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THE i-th DERIVATIVE OF THE p-ANALOGUE OF THE EXPONENTIAL INTEGRAL FUNCTION AND SOME PROPERTIES

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Abstract. In this paper, we introduce the i-th derivative of the p-analogue of the exponential integral function and further establish some analytical inequalities involving the function. We employ the Hölder and Minkowski's inequalities for integral.

Keywords: exponential integral function; p-analogue; inequality; Hölder's inequality; Minkowski's inequality.

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1. Introduction

The exponential integral function and its analogues and extensions have been an area of serious research, it has been used by many mathematicians and scientist in various aspects or areas. This function is applied in areas like, non equilibrium ground water-flow in the Theis solution (called a well function), time dependent heat transfer and evaluation of exchange integrals occurring in quantum mechanics [1] among others.

The focus of this paper is on the usual exponential integral function defined by Schloemich in

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[2] as

$$E_n(x) = \int_1^\infty t^{-n} e^{-tx} \ dt \quad x > 0, \quad n \in \mathbb{N}, \tag{1}$$

and the *i-th* derivative of (1) is given by

$$E_n^{(i)}(x) = (-1)^i \int_1^\infty t^{i-n} e^{-xt} dt, \quad i \in \mathbb{N}_0. \tag{2}$$

This special function has been investigated in diverse ways (see [3], [4], [5], [6], [7], [8], [9] and the related references therein).

The p-analogue of the exponential integral function, $E_{n,p}(x)$ is defined for $x > 0$, $p > 1$ and $n \in \mathbb{N}_0$ by [3]

$$E_{n,p}(x) = \int_1^p t^{-n} A_p^{-xt} dt, \tag{3}$$

where, $E_{n,p}(x) \longrightarrow E_n(x)$ as $p \longrightarrow \infty$ and $A_p = (1+\frac{1}{p})^p$.

The objective of this paper is to introduce the *i-th* derivative of (3) and to establish some analytical inequalities involving the function. The Hölder and the Minkowski's inequalities for integrals were used to generate the results.

2. Preliminaries

We begin with the following well known results (see for instance [10], [11], [12] or [13]).

Lemma 2.1. *(Hölder's Inequality) Let $u, v > 1$ and $\frac{1}{u} + \frac{1}{v} = 1$. If f and g are continuous real-valued functions on $[a,b]$, then the inequality*

$$\int_a^b |f(t)g(t)| dt \leq \left(\int_a^b |f(t)|^u dt \right)^{\frac{1}{u}} \left(\int_a^b |g(t)|^v dt \right)^{\frac{1}{v}}, \tag{4}$$

holds. With equality when $|g(t)| = c|f(t)|^{u-1}$. If $u = v = 2$, the inequality becomes Schwarz's inequality.

Lemma 2.2. *(Minkowski's Inequality) Let $u > 1$. If f and g are continuous real-valued functions on $[a,b]$, then the inequality*

$$\left(\int_a^b |f(x)+g(x)|^u dx \right)^{\frac{1}{u}} \leq \left(\int_a^b |f(x)|^u dx \right)^{\frac{1}{u}} + \left(\int_a^b |g(x)|^u dx \right)^{\frac{1}{u}}, \tag{5}$$

holds.

3. Main results

Proposition 3.1. *Let $x > 0$, $p > 1$, $n \in \mathbb{N}_0$, $i \in \mathbb{N}$ such that $i > n$. Then, the i-th derivative of* (3) *is given by*

$$E_{n,p}^{(i)}(x) = \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-xt} dt, \tag{6}$$

where, $E_{n,p}^{(i)}(x) \longrightarrow E_n^{(i)}(x)$ as $p \longrightarrow \infty$.

Proof. This is obtained by differentiating (3) i number of times.

Lemma 3.2. *The function $\left|E_{n,p}^{(i)}(x)\right|$ is decreasing for all $i \in \mathbb{N}$ and $x > 0$.*

Proof. Let $0 < x \leq y$. Then,

$$\begin{aligned}\left|E_{n,p}^{(i)}(x)\right| - \left|E_{n,p}^{(i)}(y)\right| &= \left|\left(\ln A_p^{-1}\right)^i\right| \int_1^p t^{i-n} \left(A_p^{-xt} - A_p^{-yt}\right) dt \\ &= \left|\ln A_p\right|^i \int_1^p t^{i-n} \left(A_p^{-xt} - A_p^{-yt}\right) dt \\ &\geq 0,\end{aligned}$$

since $A_P > 1$. This completes the proof.

Theorem 3.3. *Let $n \in \mathbb{N}_0$ and $i \in \mathbb{N}$. Then the inequality*

$$\left|E_{n,p}^{(i)}(xy)\right| \leq \left|E_{n,p}^{(i)}(\eta x)\right|^{\frac{1}{\eta}} \left|E_{n,p}^{(i)}(\mu y)\right|^{\frac{1}{\mu}}, \tag{7}$$

holds for $x > 1$, $y > 1$, $\eta > 1$, $\frac{1}{\eta} + \frac{1}{\mu} = 1$ and $x + y \leq xy$.

Proof. Using (6), the decreasing property of $\left|E_{n,p}^{(i)}(x)\right|$ and the Hölder's inequality for integrals, we obtain

$$\begin{aligned}
\left|E_{n,p}^{(i)}(xy)\right| &\leq \left|E_{n,p}^{(i)}(x+y)\right| \\
&= \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-(x+y)t} dt \\
&= \left(\ln A_p^{-1}\right)^{i\left(\frac{1}{\eta}+\frac{1}{\mu}\right)} \int_1^p t^{i\left(\frac{1}{\eta}+\frac{1}{\mu}\right)-n\left(\frac{1}{\eta}+\frac{1}{\mu}\right)} A_p^{-(x+y)t} dt \\
&= \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}} \left(\ln A_p^{-1}\right)^{\frac{i}{\mu}} \int_1^p t^{\frac{i}{\eta}-\frac{n}{\eta}} A_p^{-xt} t^{\frac{i}{\mu}-\frac{n}{\mu}} A_p^{-yt} dt \\
&\leq \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}} \left(\ln A_p^{-1}\right)^{\frac{i}{\mu}} \left(\int_1^p \left(t^{\frac{i}{\eta}-\frac{n}{\eta}} A_p^{-xt}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{\frac{i}{\mu}-\frac{n}{\mu}} A_p^{-yt}\right)^{\mu} dt\right)^{\frac{1}{\mu}} \\
&= \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-\eta xt} dt\right)^{\frac{1}{\eta}} \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-\mu yt} dt\right)^{\frac{1}{\mu}} \\
&= \left|E_{n,p}^{(i)}(\eta x)\right|^{\frac{1}{\eta}} \left|E_{n,p}^{(i)}(\mu y)\right|^{\frac{1}{\mu}},
\end{aligned}$$

which completes the proof.

Theorem 3.4. *Let $m,n \in \mathbb{N}_0$, $\alpha \in \mathbb{N}$, $p > 1$ and $i \in \mathbb{N}$. Then the inequality*

$$\left(\left|E_{m,p}^{(i)}(x)\right| + \left|E_{n,p}^{(i)}(y)\right|\right)^{\frac{1}{\alpha}} \leq \left|E_{m,p}^{(i)}(x)\right|^{\frac{1}{\alpha}} + \left|E_{n,p}^{(i)}(y)\right|^{\frac{1}{\alpha}}, \tag{8}$$

holds for $x,y > 0$.

Proof. Using (6), the Minkowski's inequality for integrals and the fact that $a^\alpha + b^\alpha \leq (a+b)^\alpha$, for $a,b \geq 0$ and $\alpha \in \mathbb{N}$, we obtain

$$\begin{aligned}
\left(\left|E_{m,p}^{(i)}(x)\right| + \left|E_{n,p}^{(i)}(y)\right|\right)^{\frac{1}{\alpha}} &= \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-m} A_p^{-xt} dt + \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-yt} dt\right)^{\frac{1}{\alpha}} \\
&= \left(\ln A_p^{-1}\right)^{\frac{i}{\alpha}} \left(\int_1^p \left[\left(t^{\frac{i-m}{\alpha}} A_p^{-\frac{xt}{\alpha}}\right)^{\alpha} + \left(t^{\frac{i-n}{\alpha}} A_p^{-\frac{yt}{\alpha}}\right)^{\alpha}\right] dt\right)^{\frac{1}{\alpha}} \\
&\leq \left(\ln A_p^{-1}\right)^{\frac{i}{\alpha}} \left(\int_1^p \left[\left(t^{\frac{i-m}{\alpha}} A_p^{-\frac{xt}{\alpha}}\right) + \left(t^{\frac{i-n}{\alpha}} A_p^{-\frac{yt}{\alpha}}\right)\right]^{\alpha} dt\right)^{\frac{1}{\alpha}}
\end{aligned}$$

$$\le \left(\ln A_p^{-1}\right)^{\frac{i}{\alpha}} \left(\left(\int_1^p \left[t^{\frac{i-m}{\alpha}} A_p^{-\frac{xt}{\alpha}} \right]^\alpha dt \right)^{\frac{1}{\alpha}} + \left(\int_1^p \left[t^{\frac{i-n}{\alpha}} A_p^{-\frac{yt}{\alpha}} \right]^\alpha dt \right)^{\frac{1}{\alpha}} \right)$$

$$= \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-m} A_p^{-xt} dt \right)^{\frac{1}{\alpha}} + \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-yt} dt \right)^{\frac{1}{\alpha}}$$

$$= \left| E_{m,p}^{(i)}(x) \right|^{\frac{1}{\alpha}} + \left| E_{n,p}^{(i)}(y) \right|^{\frac{1}{\alpha}},$$

which completes the proof.

Theorem 3.5. *Let $p > 1$, $i \in \mathbb{N}$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$. Then the inequality*

$$\left| E_{m+n,p}^{(i)}\left(\frac{x}{\eta} + \frac{y}{\mu} \right) \right| \le \left| E_{\eta m,p}^{(i)}(x) \right|^{\frac{1}{\eta}} \left| E_{\mu n,p}^{(i)}(y) \right|^{\frac{1}{\mu}}, \tag{9}$$

holds for $x, y > 0$, $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$.

Proof. Using (6) and the Hölder's inequality for integrals, we obtain

$$\left| E_{m+n,p}^{(i)}\left(\frac{x}{\eta} + \frac{y}{\mu} \right) \right| = \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-(m+n)} A_p^{-\left(\frac{x}{\eta}+\frac{y}{\mu}\right)t} dt$$

$$= \left(\ln A_p^{-1}\right)^{i\left(\frac{1}{\eta}+\frac{1}{\mu}\right)} \int_1^p t^{i\left(\frac{1}{\eta}+\frac{1}{\mu}\right)-(m+n)} A_p^{-\left(\frac{x}{\eta}+\frac{y}{\mu}\right)t} dt$$

$$= \left(\ln A_p^{-1}\right)^{i\left(\frac{1}{\eta}+\frac{1}{\mu}\right)} \int_1^p t^{\frac{i}{\eta}-m} A_p^{-\frac{xt}{\eta}} t^{\frac{i}{\mu}-n} A_p^{-\frac{yt}{\mu}} dt$$

$$= \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}} \left(\ln A_p^{-1}\right)^{\frac{i}{\mu}} \int_1^p t^{\frac{i}{\eta}-m} A_p^{-\frac{xt}{\eta}} t^{\frac{i}{\mu}-n} A_p^{-\frac{yt}{\mu}} dt$$

$$\le \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}} \left(\ln A_p^{-1}\right)^{\frac{i}{\mu}} \left(\int_1^p \left(t^{\frac{i}{\eta}-m} A_p^{-\frac{xt}{\eta}} \right)^\eta dt \right)^{\frac{i}{\eta}} \left(\int_1^p \left(t^{\frac{i}{\mu}-n} A_p^{-\frac{yt}{\mu}} \right)^\mu dt \right)^{\frac{i}{\mu}}$$

$$= \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-\eta m} A_p^{-xt} dt \right)^{\frac{i}{\eta}} \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-\mu n} A_p^{-yt} dt \right)^{\frac{1}{\mu}}$$

$$= \left| E_{\eta m,p}^{(i)}(x) \right|^{\frac{1}{\eta}} \left| E_{\mu n,p}^{(i)}(y) \right|^{\frac{1}{\eta}},$$

which completes the proof.

Corollary 3.6. *Let $m,n \in \mathbb{N}_0$, $p > 1$ and i be an even integer such that $i > m+n$. Then the inequality*

$$\left(E^{(i)}_{m+n,p}\left(\frac{x+y}{2}\right)\right)^2 \leq E^{(i)}_{2m,p}(x)E^{(i)}_{2n,p}(y), \tag{10}$$

holds for $x,y > 0$.

Proof. This follows from Theorem 3.5 by letting $\eta = \mu = 2$.

Theorem 3.7. *Let $p > 1$, $m,n \in \mathbb{N}_0$ such that $\frac{m}{\eta} + \frac{n}{\mu} \in \mathbb{N}_0$. Then the inequality*

$$\left|E^{(i)}_{\frac{m}{\eta}+\frac{n}{\mu},p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right)\right| \leq \left|E^{(i)}_{m,p}(x)\right|^{\frac{1}{\eta}} \left|E^{(i)}_{n,p}(y)\right|^{\frac{1}{\mu}}, \tag{11}$$

holds for $\eta > 1$, $x,y > 0$, $\frac{1}{\eta} + \frac{1}{\mu} = 1$.

Proof. Using (6) and Hölder's inequality for integrals, we obtain

$$\begin{aligned}
\left|E^{(i)}_{\frac{m}{\eta}+\frac{n}{\mu},p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right)\right| &= \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-\left(\frac{m}{\eta}+\frac{n}{\mu}\right)} A_p^{-(\frac{x}{\eta}+\frac{y}{\mu})t} dt \\
&= \left(\ln A_p^{-1}\right)^{i(\frac{1}{\eta}+\frac{1}{\mu})} \int_1^p t^{i(\frac{1}{\eta}+\frac{1}{\mu})-\left(\frac{m}{\eta}+\frac{n}{\mu}\right)} A_p^{-(\frac{x}{\eta}+\frac{y}{\mu})t} dt \\
&= \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}+\frac{i}{\mu}} \int_1^p t^{\frac{i}{\eta}-\frac{m}{\eta}} A_p^{-\frac{xt}{\eta}} t^{\frac{i}{\mu}-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} dt \\
&\leq \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}} \left(\ln A_p^{-1}\right)^{\frac{i}{\mu}} \left(\int_1^p \left(t^{\frac{i}{\eta}-\frac{m}{\eta}} A_p^{-\frac{xt}{\eta}}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{\frac{i}{\mu}-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}}\right)^{\mu} dt\right)^{\frac{1}{\mu}} \\
&= \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-m} A_p^{-xt} dt\right)^{\frac{1}{\eta}} \left(\left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-yt} dt\right)^{\frac{1}{\mu}} \\
&= \left|E^{(i)}_{m,p}(x)\right|^{\frac{1}{\eta}} \left|E^{(i)}_{n,p}(y)\right|^{\frac{1}{\mu}},
\end{aligned}$$

which completes the proof.

Corollary 3.8. *Let $m,n \in N_0$, $p > 1$. Then the inequality*

$$\left|E^{(i)}_{\frac{m+n}{2},p}\left(\frac{x+y}{2}\right)\right|^2 \leq E^{(i)}_{m,p}(x)E^{(i)}_{n,p}(y), \tag{12}$$

holds for $x,y > 0$.

Proof. This follows from Theorem 3.7 by letting $\eta = \mu = 2$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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