F}$, we deduce that

$$|\eta^*(\sigma_{n+1}, \sigma_n)| \leq |\eta^*(\sigma_n, \sigma_{n-1})|$$

Case 2. If

$$\max\left\{\phi(|\eta^*(\sigma_n, \sigma_{n-1})|), \phi\left(\left|\frac{\eta^*(\sigma_{n-1}, \sigma_n)(1+\eta^*(\sigma_n, \sigma_{n+1}))}{1+\eta^*(\sigma_n, \sigma_{n-1})}\right|\right)\right\} = \phi\left(\left|\frac{\eta^*(\sigma_{n-1}, \sigma_n)(1+\eta^*(\sigma_n, \sigma_{n+1}))}{1+\eta^*(\sigma_{n-1}, \sigma_n)}\right|\right) \tag{3.6}$$

Then from (3.3), we have

$$\varphi(|\eta^*(\sigma_{n+1}, \sigma_n)|) \leq \phi\left(\left|\frac{\eta^*(\sigma_{n-1}, \sigma_n)(1+\eta^*(\sigma_n, \sigma_{n+1}))}{1+\eta^*(\sigma_{n-1}, \sigma_n)}\right|\right) \tag{3.7}$$

Since $(\varphi, \phi) \in \mathfrak{F}$ we get

$$|\eta^*(\sigma_n, \sigma_{n+1})| \leq \left|\frac{\eta^*(\sigma_{n-1}, \sigma_n)(1+\eta^*(\sigma_n, \sigma_{n+1}))}{1+\eta^*(\sigma_{n-1}, \sigma_n)}\right|$$

which implies that

$$|\eta^*(\sigma_{n+1}, \sigma_n)| \leq |\eta^*(\sigma_n, \sigma_{n-1})|$$

From both cases, we conclude that the sequence $\{|\eta^*(\sigma_{n+1},\sigma_n)|\}$ is a monotone and bounded below sequence of non-negative real numbers, it is convergent and converges to a point $\mathcal{L}$, i.e. $\lim_{n\to\infty}|\eta^*(\sigma_{n+1},\sigma_n)| = \mathcal{L} \geq 0$. If $\mathcal{L} = 0$. Then we have done. Let $\mathcal{L} \succ 0$ and denote $A = \{n \in \mathbb{N} \mid n \text{ satisfies } (3.4)\}$ and $B = \{n \in \mathbb{N} \mid n \text{ satisfies } (3.6)\}$. Now, we make the following remark.

(1) If Card $A = \infty$, then from (3.3), we can find infinitely natural numbers n satisfying inequality (3.5) and since $\lim_{n\to\infty}|\eta^*(\sigma_{n+1},\sigma_n)| = \lim_{n\to\infty}|\eta^*(\sigma_n,\sigma_{n-1})| = \mathcal{L}$ and $(\varphi,\phi) \in \mathfrak{F}$, we deduce that $\mathcal{L} = 0$.

(2) If Card $B = \infty$, then from (3.3), we can find infinitely many $n \in \mathbb{N}$ satisfying inequality (3.7). Since $(\varphi,\phi) \in \mathfrak{F}$ and using the similar argument to the one used in case 2, we obtain $|\eta^*(\sigma_n,\sigma_{n+1})| \leq \left|\frac{\eta^*(\sigma_{n-1},\sigma_n)(1+\eta^*(\sigma_n,\sigma_{n+1}))}{1+\eta^*(\sigma_{n-1},\sigma_n)}\right|$ for infinitely many $n \in \mathbb{N}$. On letting the limit as $n \to \infty$ and taking into account that $\lim_{n\to\infty}|\eta^*(\sigma_{n+1},\sigma_n)| = \mathcal{L}$, we deduce that $\mathcal{L} \leq \frac{\mathcal{L}(1+\mathcal{L})}{1+\mathcal{L}}$ and consequently, we obtain $\mathcal{L} = 0$.

Therefore, in both cases we have

$$\lim_{n\to\infty}|\eta^*(\sigma_{n+1},\sigma_n)| = 0 \text{ and then } \lim_{n\to\infty}\eta^*(\sigma_{n+1},\sigma_n) = 0 \tag{3.8}$$

We use (3.1) to find the self distance $\eta^*(\sigma_n,\sigma_n)$, as follows:

$$\begin{aligned}\varphi(|\eta^*(\sigma_n,\sigma_n)|) &= \varphi(|\eta^*(\mathcal{T}\sigma_{n-1},\mathcal{T}\sigma_{n-1})|)\\ &\leq \max\left\{\phi(|\eta^*(\sigma_{n-1},\sigma_{n-1})|),\phi\left(\left|\frac{\eta^*(\sigma_{n-1},\mathcal{T}\sigma_{n-1})(1+\eta^*(\sigma_{n-1},\mathcal{T}\sigma_{n-1}))}{1+\eta^*(\sigma_{n-1},\sigma_{n-1})}\right|\right)\right\}\\ &= \max\left\{\phi(|\eta^*(\sigma_{n-1},\sigma_{n-1})|),\phi\left(\left|\frac{\eta^*(\sigma_{n-1},\sigma_n)(1+\eta^*(\sigma_{n-1},\sigma_n))}{1+\eta^*(\sigma_{n-1},\sigma_{n-1})}\right|\right)\right\}\end{aligned} \tag{3.9}$$

Put

$$C = \{n \in \mathbb{N} \mid \varphi(|\eta^*(\sigma_n,\sigma_n)|) \leq \phi(|\eta^*(\sigma_{n-1},\sigma_{n-1})|)\}$$

$$D = \left\{n \in \mathbb{N} \mid \varphi(|\eta^*(\sigma_n,\sigma_n)|) \leq \phi\left(\left|\frac{\eta^*(\sigma_{n-1},\sigma_n)(1+\eta^*(\sigma_{n-1},\sigma_n))}{1+\eta^*(\sigma_{n-1},\sigma_{n-1})}\right|\right)\right\}$$

By (3.9), we have Card $C = \infty$ or Card $D = \infty$. If Card $C = \infty$, then there exists infinitely many $n \in \mathbb{N}$ satisfying

$$\varphi(|\eta^*(\sigma_n,\sigma_n)|) \leq \phi(|\eta^*(\sigma_{n-1},\sigma_{n-1})|) \tag{3.10}$$

and since $(\varphi,\phi) \in \mathfrak{F}$, we have

$$|\eta^*(\sigma_n,\sigma_n)| \leq |\eta^*(\sigma_{n-1},\sigma_{n-1})|$$

Thus, $\{|\eta^*(\sigma_{n+1},\sigma_n)|\}$ is a non-increasing sequence of positive real numbers and arguing like case of (3.8), we have $\lim_{n\to\infty}|\eta^*(\sigma_n,\sigma_n)| = 0$. On the other hand, if Card $D = \infty$, then we can find infinitely many $n \in \mathbb{N}$ satisfying

$$\varphi(|\eta^*(\sigma_n,\sigma_n)|) \leq \phi\left(\left|\frac{\eta^*(\sigma_{n-1},\sigma_n)(1+\eta^*(\sigma_{n-1},\sigma_n))}{1+\eta^*(\sigma_{n-1},\sigma_{n-1})}\right|\right) \tag{3.11}$$

and since $(\varphi,\phi) \in \mathfrak{F}$, we infer

$$|\eta^*(\sigma_n,\sigma_n)| \leq \left|\frac{\eta^*(\sigma_{n-1},\sigma_n)(1+\eta^*(\sigma_{n-1},\sigma_n))}{1+\eta^*(\sigma_{n-1},\sigma_{n-1})}\right| \tag{3.12}$$

taking the $\lim_{n\to\infty}$ on (3.12) and using (3.8), we obtain that $\lim_{n\to\infty}|\eta^*(\sigma_n,\sigma_n)| \leq 0$ and then $\lim_{n\to\infty}|\eta^*(\sigma_n,\sigma_n)| = 0$. Thus, in both cases, we infer that $\lim_{n\to\infty}|\eta^*(\sigma_n,\sigma_n)| = 0$ and then

$$\lim_{n\to\infty}\eta^*(\sigma_n,\sigma_n) = 0 \tag{3.13}$$

We deduce from (2.2) that

$$d_{\eta^*}(\sigma_n,\sigma_{n+1}) = \eta^*(\sigma_n,\sigma_{n+1}) - \eta^*(\sigma_n,\sigma_n)$$

So using (3.8) and (3.13), we get

$$\lim_{n\to\infty} d_{\eta^*}(\sigma_n,\sigma_{n+1}) = 0 \tag{3.14}$$

Next step is to show that $\{\sigma_n\}$ is a Cauchy sequence in $(\mathfrak{D}, d^s_{\eta^*})$. For this we have to show that

$$\lim_{m,n\to\infty} d^s_{\eta^*}(\sigma_n,\sigma_m) = \lim_{m,n\to\infty}\max\{d_{\eta^*}(\sigma_n,\sigma_m), d_{\eta^*}(\sigma_m,\sigma_n)\} = 0$$

Suppose on contrary that $\{\sigma_n\}$ is not a Cauchy sequence, that is $\lim_{m,n\to\infty} d_{\eta^*}(\sigma_n,\sigma_m) \neq 0$. Then given $\epsilon > 0$, we will construct a pair of subsequences $\{\sigma_{n_k}\}$ and $\{\sigma_{m_k}\}$ of $\{\sigma_n\}$ such that n_k is smallest index for which for all $n_k > m_k > k$, where $k \in \mathbb{N}$

$$d_{\eta^*}(\sigma_{n_k},\sigma_{m_k}) \geq \epsilon \tag{3.15}$$

It follows directly that

$$d_{\eta^*}(\sigma_{n_k-1},\sigma_{m_k}) < \epsilon \tag{3.16}$$

By (3.15) and (3.16), we have

$$\begin{aligned}\epsilon &\leq d_{\eta^*}(\sigma_{n_k},\sigma_{m_k})\\ &\leq d_{\eta^*}(\sigma_{n_k},\sigma_{n_k-1}) + d_{\eta^*}(\sigma_{n_k-1},\sigma_{m_k})\\ &< d_{\eta^*}(\sigma_{n_k},\sigma_{n_k-1}) + \epsilon\end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides in above inequality and from (3.14), we obtain

$$\lim_{k\to\infty} d_{\eta^*}(\sigma_{n_k},\sigma_{m_k}) = \epsilon \tag{3.17}$$

Using triangle inequality, we have

$$d_{\eta^*}(\sigma_{n_k},\sigma_{m_k}) \leq d_{\eta^*}(\sigma_{n_k},\sigma_{n_k-1}) + d_{\eta^*}(\sigma_{n_k-1},\sigma_{m_k})$$
$$\leq d_{\eta^*}(\sigma_{n_k},\sigma_{n_k-1}) + d_{\eta^*}(\sigma_{n_k-1},\sigma_{m_k-1}) + d_{\eta^*}(\sigma_{m_k-1},\sigma_{m_k})$$

and

$$d_{\eta^*}(\sigma_{n_k-1},\sigma_{m_k-1}) \leq d_{\eta^*}(\sigma_{n_k-1},\sigma_{n_k}) + d_{\eta^*}(\sigma_{n_k},\sigma_{m_k-1})$$
$$\leq d_{\eta^*}(\sigma_{n_k-1},\sigma_{n_k}) + d_{\eta^*}(\sigma_{n_k},\sigma_{m_k}) + d_{\eta^*}(\sigma_{m_k},\sigma_{m_k-1})$$

Taking the limit as $k \to \infty$ in the above two inequalities and using (3.14) and (3.17), we get

$$\lim_{k\to\infty} d_{\eta^*}(\sigma_{n_k-1},\sigma_{m_k-1}) = \epsilon \tag{3.18}$$

Now applying contractive condition (3.1), for $\sigma_{n_k} \neq \sigma_{m_k}$, we have

$$\varphi(|\eta^*(\sigma_{n_k},\sigma_{m_k})|) = \varphi(|\eta^*(\mathcal{T}\sigma_{n_k-1},\mathcal{T}\sigma_{m_k-1})|)$$
$$\leq \max\left\{\phi(|\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})|), \phi\left(\left|\frac{\eta^*(\sigma_{n_k-1},\mathcal{T}\sigma_{n_k-1})(1+\eta^*(\sigma_{m_k-1},\mathcal{T}\sigma_{m_k-1}))}{1+\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})}\right|\right)\right\}$$
$$= \max\left\{\phi(|\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})|), \phi\left(\left|\frac{\eta^*(\sigma_{n_k-1},\sigma_{n_k})(1+\eta^*(\sigma_{m_k-1},\sigma_{m_k}))}{1+\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})}\right|\right)\right\} \tag{3.19}$$

Let us put

$$E = \{k \in \mathbb{N} \mid \varphi(|\eta^*(\sigma_{n_k},\sigma_{m_k})|) \leq \phi(|\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})|)\}$$
$$F = \left\{k \in \mathbb{N} \mid \varphi(|\eta^*(\sigma_{n_k},\sigma_{m_k})|) \leq \phi\left(\left|\frac{\eta^*(\sigma_{n_k-1},\sigma_{n_k})(1+\eta^*(\sigma_{m_k-1},\sigma_{m_k}))}{1+\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})}\right|\right)\right\}$$

By (3.19), we have Card $E = \infty$ or Card $F = \infty$. Let us suppose that Card $E = \infty$, then there exists infinitely many $k \in \mathbb{N}$ satisfying

$$\varphi(|\eta^*(\sigma_{n_k},\sigma_{m_k})|) \leq \phi(|\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})|) \tag{3.20}$$

and since $(\varphi,\phi) \in \mathfrak{F}$, by letting the limit as $k \to \infty$, we have

$$\lim_{k\to\infty}|\eta^*(\sigma_{n_k},\sigma_{m_k})| \leq \lim_{k\to\infty}|\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})|$$

In the view of (3.17) and (3.18), we get $\epsilon = 0$ a contradiction. On the other hand, if Card $F = \infty$, then we can find infinitely many $k \in \mathbb{N}$ satisfying

$$\varphi(|\eta^*(\sigma_{n_k},\sigma_{m_k})|) \leq \phi\left(\left|\frac{\eta^*(\sigma_{n_k-1},\sigma_{n_k})(1+\eta^*(\sigma_{m_k-1},\sigma_{m_k}))}{1+\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})}\right|\right) \tag{3.21}$$

and since $(\varphi,\phi) \in \mathfrak{F}$, we infer

$$|\eta^*(\sigma_{n_k},\sigma_{m_k})| \leq \left|\frac{\eta^*(\sigma_{n_k-1},\sigma_{n_k})(1+\eta^*(\sigma_{m_k-1},\sigma_{m_k}))}{1+\eta^*(\sigma_{n_k-1},\sigma_{m_k-1})}\right|$$

Taking the limit as $k \to \infty$ and in the view of (3.14) and (3.17), it follows that $\epsilon \leq 0$ and we reach a contradiction. Therefore, in both the possibilities, we reach a contradiction and therefore $\lim_{m,n\to\infty} d_{\eta^*}(\sigma_n,\sigma_m) = 0$. Similarly we can prove that $\lim_{m,n\to\infty} d_{\eta^*}(\sigma_m,\sigma_n) = 0$. Hence $\lim_{m,n\to\infty} d^s_{\eta^*}(\sigma_n,\sigma_m) = 0$, which ensures that $\{\sigma_n\}$ is a Cauchy sequence in $\left(\mathfrak{D}, d^s_{\eta^*}\right)$. Since $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space, by Lemma 2.9(2), $\left(\mathfrak{D}, d^s_{\eta^*}\right)$ is a complete metric space. Thus, there exists $\nu \in \left(\mathfrak{D}, d^s_{\eta^*}\right)$ such that $\sigma_n \to \nu$ as $n\to\infty$, that is $\lim_{n\to\infty} d_{\eta^*}(\sigma_n,\nu) = 0$ and by Lemma 2.9 (3), we know that

$$\eta^*(\nu,\nu) = \lim_{n\to\infty} \eta^*(\sigma_n,\nu) = \lim_{n\to\infty} \eta^*(\sigma_n,\sigma_m) \tag{3.22}$$

Since, $\lim_{n\to\infty} d_{\eta^*}(\sigma_n,\nu) = 0$, by (2.2) and (3.12), we have

$$\eta^*(\nu,\nu) = \lim_{n\to\infty} \eta^*(\sigma_n,\nu) = \lim_{n\to\infty} \eta^*(\sigma_n,\sigma_m) = 0 \tag{3.23}$$

This shows that $\{\sigma_n\}$ is a Cauchy sequence converging to $\nu \in (\mathfrak{D},\eta^*)$. We are left to prove that ν is a fixed point of $\mathcal{T}$. We have to deal with two cases:

Case 1. If $\mathcal{T}$ is continuous, then

$$\nu = \lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \mathcal{T}\sigma_{n-1} = \mathcal{T}\left(\lim_{n\to\infty} \sigma_{n-1}\right) = \mathcal{T}\nu \tag{3.24}$$

Hence $\mathcal{T}\nu = \nu$ that is ν is fixed point of $\mathcal{T}$.

Case 2. If $\{\sigma_n\} \subset \mathfrak{D}$ is non-decreasing sequence such that $\sigma_n \to \nu$, then $\sigma_n \preceq \nu, \forall n \in \mathbb{N}$. Now applying (3.1), we have

$$\varphi(|\eta^*(\mathcal{T}\nu,\mathcal{T}\sigma_n)|) \le \max\left\{\phi(|\eta^*(\nu,\sigma_n)|), \phi\left(\left|\frac{\eta^*(\sigma_n,\mathcal{T}\sigma_n)\left(1+\eta^*(\nu,\mathcal{T}\nu)\right)}{1+\eta^*(\sigma_n,\nu)}\right|\right)\right\} \tag{3.25}$$

Denote

$$G = \left\{n \in \mathbb{N} \,\middle|\, \varphi(|\eta^*(\mathcal{T}\nu,\mathcal{T}\sigma_n)|) \le \phi(|\eta^*(\nu,\sigma_n)|)\right\}$$

$$H = \left\{n \in \mathbb{N} \,\middle|\, \varphi(|\eta^*(\mathcal{T}\nu,\mathcal{T}\sigma_n)|) \le \phi\left(\left|\frac{\eta^*(\sigma_n,\mathcal{T}\sigma_n)\left(1+\eta^*(\nu,\mathcal{T}\nu)\right)}{1+\eta^*(\sigma_n,\nu)}\right|\right)\right\}$$

We have Card $G = \infty$ or Card $H = \infty$. If Card $G = \infty$, then there exists infinitely many $n \in \mathbb{N}$ such that

$$\varphi(|\eta^*(\mathcal{T}\nu,\mathcal{T}\sigma_n)|) \le \phi(|\eta^*(\nu,\sigma_n)|) \tag{3.26}$$

and since $(\varphi,\phi) \in \mathfrak{F}$, by taking the limit as $n\to\infty$, we have

$$\lim_{n\to\infty} |\eta^*(\mathcal{T}\nu,\mathcal{T}\sigma_n)| \le \lim_{n\to\infty} |\eta^*(\nu,\sigma_n)|$$

To simplify our consideration, we will denote the subsequence by the same symbol $\{\mathcal{T}\sigma_n\}$. Since $\mathcal{T}\sigma_n = \sigma_{n+1}$ and $\sigma_n \to \nu \in \mathfrak{D}$, this means that $\limsup_{n\to\infty} \eta^*(\nu,\sigma_n) = 0$ and consequently $\lim_{n\to\infty} \sigma_{n+1} = \nu$. We infer $|\eta^*(\mathcal{T}\nu,\nu)| \le 0$ and then $|\eta^*(\mathcal{T}\nu,\nu)| = 0$.On the other hand, if Card $H = \infty$, then we can find infinitely many $n \in \mathbb{N}$ such that

$$\varphi(|\eta^*(\mathcal{T}\nu,\mathcal{T}\sigma_n)|) \leq \phi\left(\left|\frac{\eta^*(\nu,\mathcal{T}\nu)(1+\eta^*(\sigma_n,\mathcal{T}\sigma_n))}{1+\eta^*(\sigma_n,\nu)}\right|\right) \tag{3.27}$$

Since $(\varphi,\phi) \in \mathfrak{F}, \mathcal{T}\sigma_n = \sigma_{n+1}$ and $\lim_{n\to\infty} \sigma_n = \nu$, on letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty}|\eta^*(\mathcal{T}\nu,\sigma_{n+1})| \leq \lim_{n\to\infty}\left|\frac{\eta^*(\sigma_n,\mathcal{T}\sigma_n)(1+\eta^*(\nu,\mathcal{T}\nu))}{1+\eta^*(\sigma_n,\nu)}\right| \tag{3.28}$$

In the view of (3.8), arguing like above, we conclude that $|\eta^*(\mathcal{T}\nu,\nu)| = 0$. Therefore, in both the cases, we obtain $|\eta^*(\mathcal{T}\nu,\nu)| = 0$ and then $\eta^*(\mathcal{T}\nu,\nu) = 0$. Since $\mathcal{T}$ has (CCP), we get

$$0 = \eta^*(\nu,\nu) \leq k\eta^*(\mathcal{T}\nu,\mathcal{T}\nu). \tag{3.29}$$

On the other hand, by axiom (η_4^*) we have $\eta^*(\nu,\nu) \leq \eta^*(\nu,\mathcal{T}\nu) + \eta^*(\mathcal{T}\nu,\nu) - \eta^*(\mathcal{T}\nu,\mathcal{T}\nu)$ which implies that

$$\eta^*(\mathcal{T}\nu,\mathcal{T}\nu) \leq 0 \tag{3.30}$$

The inequalities (3.29) and (3.30) imply that $\eta^*(\mathcal{T}\nu,\mathcal{T}\nu) = 0$. Thus

$$\eta^*(\mathcal{T}\nu,\mathcal{T}\nu) = \eta^*(\nu,\nu) = \eta^*(\nu,\mathcal{T}\nu) \tag{3.31}$$

By using axiom (η_1^*), we have $\mathcal{T}\nu = \nu$ and hence ν is a fixed point of $\mathcal{T}$. This finishes the proof.

Note that in the above result fixed point may not be unique, in order to prove uniqueness of the fixed point, we need some more conditions and for this purpose, we have following Theorem.

Theorem 3.2 Let $(\mathfrak{D},\eta^*,\preceq)$ be an ordered complete dualistic partial metric space. Let $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$ be a mapping satisfying all the conditions of Theorem 3.1. Besides, if for each $\sigma,\varsigma \in \mathfrak{D}$, there exists $\omega \in \mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Proof Following the proof of Theorem 3.1, we are only left to prove the uniqueness of the fixed point. Let ν^* be another fixed point of $\mathcal{T}$, then $\mathcal{T}\nu^* = \nu^*$ and $\eta^*(\nu^*,\nu^*) = 0$. We distinguish two cases:

Case 1. ν and ν^* are comparable.

Suppose $\nu \preceq \nu^*$(the same argument works for $\nu^* \preceq \nu$). By applying the contractive condition (3.1), we get

$$\begin{aligned}\varphi(|\eta^*(\nu,\nu^*)|) &= \varphi(|\eta^*(\mathcal{T}\nu,\mathcal{T}\nu^*)|)\\ &\leq \max\left\{\phi(|\eta^*(\nu,\nu^*)|), \phi\left(\left|\frac{\eta^*(\nu^*,\mathcal{T}\nu^*)(1+\eta^*(\nu,\mathcal{T}\nu))}{1+\eta^*(\nu,\nu^*)}\right|\right)\right\}\\ &= \max\left\{\phi(|\eta^*(\nu,\nu^*)|), \phi\left(\left|\frac{\eta^*(\nu^*,\nu^*)(1+\eta^*(\nu,\nu))}{1+\eta^*(\nu,\nu^*)}\right|\right)\right\}\\ &= \max\{\phi(|\eta^*(\nu,\nu^*)|), \phi(0)\}\end{aligned} \tag{3.32}$$

If $\max\{\phi(|\eta^*(\nu,\nu^*)|),\phi(0)\}=\phi(|\eta^*(\nu,\nu^*)|)$, in this case from (3.32), $\varphi(|\eta^*(\nu,\nu^*)|)\le\phi(|\eta^*(\nu,\nu^*)|)$. Since $(\varphi,\phi)\in\mathfrak{F}$ and by Remark 2.18, we deduce that $|\eta^*(\nu,\nu^*)|=0$. Similarly, if $\max\{\phi(|\eta^*(\nu,\nu^*)|),\phi(0)\}=\phi(0)$, then from (3.32), $\varphi(|\eta^*(\nu,\nu^*)|)\le\phi(0)$. We infer that $|\eta^*(\nu,\nu^*)|\le 0$ and then $|\eta^*(\nu,\nu^*)|=0$. Hence in the both possibilities, $|\eta^*(\nu,\nu^*)|=0$ and then $\eta^*(\nu,\nu^*)=0$. Thus $\eta^*(\nu,\nu^*)=\eta^*(\nu,\nu)=\eta^*(\nu^*,\nu^*)$, by using axiom (η_1^*), we have $\nu=\nu^*$ and hence ν is a unique fixed point of $\mathcal{T}$.

Case 2. ν and ν^* are not comparable.

Then there exists $\omega\in\mathfrak{D}$ which is comparable to both ν and ν^*, that is, $\omega\preceq\nu$ and $\omega\preceq\nu^*$. Since $\omega\preceq\nu$, the non-decreasing character of $\mathcal{T}$ gives us $\mathcal{T}^n\omega\preceq\mathcal{T}^n\nu=\nu, \forall\, n\in\mathbb{N}$. By using (3.1), we have

$$\begin{aligned}\varphi(|\eta^*(\mathcal{T}^n\omega,\nu)|)&=\varphi(|\eta^*(\mathcal{T}^n\omega,\mathcal{T}^n\nu)|)\\&\le\max\left\{\phi(|\eta^*(\mathcal{T}^{n-1}\omega,\mathcal{T}^{n-1}\nu)|),\phi\left(\left|\frac{\eta^*(\mathcal{T}^{n-1}\nu,\mathcal{T}^n\nu)\left(1+\eta^*(\mathcal{T}^{n-1}\omega,\mathcal{T}^n\omega)\right)}{1+\eta^*(\mathcal{T}^{n-1}\omega,\mathcal{T}^{n-1}\nu)}\right|\right)\right\}\\&=\max\left\{\phi(|\eta^*(\mathcal{T}^{n-1}\omega,\nu)|),\phi\left(\left|\frac{\eta^*(\nu,\nu)\left(1+\eta^*(\mathcal{T}^{n-1}\omega,\mathcal{T}^n\omega)\right)}{1+\eta^*(\mathcal{T}^{n-1}\omega,\nu)}\right|\right)\right\}\\&=\max\{\phi(|\eta^*(\mathcal{T}^{n-1}\omega,\nu)|),\phi(0)\}\end{aligned}\tag{3.33}$$

Suppose

$$I=\left\{n\in\mathbb{N}\,\middle|\,\varphi(|\eta^*(\nu,\mathcal{T}^n\omega)|)\le\phi(|\eta^*(\nu,\mathcal{T}^{n-1}\omega)|)\right\}$$
$$J=\left\{n\in\mathbb{N}\,\middle|\,\varphi(|\eta^*(\mathcal{T}^n\omega,\nu)|)\le\phi(0)\right\}$$

We have Card $I=\infty$ or Card $J=\infty$. If Card $I=\infty$, then there exists infinitely many $n\in\mathbb{N}$ such that $\varphi(|\eta^*(\mathcal{T}^n\omega,\nu)|)\le\phi(|\eta^*(\mathcal{T}^{n-1}\omega,\nu)|)$ and since $(\varphi,\phi)\in\mathfrak{F}$, we have $\eta^*(\mathcal{T}^n\omega,\nu)\le|\eta^*(\mathcal{T}^{n-1}\omega,\nu)|$. On letting limit as $n\to\infty$ and by (3.26), we have $\lim_{n\to\infty}|\eta^*(\mathcal{T}^n\omega,\nu)|=0$. If Card $J=\infty$, then there exists infinitely many $n\in\mathbb{N}$ such that $\varphi(|\eta^*(\mathcal{T}^n\omega,\nu)|)\le\phi(0)$. Since $(\varphi,\phi)\in\mathfrak{F}$, taking $\lim_{n\to\infty}$ and by Remark 2.18, we infer that $\lim_{n\to\infty}|\eta^*(\mathcal{T}^n\omega,\nu)|=0$. In both cases, we have $\lim_{n\to\infty}|\eta^*(\mathcal{T}^n\omega,\nu)|=0$ and then $\lim_{n\to\infty}\eta^*(\mathcal{T}^n\omega,\nu)=0$. Similarly, we can show that $\lim_{n\to\infty}\eta^*(\mathcal{T}^n\omega,\nu^*)=0$. By axiom (η_4^*) we have

$$\begin{aligned}\eta^*(\nu,\nu^*)&\le\eta^*(\nu,\mathcal{T}^n\omega)+\eta^*(\mathcal{T}^n\omega,\nu^*)-\eta^*(\mathcal{T}^n\omega,\mathcal{T}^n\omega)\\&\le\eta^*(\nu,\mathcal{T}^n\omega)+\eta^*(\mathcal{T}^n\omega,\nu^*)-\eta^*(\mathcal{T}^n\omega,\nu)-\eta^*(\nu,\mathcal{T}^n\omega)+\eta^*(\nu,\nu)\end{aligned}\tag{3.34}$$

Letting $n\to\infty$, we obtain that $\eta^*(\nu,\nu^*)\le 0$. Again by axiom (η_4^*), we have

$$0=\eta^*(\nu,\nu)\le\eta^*(\nu,\nu^*)+\eta^*(\nu^*,\nu)-\eta^*(\nu^*,\nu^*)\tag{3.35}$$

which implies that $\eta^*(\nu,\nu^*) \geq 0$. We infer that $\eta^*(\nu,\nu^*) = 0$. Therefore $\eta^*(\nu,\nu^*) = \eta^*(\nu,\nu) = \eta^*(\nu^*,\nu^*)$ and by using axiom (η_1^*), we have $\nu = \nu^*$. This completes the proof.

From Theorem 3.1 and Theorem 3.2, we obtain the following corollaries.

Corollary 3.3 Let $(\mathfrak{D},\preceq)$ is a partially ordered set and suppose that $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}:\mathfrak{D}\to\mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that there exists a pair of functions $(\varphi,\phi)\in\mathfrak{F}$ satisfying

$$\varphi(|\eta^*(\mathcal{T}\sigma,\mathcal{T}\varsigma)|) \leq \phi(|\eta^*(\sigma,\varsigma)|) \tag{3.36}$$

for all comparable elements $\sigma,\varsigma\in\mathfrak{D}$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\}\subset\mathfrak{D}$ is non-decreasing sequence such that $\sigma_n\to\nu$, then $\sigma_n\preceq\nu, \forall n\in\mathbb{N}$. If $\exists\ \sigma_0\in\mathfrak{D}$ such that $\sigma_0\preceq\mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma,\varsigma\in\mathfrak{D}$, there exists $\omega\in\mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Corollary 3.4 Let $(\mathfrak{D},\preceq)$ is a partially ordered set and suppose that $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}:\mathfrak{D}\to\mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that there exists a pair of functions $(\varphi,\phi)\in\mathfrak{F}$ satisfying

$$\varphi(|\eta^*(\mathcal{T}\sigma,\mathcal{T}\varsigma)|) \leq \phi\left(\left|\frac{\eta^*(\varsigma,\mathcal{T}\varsigma)\left(1+\eta^*(\sigma,\mathcal{T}\sigma)\right)}{1+\eta^*(\sigma,\varsigma)}\right|\right) \tag{3.37}$$

for all comparable elements $\sigma,\varsigma\in\mathfrak{D}$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\}\subset\mathfrak{D}$ is non-decreasing sequence such that $\sigma_n\to\nu$, then $\sigma_n\preceq\nu, \forall n\in\mathbb{N}$. If $\exists\ \sigma_0\in\mathfrak{D}$ such that $\sigma_0\preceq\mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma,\varsigma\in\mathfrak{D}$, there exists $\omega\in\mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Taking into account Example 2.21, we have the following corollary.

Corollary 3.5 Let $(\mathfrak{D},\preceq)$ is a partially ordered set and suppose that $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}:\mathfrak{D}\to\mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that there exists two functions $\varphi,\phi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ satisfying

$$\varphi(|\eta^*(\mathcal{T}\sigma,\mathcal{T}\varsigma)|) \leq \max\left\{\varphi\left(|\eta^*(\sigma,\varsigma)| - \phi(|\eta^*(\sigma,\varsigma)|)\right), \varphi\left(\left|\frac{\eta^*(\varsigma,\mathcal{T}\varsigma)\left(1+\eta^*(\sigma,\mathcal{T}\sigma)\right)}{1+\eta^*(\sigma,\varsigma)}\right| - \phi\left(\left|\frac{\eta^*(\varsigma,\mathcal{T}\varsigma)\left(1+\eta^*(\sigma,\mathcal{T}\sigma)\right)}{1+\eta^*(\sigma,\varsigma)}\right|\right)\right)\right\} \tag{3.38}$$

for all comparable elements $\sigma,\varsigma\in\Delta$, where φ is an increasing function and ϕ is a non-decreasing function and they satisfy $\varphi(\kappa)=\phi(\kappa)=0$ if and only if $\kappa=0$ and ' is continuous with $\phi\leq\varphi$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\}\subset\mathfrak{D}$ is non-decreasing sequence such that $\sigma_n\to\nu$, then $\sigma_n\preceq\nu, \forall n\in\mathbb{N}$. If $\exists\ \sigma_0\in\mathfrak{D}$ such that $\sigma_0\preceq\mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma,\varsigma\in\mathcal{A}$, there exists $\omega\in\mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Corollary 3.5 has the following consequences.

Corollary 3.6 Let $(\mathfrak{D}, \preceq)$ is a partially ordered set and suppose that $(\mathfrak{D}, \eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that there exists two functions $\varphi, \phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the same conditions as in Corollary 3.6 such that

$$\varphi(|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)|) \leq \varphi\big(|\eta^*(\sigma, \varsigma)| - \phi(|\eta^*(\sigma, \varsigma)|)\big) \tag{3.39}$$

for all comparable elements $\sigma, \varsigma \in \mathfrak{D}$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\} \subset \mathfrak{D}$ is non-decreasing sequence such that $\sigma_n \to \nu$, then $\sigma_n \preceq \nu, \forall n \in \mathbb{N}$. If $\exists\ \sigma_0 \in \mathfrak{D}$ such that $\sigma_0 \preceq \mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma, \varsigma \in \mathfrak{D}$, there exists $\omega \in \mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Corollary 3.7 Let $(\mathfrak{D}, \preceq)$ is a partially ordered set and suppose that $(\mathfrak{D}, \eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that there exists two functions $\varphi, \phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the same conditions as in Corollary 3.6 such that

$$\varphi(|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)|) \leq \varphi\left(\left|\frac{\eta^*(\varsigma, \mathcal{T}\varsigma)\big(1+\eta^*(\sigma, \mathcal{T}\sigma)\big)}{1+\eta^*(\sigma, \varsigma)}\right| - \phi\left(\left|\frac{\eta^*(\varsigma, \mathcal{T}\varsigma)\big(1+\eta^*(\sigma, \mathcal{T}\sigma)\big)}{1+\eta^*(\sigma, \varsigma)}\right|\right)\right) \tag{3.40}$$

for all comparable elements $\sigma, \varsigma \in \Delta$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\} \subset \mathfrak{D}$ is non-decreasing sequence such that $\sigma_n \to \nu$, then $\sigma_n \preceq \nu, \forall n \in \mathbb{N}$. If $\exists\ \sigma_0 \in \mathfrak{D}$ such that $\sigma_0 \preceq \mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma, \varsigma \in \mathfrak{D}$, there exists $\omega \in \mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Remark 3.8 The main result of [16] is Theorem 2.12. Notice that the rational contractive condition appearing in this theorem

$$|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)| \leq \alpha\left|\frac{\eta^*(\varsigma, \mathcal{T}\varsigma)\big(1+\eta^*(\sigma, \mathcal{T}\sigma)\big)}{1+\eta^*(\sigma, \varsigma)}\right| + \beta|\eta^*(\sigma, \varsigma)|$$

for any $\sigma, \varsigma \in \mathfrak{D}$, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$ implies that

$$\begin{aligned}|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)| &\leq (\alpha+\beta)\max\left\{\left|\frac{\eta^*(\varsigma, \mathcal{T}\varsigma)\big(1+\eta^*(\sigma, \mathcal{T}\sigma)\big)}{1+\eta^*(\sigma, \varsigma)}\right|, |\eta^*(\sigma, \varsigma)|\right\} \\ &\leq \max\left\{(\alpha+\beta)\left|\frac{\eta^*(\varsigma, \mathcal{T}\varsigma)\big(1+\eta^*(\sigma, \mathcal{T}\sigma)\big)}{1+\eta^*(\sigma, \varsigma)}\right|, (\alpha+\beta)|\eta^*(\sigma, \varsigma)|\right\}\end{aligned}$$

This condition is a particular case of the contractive condition appearing in Theorem 3.1 with the pair of functions $(\varphi, \phi) \in \mathfrak{F}$ given by $\varphi = 1_{\mathbb{R}_0^+}$ and $\phi = (\alpha+\beta)1_{\mathbb{R}_0^+}$. Therefore, Theorem 2.12 is a particular case of the following corollary and considered as an extension and generalizations of Theorem 2.12 in the setting of complete dualistic partial metric spaces.

Corollary 3.9 Let $(\mathfrak{D},\preceq)$ is a partially ordered set and suppose that $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}:\mathfrak{D}\to\mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that

$$|\eta^*(\mathcal{T}\sigma,\mathcal{T}\varsigma)| \leq \max\left\{(\alpha+\beta)\left|\frac{\eta^*(\varsigma,\mathcal{T}\varsigma)\big(1+\eta^*(\sigma,\mathcal{T}\sigma)\big)}{1+\eta^*(\sigma,\varsigma)}\right|, (\alpha+\beta)|\eta^*(\sigma,\varsigma)|\right\} \qquad (3.41)$$

for all comparable elements $\sigma,\varsigma\in\Delta$, where $\alpha,\beta\geq 0$ and $\alpha+\beta < 1$ Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\}\subset\mathfrak{D}$ is non-decreasing sequence such that $\sigma_n\to\nu$, then $\sigma_n\preceq\nu,\forall n\in\mathbb{N}$. If $\exists\ \sigma_0\in\mathfrak{D}$ such that $\sigma_0\preceq\mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma,\varsigma\in\mathfrak{D}$, there exists $\omega\in\mathfrak{D}$which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Observations 3.10

1. If in Corollary 3.9, we put $\alpha+\beta=c$ and $\max\left\{\left|\frac{\eta^*(\varsigma,\mathcal{T}\varsigma)\big(1+\eta^*(\sigma,\mathcal{T}\sigma)\big)}{1+\eta^*(\sigma,\varsigma)}\right|, |\eta^*(\sigma,\varsigma)|\right\} = |\eta^*(\sigma,\varsigma)|$, then we get Corollary 1 of Nazam et al. [16] and Theorem 2.3 of Oltra and Valero [22].
2. Usually the range of a dualistic partial metric η^* is $\mathbb{R}$ but if we replace $\mathbb{R}$ by $\mathbb{R}_0^+$, then η^* is identical to a partial metric η and hence Theorem 3.1 is applicable in the setting of partial metric space. The Theorems 3.1 and 3.2 also generalizes the results in [18] and [27].
3. If we set $\eta^*(\sigma,\sigma)=0$ in Theorems 3.1 and 3.2, we retrieve corresponding theorems in metric spaces (see [2], [5], [7], [8], [26]).

Taking into account Example 2.20, we have the following corollary.

Corollary 3.11 Let $(\mathfrak{D},\preceq)$ is a partially ordered set and suppose that $(\mathfrak{D},\eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}:\mathfrak{D}\to\mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that there exist $\ell\in\mathcal{S}$ (see Example 2.20) satisfying

$$\varphi(|\eta^*(\mathcal{T}\sigma,\mathcal{T}\varsigma)|) \leq$$
$$\max\left\{\ell(|\eta^*(\sigma,\varsigma)|)|\eta^*(\sigma,\varsigma)|, \ell\left(\left|\frac{\eta^*(\varsigma,\mathcal{T}\varsigma)\big(1+\eta^*(\sigma,\mathcal{T}\sigma)\big)}{1+\eta^*(\sigma,\varsigma)}\right|\right)\left|\frac{\eta^*(\varsigma,\mathcal{T}\varsigma)\big(1+\eta^*(\sigma,\mathcal{T}\sigma)\big)}{1+\eta^*(\sigma,\varsigma)}\right|\right\} \qquad (3.42)$$

for all comparable elements $\sigma,\varsigma\in\Delta$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\}\subset\mathfrak{D}$ is non-decreasing sequence such that $\sigma_n\to\nu$, then $\sigma_n\preceq\nu,\forall n\in\mathbb{N}$. If $\exists\ \sigma_0\in\mathfrak{D}$ such that $\sigma_0\preceq\mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma,\varsigma\in\mathfrak{D}$, there exists $\omega\in\mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

Following Corollary is a generalization of Theorem 2.3 of Oltra and Valero [22], Corollary 2.9 of Nazam *et al*. [15] and main result of Geraghty [8].

Corollary 3.12 Let $(\mathfrak{D}, \preceq)$ is a partially ordered set and suppose that $(\mathfrak{D}, \eta^*)$ is a complete dualistic partial metric space. Let $\mathcal{T}: \mathfrak{D} \to \mathfrak{D}$ be a non-decreasing mapping and satisfies (CCP) such that there exist $\ell \in \mathcal{S}$ (see Example 2.20) satisfying

$$\varphi(|\eta^*(\mathcal{T}\sigma, \mathcal{T}\varsigma)|) \leq \ell(|\eta^*(\sigma, \varsigma)|)|\eta^*(\sigma, \varsigma)| \tag{3.43}$$

for all comparable elements $\sigma, \varsigma \in \mathfrak{D}$. Assume that either $\mathcal{T}$ is continuous or if $\{\sigma_n\} \subset \mathfrak{D}$ is non-decreasing sequence such that $\sigma_n \to \nu$, then $\sigma_n \preceq \nu, \forall n \in \mathbb{N}$. If $\exists\, \sigma_0 \in \mathfrak{D}$ such that $\sigma_0 \preceq \mathcal{T}\sigma_0$, then $\mathcal{T}$ has a fixed point. Besides, if for each $\sigma, \varsigma \in \mathfrak{D}$, there exists $\omega \in \mathfrak{D}$ which is comparable to both σ and ς. Then $\mathcal{T}$ has a unique fixed point.

4. EXAMPLES

In this section, we give an example in support of our main result.

Example 4.1 Define η^* on $(-\infty, 0]^2$ as $\eta^*(x, y) = \max\{\sigma_1, \varsigma_1\}$, where $x = (\sigma_1, \varsigma_1)$ and $y = (\sigma_2, \varsigma_2)$. It is easy to check that $\left((-\infty, 0]^2, \eta^*\right)$ is a complete dualistic partial metric space. Define $\mathcal{T}: (-\infty, 0]^2 \to (-\infty, 0]^2$ as $\mathcal{T}x = \frac{x}{2}, \forall\, x \in (-\infty, 0]^2$. In $(-\infty, 0]^2$, we define the relation $\preceq$ in the following way: $x \preceq y$ if and only if $\sigma_1 \leq \varsigma_1$, where $x = (\sigma_1, \varsigma_1)$ and $y = (\sigma_2, \varsigma_2)$. Obviously, $\preceq$ is a partial order on $(-\infty, 0]^2$ and $\mathcal{T}$ is a non-decreasing mapping. Moreover, $\sigma_0 = (-1,0) \in (-\infty, 0]^2$ and $\sigma_0 \preceq \sigma_0$. Since

$$\max\{\sigma_1, \varsigma_1\} \leq \max\left\{\frac{\sigma_1}{2}, \frac{\varsigma_1}{2}\right\} \Longrightarrow \eta^*(x, y) \leq \eta^*(\mathcal{T}x, \mathcal{T}y), \forall\, x, y \in (-\infty, 0]^2$$

Hence $\mathcal{T}$ satisfies (CCP) with respect to $\preceq$. Define the function $\varphi, \phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as follows:

$$\varphi(\kappa) = \ln\left(\frac{5\kappa+1}{12}\right) \text{ and } \phi(\kappa) = \ln\left(\frac{3\kappa+1}{12}\right), \forall \kappa \in \mathbb{R}_0^+.$$

Clearly, $(\varphi, \phi) \in \mathfrak{F}$. We shall show that for all $x, y \in (-\infty, 0]^2$ with $x \preceq y$, (3.1) is satisfied. For this, consider $\forall\, \sigma_1 \leq \varsigma_1$,

$$\varphi(|\eta^*(\mathcal{T}x, \mathcal{T}y)|) = \ln\left(\frac{5|\eta^*(\mathcal{T}x,\mathcal{T}y)|+1}{12}\right) = \ln\left(\frac{5\left|\frac{\varsigma_1}{2}\right|+1}{12}\right) = \ln\left(\frac{5}{24}|\varsigma_1| + \frac{1}{12}\right)$$

On the other hand,

$$\phi(|\eta^*(x, y)|) = \ln\left(\frac{3|\eta^*(x,y)|+1}{12}\right) = \ln\left(\frac{3|\varsigma_1|+1}{12}\right) = \ln\left(\frac{3}{12}|\varsigma_1| + \frac{1}{12}\right)$$

$$\phi\left(\left|\frac{\eta^*(y,\mathcal{T}y)\left(1+\eta^*(x,\mathcal{T}x)\right)}{1+\eta^*(x,y)}\right|\right) = \phi\left(\left|\frac{\frac{\varsigma_1}{2}\left(1+\frac{\sigma_1}{2}\right)}{1+\varsigma_1}\right|\right) = \phi\left(\left|\frac{\varsigma_1(2+\sigma_1)}{4(1+\varsigma_1)}\right|\right)$$

$$= \ln\left(\frac{3\left|\frac{\varsigma_1(2+\sigma_1)}{4(1+\varsigma_1)}\right|+1}{12}\right) = \ln\left(\frac{3|\varsigma_1(2+\sigma_1)|+4|1+\varsigma_1|}{24}\right)$$

Combining the observations above, we get

$$\begin{aligned}\varphi(|\eta^*(\mathcal{T}x,\mathcal{T}y)|) &= \ln\left(\frac{5}{24}|\varsigma_1| + \frac{1}{12}\right) \leq \ln\left(\frac{3}{12}|\varsigma_1| + \frac{1}{12}\right)\\ &\leq \max\left\{\ln\left(\frac{3}{12}|\varsigma_1| + \frac{1}{12}\right), \ln\left(\frac{3|\varsigma_1(2+\sigma_1)|+4|1+\varsigma_1|}{24}\right)\right\}\\ &= \max\left\{\phi(|\eta^*(x,y)|), \phi\left(\left|\frac{\eta^*(y,\mathcal{T}y)(1+\eta^*(x,\mathcal{T}x))}{1+\eta^*(x,y)}\right|\right)\right\}\end{aligned}$$

Thus all the conditions of Theorem 3.1 are satisfied. Hence $\mathcal{T}$ has a fixed point, indeed $\nu = (0,0)$ is a fixed point.

AUTHORS' CONTRIBUTIONS

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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CONFLICTS OF INTEREST

The authors declare that they have no con³icts of interest.

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