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A NOVEL TWO-STEP ITERATIVE SCHEME BASED ON COMPOSITE SIMPSON RULE FOR SOLVING NONLINEAR EQUATIONS

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Abstract: This article proposes a novel two-step iterative scheme of order three based on fundamental theorem of calculus and composite Simpson rule for solving nonlinear equations. The efficiency of method is illustrated with the help of numerical examples and a comparative study is done with some well-known existing iterative methods.

Keywords: nonlinear equations; iterative methods; composite Simpson rule; Newton's method; convergence.

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1. INTRODUCTION

A number of nonlinear equations arises in many problems of applied mathematics and engineering where determining their roots is of great importance. The nonlinear equation $f(x)=0$ can be solved by direct and indirect method. Direct methods give the exact root in a finite number of steps but are not suitable in most of the cases and hence the indirect methods, i.e. iterative methods, comes into role which gives appropriate approximate solution to a problem. Iterative methods are based on the successive approximations which starts with one or more initial approximations to the root and converges to the root after a sequence of iterations, when the desired degree of accuracy is achieved. The convergence of most iterative methods depends on an initial

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approximation and local behavior of the function $f(x)$ near a root. Well-known conventional root finding methods and their convergence analysis can be found in the literature Ref. [1-2]. In recent past more emphasis were given on developing many new iterative methods to solve nonlinear equations Ref. [3-13].

One of the classical method to find roots of nonlinear equation $f(x)=0$ is Newton's method [1].

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)} \tag{1}$$

let ξ (say) be the root of $f(x)=0$ and f is C^2 function in the neighbourhood of the ξ and $|f'(\xi)|>0$.

Another method, which is free from derivative is Steffensen's method [2].

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)} \tag{2}$$

Newton's method and Steffensen's method, both have quadratic convergence.

2. DESCRIPTION OF THE METHOD

Consider a nonlinear equation

$$f(x)=0 \tag{3}$$

If $f(x)$ is continuous at every point in $[a,b]$ and $F(x)$ is anti-derivative of $f(x)$ in $[a,b]$, then by Fundamental Theorem of Calculus, we have

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{4}$$

which on differentiating both sides, we get

$$f(x) = f(b) - f(a) \tag{5}$$

where $f(a)$ and $f(b)$ are derivatives of $F(a)$ and $F(b)$ respectively.

Recall composite Simpson rule i.e.

$$\int_a^b f(x)\,dx = \frac{(b-a)}{3n}\left[f(a) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + f(b) \right] \tag{6}$$

where $f \in C^4[a,b]$, n be an even integer, $h = \dfrac{b-a}{n}$, and $x_i = a + i\,h$ for each $i = 0,1,2,\ldots\ldots,n$.

If $n = 4$ in expression (6), we have

$$\int_a^b f(x)\,dx = \frac{(b-a)}{12}\left[f(a) + 2\,f\left(\frac{a+b}{2}\right) + 4\,f\left(\frac{3a+b}{4}\right) + 4\,f\left(\frac{a+3b}{4}\right) + f(b) \right]$$

$$\therefore\quad f(x) = \frac{b-a}{12}\left[f'(a) + 2\,f'\left(\frac{a+b}{2}\right) + 4\,f'\left(\frac{3a+b}{4}\right) + 4\,f'\left(\frac{a+3b}{4}\right) + f'(b) \right] \tag{7}$$

From (5) and (7), we have

$$f(b) - f(a) = \frac{b-a}{12}\left[f'(a) + 2\,f'\left(\frac{a+b}{2}\right) + 4\,f'\left(\frac{3a+b}{4}\right) + 4\,f'\left(\frac{a+3b}{4}\right) + f'(b) \right] \tag{8}$$

The following fixed point formula can be derived using equations (3) and (8):

$$x = a - \frac{12\,f(a)}{f'(a) + 2\,f'\left(\frac{a+b}{2}\right) + 4\,f'\left(\frac{3a+b}{4}\right) + 4\,f'\left(\frac{a+3b}{4}\right) + f'(b)} \tag{9}$$

The expression (9) enable us to suggest a new iterative approach for solving nonlinear equations. The approximate solution x_{n+1} by two step iterative scheme, for a given initial approximation close to the root of equation (3), is given by

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{10}$$

$$x_{n+1} = x_n - \frac{12\,f(x_n)}{f'(x_n) + 2\,f'\left(\frac{x_n+z_n}{2}\right) + 4\,f'\left(\frac{3x_n+z_n}{4}\right) + 4\,f'\left(\frac{x_n+3z_n}{4}\right) + f'(z_n)} \tag{11}$$

3. CONVERGENCE ANALYSIS

Theorem: Let $\xi \in I$ be a simple root of sufficiently differentiable function $f : R \to R$ in an open interval I. If an initial approximation of f is sufficiently close to ξ, then the iterative method defined by (11) is of order three and satisfies the following error equation:

$$e_{n+1} = c_2^2 e_n^3 + O(e_n^4)$$

Proof: Let ξ be a simple root of $f(x) = 0$ and $e_n = x_n - \xi$. Using Taylor series expansion around $x = \xi$, we get

$$f(x_n) = f'(\xi)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + ...\right] \tag{12}$$

$$f'(x_n) = f'(\xi)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + ...\right] \tag{13}$$

where $c_k = \dfrac{f^k(\xi)}{k!\, f'(\xi)}$, $k = 2, 3, 4, ...$

From (12) and (13), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 - (4c_2^3 - 7c_2 c_3 + 3c_4) e_n^4 + ... \tag{14}$$

But $z_n = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= \xi + c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 + (4c_2^3 - 7c_2 c_3 + 3c_4) e_n^4 + ...$$

Hence,

$$f'(z_n) = f'(\xi)\left[1 + 2c_2^2 e_n^2 - 4(c_2^3 - c_2 c_3) e_n^3 + (8c_2^4 - 14c_2^2 c_3 + 3c_2 c_4 + 3c_3 c_2^2) e_n^4 + ... + O(e_n^6)\right] \tag{15}$$

Similarly, we have

$$f'\left(\frac{x_n+z_n}{2}\right)=f'(\xi)\left[1+c_2e_n+\left(c_2^2+\frac{3c_3}{4}\right)e_n^2+\left(-2c_2^3+\frac{7c_2c_3}{2}+\frac{c_4}{2}\right)e_n^3\right.$$
$$\left.+\left(\frac{9}{2}c_2c_4+c_2^4-\frac{37}{4}c_2^2c_3+3c_3^2+\frac{5}{16}c_5\right)e_n^4+......+O(e_n^6)\right] \quad (16)$$

$$f'\left(\frac{3x_n+z_n}{4}\right)=f'(\xi)\left[1+\frac{3c_2}{2}e_n+\left(\frac{c_2^2}{2}+\frac{27c_3}{16}\right)e_n^2\right.$$
$$\left.+\left(-c_2^3+\frac{11c_2c_3}{2}+\frac{27c_4}{16}\right)e_n^3+....+O(e_n^6)\right] \quad (17)$$

$$f'\left(\frac{x_n+3z_n}{4}\right)=f'(\xi)\left[1+\frac{c_2}{2}e_n+\left(\frac{3c_2^2}{2}+\frac{3c_3}{16}\right)e_n^2\right.$$
$$\left.+\left(-3c_2^3+\frac{15c_2c_3}{2}+\frac{c_4}{16}\right)e_n^3+....+O(e_n^6)\right] \quad (18)$$

From (13), (15), (16), (17) and (18), we get

$$f'(x_n)+2f'\left(\frac{x_n+z_n}{2}\right)+4f'\left(\frac{3x_n+z_n}{4}\right)+4f'\left(\frac{x_n+3z_n}{4}\right)$$
$$+f'(z_n)=12\left[1+c_2e_n+\left(c_2^2+c_3\right)e_n^2\right. \quad (19)$$
$$\left.-\left(2c_2^3-\frac{63}{12}c_2c_3+c_4\right)e_n^4+....+O(e_n^6)\right]$$

From (11), (12) and (19), we get

$$x_{n+1}=x_n-\left[e_n+c_2^2e_n^3+\left(2c_2^3-\frac{63}{12}c_2c_3+5c_4\right)e_n^4+.........\right]$$

$$\Rightarrow e_{n+1}=c_2^2e_n^3+O(e_n^4)$$

Which shows that the proposed method is of order three. This completes the proof.

4. NUMERICAL EXAMPLES

In this section, the efficiency of the new developed method is illustrated through some numerical examples. The comparison of performance of new method with Newton's method and Steffensen's method has been shown. All computations are performed using C. Table 1 shows the comparison of number of iterations to get a appropiate approximate root.

The used stopping criteria are as under:

i. $|x_{n+1} - x_n| < \varepsilon$

ii. $|f(x_{n+1})| < \varepsilon$

where $\varepsilon = 10^{-15}$.

The following test functions, with initial approximation x_0, have been used to show the efficiency of the method.

$$f_1(x) = \cos(x) - x$$

$$f_2(x) = \sin(x) + x - 1$$

$$f_3(x) = e^x - 3x$$

$$f_4(x) = x\ \tan(x) + 1$$

$$f_5(x) = 2\sin(x) - x$$

$$f_6(x) = x + \sin(x) - x^3$$

$$f_7(x) = \cos(x) - x e^x$$

$$f_8(x) = x^2 - 9$$

$$f_9(x) = e^x - \tan^{-1} x - 1.5$$

$$f_{10}(x) = x^3 - 6x + 4$$

Table 1: Comparison b/w methods depending upon the number of iterations

$f(x)$	x_0	Number of Iterations (NT)			Root (ξ)
		NM	SM	New Method	
$f_1(x)$	-2	7	6	5	0.739085633215161
$f_2(x)$	1	4	5	3	0.510973429388569
$f_3(x)$	0	5	4	4	0.619061286735945
$f_4(x)$	2.5	3	6	3	2.798386045783889
$f_5(x)$	2.9	5	5	4	1.895494267033981
$f_6(x)$	1	6	7	4	1.317162961006034
$f_7(x)$	1	6	6	4	0.517757363682458
$f_8(x)$	2.5	4	4	3	3
$f_9(x)$	1	5	5	3	0.767653266201279
$f_{10}(x)$	1	5	5	4	0.732050807568677

5. CONCLUSION

In this article, a novel two-step iterative method, based on fundamental theorem of calculus and composite Simpson rule, has been derived for solving nonlinear equations. Through proof it has been shown that the rate of convergence of developed method is three. From Table 1, we observe that the new method is advancement of Newton's and Steffensen's method which can be used as a better substitute to other second and third order convergent methods to solve nonlinear equations.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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