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COMPRESSED INTERSECTION ANNIHILATOR GRAPH

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Abstract. Let R be a commutative ring with a non-zero identity. In this paper, we define a new graph, the compressed intersection annihilator graph, denoted by $IA(R)$, and investigate some of its properties and its relation with the structure of the ring. It is a generalization of the torsion graph $\Gamma_R(R)$ and has some advantages over the torsion graph and some other graphs. We study classes of rings for which the equivalence between the set of zero-divisors of R, $Z(R)$, being an ideal and the completeness of $IA(R)$ holds. We also study the relation between $\Gamma_R(R)$ and $IA(R)$. Besides, we show that if the compressed intersection annihilator graph of a ring R is finite, then there exists a subring S of R such that $IA(S) \cong IA(R)$. Also, we show that the compressed intersection annihilator graph will never be a complete bipartite graph. Besides, we show that the graph $IA(R)$ with at least three vertices is connected and its diameter is less than or equal to three. Finally, we determine the properties of the graph in the cases when R is the ring of integers modulo n, the direct product of integral domains, the direct product of Artinian local rings and the direct product of two rings such that one of them is not an integral domain.

Keywords: annihilator; annihilator graph; torsion graph; compressed annihilator graph.

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1 Introduction and Preliminaries

The study of zero-divisors plays an important role in ring theory, for example, to find solutions to equations. However, the set of zero-divisors lacks an algebraic structure. The set of

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zero-divisors of a ring R, denoted by $Z(R)$, is always closed under multiplication but it is not always closed under addition. Besides, we assume throughout this paper that R is a commutative ring with non-zero identity.

The interchange between ring theory and graph theory began from the work of I. Beck in [11] (1988). He defined the zero-divisor graph $\Gamma(R)$ as the undirected simple graph with vertices represented by all elements of R and two distinct vertices are adjacent if their product is zero. In 1999, D. F. Anderson and P. S. Livingston in [5] modified Beck's graph by considering the set of vertices to be only the set of non-zero zero-divisors of the ring R and the adjacency kept as before. They were interested in determining some important properties of the graph and their relation to R. Inspired by ideas of S. B. Mulay in [18], S. Spiroff and C. Wickham in [22] (2011), modified again the set of vertices by introducing the concept of the compressed zero-divisor graph denoted by $\Gamma_E(R)$. The set of vertices was constructed from equivalence classes of zero-divisors determined by the following equivalence relation $\sim$ on R: $x \sim y$ if and only if $\operatorname{ann}_R(x) = \operatorname{ann}_R(y)$, for any $x, y \in R$, where $\operatorname{ann}_R(x) = \{r \in R \mid rx = 0\}$. E. Lewis in [15] showed that $\sim$ is a multiplicative congruence relation on R. By the definition of the relation, we get $[0]_\sim = \{x \in R \mid \operatorname{ann}_R(x) = R\} = \{0\}$, $[1]_\sim = \{x \in R \mid \operatorname{ann}_R(x) = \{0\}\} = R \setminus Z(R)$ and thus we have $[x]_\sim \subseteq Z^*(R) = Z(R) \setminus \{0\}$ for each $x \in Z^*(R)$. She also showed that the set $R/\sim = \{[x]_\sim \mid x \in R\}$ of all congruence classes with the well-defined multiplication, given by $[x]_\sim [y]_\sim = [xy]_\sim$ for all $x, y \in R$, is a commutative monoid with identity $[1]_\sim$ and zero $[0]_\sim$. S. Spiroff and C. Wickham defined $\Gamma_E(R)$ as the undirected simple graph with the set of vertices $Z(R/\sim)^* = \{[x]_\sim \mid x \in Z^*(R)\}$ and two distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if $xy = 0$. There are many different ways of generalizations of the zero-divisor graph, see for example [2, 12, 21].

This is not the only way to associate a graph to a ring and vise versa. One of those important ways introduced by D. F. Anderson and A. Badawi in [3] (2008), is the total graph of a commutative ring denoted by $T(\Gamma(R))$. It is defined as the undirected simple graph with R as the set of vertices and two distinct vertices are adjacent if and only if their sum is a zero-divisor. They characterized the properties of the graph when $Z(R)$ is an ideal and when it is not an ideal. In 2013, D. F. Anderson and A. Badawi in [4] generalized the total graph over a commutative ring R with respect to a multiplicative prime subset H of R and there are other ways of generalizations of the total graph. For surveys on this topic see also [9, 19].

Another way to associate a graph with a ring was given by A. Badawi in [8] (2014), where he introduced the annihilator graph denoted by $AG(R)$. The set of vertices is the same as the set of vertices of zero-divisor graph $Z^*(R)$, but two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$. It is obvious that $\mathrm{ann}_R(x) \cup \mathrm{ann}_R(y) \subset \mathrm{ann}_R(xy)$ but equality does not hold in general. In 2018, Sh. Payrovi and S. Babaei in [20] generalized $AG(R)$ to be the compressed annihilator graph $AG_E(R)$. The set of vertices is the set of equivalent classes of zero-divisors of R, $Z(R/\sim)^*$, and two distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if $\mathrm{ann}_R(x) \cup \mathrm{ann}_R(y) \subsetneq \mathrm{ann}_R(xy)$. For a survey related to this topic see [10].

There are also generalizations of graphs over modules. One of those graphs associated with a module, the torsion graph, denoted by $\Gamma_R(M)$, was introduced by P. Malakooti Rad, S. Yassemi, Sh. Ghalandarzadeh and P. Safari in [17], with an R-module M. The set of vertices of $\Gamma_R(M)$ is the set of non-zero torsion elements $T(M)^*$, where $T(M)^* = \{m \in M \mid \mathrm{ann}_R(m) \neq \{0\}\}$. Two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) \neq \{0\}$. In their work, they studied in which case $\Gamma_R(M)$ is connected with $\mathrm{diam}(\Gamma_R(M))$ less than or equal to three, the relationship between the diameter of $\Gamma_R(M)$ and $\Gamma_R(R)$, and proved that the girth of $\Gamma_R(M)$ belongs to $\{3, \infty\}$. Note that, $\Gamma_R(R)$ is a special case of the torsion graph $\Gamma_R(M)$ when we consider R as an R-module. For other graphs over modules see also, [1, 12, 13, 16, 23]

In this work, we define a new graph, which we call the compressed intersection annihilator graph $IA(R)$, as the undirected graph whose set of vertices is $Z(R/\sim)^* = Z(R/\sim) \setminus \{[0]_\sim\}$ and two distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) \neq \{0\}$ for some representatives x and y. This graph is a generalization of the graph $\Gamma_R(R)$. Note that if $r \in \mathrm{ann}_R(x) \cap \mathrm{ann}_R(y)$ for some x and y, then $r \in \mathrm{ann}_R(x+y)$. Thus if $[x]_\sim$ is adjacent to $[y]_\sim$

A non-empty proper subset H of R is said to be a *multiplicative-prime subset* of R if it satisfies the following two conditions:

(1) $ab \in H$ for every $a \in H$ and $b \in R$,

(2) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$.

As an example of a multiplicative-prime subset, we take $H = Z(R)$.

in $IA(R)$, then x is adjacent to y in $T(\Gamma(R))$ for all representatives x and y. Therefore we may consider $IA(R)$ as a way to compress the total graph $T(\Gamma(R))$. The compressed intersection annihilator graph has some advantages over the torsion graph and some other graphs. In many cases, the compressed intersection annihilator graph is finite when the torsion graph is infinite. To find finite non-empty torsion graphs, we have to restrict ourselves to the class of finite rings; but there is a wide class of rings for which a finite compressed intersection annihilator graph can be found when other approaches might fail. It is known that different elements from the ring may give the same annihilator ideal. The equivalence class of all vertices of the torsion graph that give the same non-zero annihilator ideal represents one vertex in $IA(R)$. Then the compressed intersection annihilator graph helps us to illuminate the structure of annihilator ideals and the relation between them. The adjacency relation is always symmetric and reflexive but is not usually transitive. So the compressed intersection annihilator graph measures this lack of transitivity. Indeed, the adjacency relation is transitive if and only if the graph $IA(R)$ is complete. This study illustrates a more brief description of the annihilator ideals. Another essential point of this graph is its connection to the associated primes of the ring. For instance, in Noetherian rings, $Z(R)$ is the union of the associated primes. Specifically, we have a natural injective map from the set of associated primes to the set of vertices of the compressed intersection annihilator graph.

The second section is divided into four subsections. In the first subsection, we study classes of rings for which the equivalence between $Z(R)$ being an ideal and the completeness of $IA(R)$ holds. Besides, we show that if the compressed intersection annihilator graph of a ring R is finite, then there exists a subring S of R such that $IA(S) \cong IA(R)$. In the second subsection, we generalize some results from [17]. Besides, we show that the graph $IA(R)$ with at least three vertices is connected and its diameter is less than or equal to three. Also, we show that the compressed intersection annihilator graph will never be a complete bipartite graph. Additionally, we study the relation between $\Gamma_R(R)$ and $IA(R)$. In the third subsection, we investigate the properties of the graph when $R = \mathbb{Z}_n$ and show that if n is divisible by at least three primes, then the graph $IA(\mathbb{Z}_n)$ is connected and determine its diameter and girth. In the last subsection, we study the graph when R is the finite direct product of integral domains with non-zero identities.

This case shows an example of a finite graph of an infinite ring and is an example of isomorphic graphs of non-isomorphic rings. Also, we show that the graph is connected with a diameter equal to two and girth equal to three when R is the direct product of Artinian local rings with non-zero identities. Finally, we show that the graph $IA(R)$ is connected and not complete with a diameter less than or equal to three when R is the direct product of two rings such that one of them is not an integral domain.

Let G be a simple undirected graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and set of edges of G respectively. We also use $x-y$ to denote the adjacency between two vertices x and y. Two vertices in G are said to be *connected* if there is a path between them. If every two vertices in G are connected, then G is said to be *connected*. A *complete bipartite graph* is a graph whose set of vertices can be divided into two disjoint sets, say V_1 and V_2, in which every vertex in V_1 is adjacent to every vertex in V_2 and no vertices within V_1 or V_2 are adjacent. Such a graph is denoted by $K^{m,n}$, where $|V_1| = m$ and $|V_2| = n$. A *complete graph* is a graph such that every two different vertices are adjacent. It is denoted by K^n, where n is the number of vertices. G is said to be *totally disconnected* if there are no adjacent vertices. The *distance*, $d(x,y)$, between two vertices x and y in G is the length of the shortest path from x to y, if there is a path, $d(x,x) = 0$ and $d(x,y) = \infty$, if x and y are not connected. The *diameter* of G is defined by $\mathrm{diam}(G) = \sup\{d(x,y) \mid x,y \in V(G)\}$. A *cycle* is a closed path consisting of more than or equal to three vertices that starts and ends at the same vertex. The length of the shortest cycle in G is said to be the *girth* of G, denoted by $\mathrm{gr}(G)$ ($\mathrm{gr}(G) = \infty$ if G has no cycles).

An element $x \in R$ is said to be a *nilpotent element* if there is an integer $n \geq 2$ such that $x^n = 0$ and $x^{n-1} \neq 0$. Clearly, any nilpotent element is a zero divisor. An element $a \in R$ is an *idempotent element* if $a^2 = a$. R is called a *local ring* if it has a unique maximal ideal. R is called a *von Neumann regular ring* if for every element $x \in R$ there exists $a \in R$ such that $x = x^2a$. R is said to be a *Noetherian ring* if it satisfies the ascending chain condition on ideals. This means that there is no infinite ascending sequence of ideals. R is said to be an *Artinian ring* if it satisfies the descending chain condition on ideals. This means that there is no infinite descending sequence of ideals. Note that any Artinian ring is a Noetherian ring, but the converse is not generally true.

R is said to have a *finite Goldie dimension* if it does not contain infinite direct sums of non-zero ideals.

2 Main Results

2.1 Compressed intersection annihilator graph

Let $\sim$ be the multiplicative congruence relation defined on R by $x \sim y$ if and only if $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$. Let $[x_1]_\sim = [x_2]_\sim$ and $[y_1]_\sim = [y_2]_\sim$, which means that $\mathrm{ann}_R(x_1) = \mathrm{ann}_R(x_2)$ and $\mathrm{ann}_R(y_1) = \mathrm{ann}_R(y_2)$. If $r \in \mathrm{ann}_R(x_1) \cap \mathrm{ann}_R(y_1)$, then $r \in \mathrm{ann}_R(x_2) \cap \mathrm{ann}_R(y_2)$. This shows that the adjacency is well defined in the following

Definition 2.1. The compressed intersection annihilator graph, denoted by $IA(R)$, is a simple undirected graph, where the set of vertices is $Z(R/\sim)^* = Z(R/\sim) \setminus \{[0]_\sim\}$ and the adjacency between any two different vertices $[x]_\sim$ and $[y]_\sim$, $[x]_\sim - [y]_\sim$, if and only if $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) \neq \{0\}$.

Note that if $r \in \mathrm{ann}_R(x) \cap \mathrm{ann}_R(y)$ for some x and y, then $r \in \mathrm{ann}_R(x+y)$. The converse is not always true. Consider the ring $R = \mathbb{Z}_6$. We have that $4+5 = 3 \in Z^*(R)$ and $2 \in \mathrm{ann}_R(4+5)$. But $2 \notin \mathrm{ann}_R(4)$ and $2 \notin \mathrm{ann}_R(5)$. It follows that, if $[x]_\sim$ is adjacent to $[y]_\sim$ in $IA(R)$, then x is adjacent to y in $T(\Gamma(R))$ for all representatives x and y. We can find injective maps from $V(IA(R))$ to $V(T(\Gamma(R)))$ and from $E(IA(R))$ to $E(T(\Gamma(R)))$. For example, let $f : V(IA(R)) \to V(T(\Gamma(R)))$ defined by $[x]_\sim \mapsto x$ for some representative x and define the map $g_f : E(IA(R)) \to E(T(\Gamma(R)))$ by $\{[x]_\sim, [y]_\sim\} \mapsto \{f(x), f(y)\}$.

Example 1. *Figure 1 represents the graph $IA(R)$, where $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Figure 2 shows the graph $IA(R)$, for $R = \mathbb{Z}_{12}$. In the latter case, we can easily check that* $\mathrm{ann}_R(2) = \mathrm{ann}_R(10)$, $\mathrm{ann}_R(3) = \mathrm{ann}_R(9)$ *and* $\mathrm{ann}_R(4) = \mathrm{ann}_R(8)$. *Thus the set of vertices is* $Z(R/\sim)^* = \{[2]_\sim, [3]_\sim, [4]_\sim, [6]_\sim\}$.

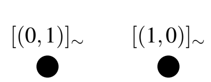

FIGURE 1. $IA(R)$

$R = \mathbb{Z}_3 \times \mathbb{Z}_3$

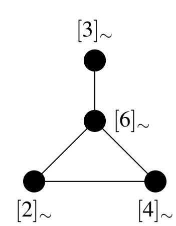

FIGURE 2. $IA(R)$, $R = \mathbb{Z}_{12}$

We notice from the definition of the graph $IA(R)$ that it is an empty graph if and only if $Z(R) = \{0\}$. Clearly, for the extreme case, if $\mathrm{ann}_R(R) \neq \{0\}$, then $IA(R)$ is complete. But the converse is not always true.

We study an essential property to a ring R when the set of its zero divisors is an ideal of R. This gives us a lot of information about the total graph $T(\Gamma(R))$ since D. F. Anderson and A. Badawi in [3], breaks its study into two cases depending on whether $Z(R)$ is an ideal or not.

In the next theorem, we show that if the graph $IA(R)$ is complete, then $Z(R)$ is an ideal. In theorem 2.2 and theorem 2.4, we show that the converse is true when any one of the following two conditions holds:

(1) The set of zero-divisors $Z(R)$ has a non-zero annihilator.

(2) R has a finite Goldie dimension.

Theorem 2.1. *If the graph $IA(R)$ is complete, then $Z(R)$ is an ideal.*

Proof. Assume that $IA(R)$ is a complete graph. Let $x, y \in Z(R)$. We have two cases, the first one when $[x]_\sim$ and $[y]_\sim$ are two elements in $Z(R/\sim)^*$, then there is $r \neq 0$ such that $r \in \mathrm{ann}_R(x) \cap \mathrm{ann}_R(y)$. So, $r(x+y) = 0$. Therefore $x+y \in Z(R)$. In the second case, we have either $[x]_\sim$ or $[y]_\sim$ is zero, and thus $x+y \in Z(R)$. Therefore $Z(R)$ is an ideal. □

The last theorem implies, from [3], that if the graph $IA(R)$ is complete, then $T(\Gamma(R))$ is a disconnected graph that breaks into two components. Those components are the complete induced subgraph $Z(\Gamma(R))$ of $T(\Gamma(R))$ with vertices $Z(R)$ and the induced subgraph $Reg(\Gamma(R))$ of $T(\Gamma(R))$ with vertices $Reg(R) = R \setminus Z(R)$, the set of regular elements of R. Also, they characterize the graph $Reg(\Gamma(R))$ depending on whether 2 is an element in $Z(R)$ or not.

Theorem 2.2. *If $Z(R)$ has a non-zero annihilator, then $IA(R)$ is a complete graph.*

Proof. Let $a \in \mathrm{ann}_R(Z(R))$ for some $a \neq 0$. Let $x, y \in Z^*(R)$ be such that $[x]_\sim \neq [y]_\sim$. Then $xa = 0$ and $ya = 0$. Thus $[x]_\sim$ is adjacent to $[y]_\sim$ which implies the completeness of $IA(R)$. □

Recall from [7], that $Z(R)$ is an ideal if and only if $Z(R)$ is a prime ideal. Specifically, when R is an Artinian ring, we have the following results:

- Every prime ideal is a maximal ideal which is a minimal prime ideal. Besides, each minimal prime ideal in a Noetherian ring has a non-zero annihilator. Therefore, each prime ideal of R has a non-zero annihilator.
- Each non-unit of R is nilpotent if and only if R is local.

Moreover, as we know for the Artinian local ring R, $Z(R)$ is the maximal ideal, and it is an annihilator ideal. Therefore, from the previous theorem, $IA(R)$ is complete.

Theorem 2.4 is a direct result from theorem 2.3, which was proved by M. Filipowicz and M. Kępczyk in [14].

Theorem 2.3 (Theorem 3.4, [14])**.** *If a proper ring R has a finite Goldie dimension, then every finitely generated ideal of R, consisting of zero-divisors, has a non-zero annihilator.*

One interpretation of theorem 2.3 is the following statement:

> "If a proper ring R has a finite Goldie dimension and $Z(R)$ is an ideal of R, then every finite set of zero divisors of R has a non-zero annihilator".

The following theorem is a consequence of theorems 2.3 and 2.1 and the definition of the graph.

Theorem 2.4. *Let R be a ring with finite Goldie dimension. $Z(R)$ is an ideal if and only if $IA(R)$ is a complete graph.*

It is clear that theorem 2.4 holds for Noetherian rings as they have a finite Goldie dimension.

The following theorem shows that we may consider only Noetherian rings to find all finite compressed intersection annihilator graphs. The proof is analogous to the proof of theorem 3.3 in [6].

Theorem 2.5. *If $IA(R)$ is a finite graph, then there exists a Noetherian subring S of R such that $IA(S) \cong IA(R)$.*

Proof. Assume that $IA(R)$ is finite. Let $|V(IA(R))| = n < \infty$ and $V(IA(R)) = Z(R/\sim)^* = \{[x_i]_\sim | x_i \in Z^*(R), 1 \le i \le n\}$. Then $\mathrm{ann}_R(x_i) \neq \mathrm{ann}_R(x_j)$ for all $i \neq j$, $1 \le i, j \le n$. Hence for every $i \neq j$, $1 \le i, j \le n$, there exist $y_{ij} \in \mathrm{ann}_R(x_i)$ and $y_{ij} \notin \mathrm{ann}_R(x_j)$ or $y_{ij} \notin \mathrm{ann}_R(x_i)$ and $y_{ij} \in \mathrm{ann}_R(x_j)$. However, $\mathrm{ann}_R(y_{ij}) = \mathrm{ann}_R(x_k)$ for some k. So we can choose $y_{ij} \in \{x_l | 1 \le l \le n\}$.

Let S be the subring generated by $\{x_l | 1 \leq l \leq n\} \cup \{1\}$. Then S is Noetherian. Clearly, $[x_i]_\sim = [x_j]_\sim$ if and only if $i = j$. Then $V(IA(S)) = Z(S/\sim)^* = \{[x_i]_\sim | 1 \leq i \leq n\} = V(IA(R))$. Also, for all i and j, $\text{ann}_S(x_i) \cap \text{ann}_S(x_j) \neq \{0\}$ if and only if $\text{ann}_R(x_i) \cap \text{ann}_R(x_j) \neq \{0\}$. Therefore, $IA(S) \cong IA(R)$. □

The next theorem shows that the compressed intersection annihilator graph cannot be a complete bipartite graph for any ring.

Theorem 2.6. *The compressed intersection annihilator graph cannot be a complete bipartite graph for any ring, i.e.* $IA(R) \neq K^{m,n}$ *for all* $m,n > 1$.

Proof. For the sake of contradiction, suppose that there is R such that $IA(R) = K^{m,n}$ for some $m,n > 1$. Let $A = \{[a_i]_\sim | 1 \leq i \leq m\}$ and $B = \{[b_i]_\sim | 1 \leq i \leq n\}$ such that $Z(R/\sim)^* = A \cup B$ and $A \cap B = \emptyset$. Since that every element in A is adjacent to every element in B, then $\text{ann}_R(a_1) \cap \text{ann}_R(b_1) \neq \{0\}$. Hence $a_1 + b_1 \in Z(R)$. If $a_1 + b_1 = 0$, then $a_1 = -b_1$ which implies that $\text{ann}_R(a_1) = \text{ann}_R(b_1)$ and this is a contradiction. It follows that $[a_1 + b_1]_\sim \in Z(R/\sim)^*$. Then $[a_1 + b_1]_\sim \in A$ or $[a_1 + b_1]_\sim \in B$. Assume that $[a_1 + b_1]_\sim \in A$. Then $[a_1 + b_1]_\sim = [a_i]_\sim$ for some i. Since $IA(R)$ is a complete bipartite graph, then there is $r \neq 0$ such that $r \in \text{ann}_R(b_1) \cap \text{ann}_R(a_1)$. Thus $r \in \text{ann}_R(a_i)$. Therefore $i = 1$ which means that $[a_1 + b_1]_\sim = [a_1]_\sim$. Since there exists $s \neq 0$ such that $s \in \text{ann}_R(b_2) \cap \text{ann}_R(a_1)$, then $sb_2 = 0$ and $0 = sa_1 = s(a_1 + b_1) = sa_1 + sb_1 = sb_1$. Therefore $[b_1]_\sim$ is adjacent to $[b_2]_\sim$ and this is a contradiction. Similarly, we can treat the case when $[a_1 + b_1]_\sim \in B$. So $IA(R) \neq K^{m,n}$ for all $m,n > 1$. □

2.2 Diameter and girth

In this subsection, we determine the diameter and girth of the compressed intersection annihilator graph. If we consider R as an R-module in the graph $\Gamma_R(R)$, then the following theorems 2.7, 2.8, and corollary 2.9 are generalizations of parts 2, 3 of theorems 3.1, 4.1, and corollary 4.2 from [17] respectively, with analogous proofs. We have to replace $T(M)^*$ by $Z(R/\sim)^*$, $Ann(x)$ by $\text{ann}_R(x)$, and the vertices in $\Gamma_R(M)$ by the vertices in $IA(R)$. Theorem 2.7 shows that if either the ring R is a von Neumann regular ring and $R \ncong \text{ann}_R(x) \oplus \text{ann}_R(y)$ for any two distinct $x, y \in Z^*(R)$ or $\text{Nil}(R) \neq \{0\}$, then $IA(R)$ is connected with a diameter less than or equal to three. The proof of the theorem depends on an equivalent definition of the von Neumann regular ring, namely every principal ideal is generated by an idempotent element.

Theorem 2.7. *$IA(R)$ is connected with* $\operatorname{diam}(IA(R)) \leq 3$ *if any one of the following conditions holds:*

(1) R is a von Neumann regular ring and $R \ncong \operatorname{ann}_R(x) \oplus \operatorname{ann}_R(y)$ for any two distinct $x, y \in Z^*(R)$.

(2) $\operatorname{Nil}(R) \neq \{0\}$ *(i.e. R is not reduced).*

The following theorem shows that the girth of the graph $IA(R)$ belongs to $\{3, \infty\}$. In corollary 2.9, we show that if $IA(R)$ is a connected graph with more than or equal to three vertices, then $IA(R)$ contains a cycle.

Theorem 2.8. *If $IA(R)$ contains a cycle, then* $\operatorname{gr}(IA(R)) = 3$.

Corollary 2.9. *If $IA(R)$ is a connected graph with $\mid Z(R/\sim)^* \mid > 2$, then $IA(R)$ contains a cycle and* $\operatorname{gr}(IA(R)) = 3$.

The following theorem shows that the graph $IA(R)$ with at least three vertices is connected, and its diameter is less than or equal to three. It follows from the previous corollary that its girth is equal to three.

Theorem 2.10. *Let $\mid Z(R/\sim)^* \mid > 2$. Then $IA(R)$ is connected and* $\operatorname{diam}(IA(R)) \leq 3$.

Proof. Let $[x]_\sim$ and $[y]_\sim$ be two distinct non-adjacent vertices in $Z(R/\sim)^*$. Let $[z]_\sim \in Z(R/\sim)^*$ such that $[z]_\sim \neq [x]_\sim$ and $[z]_\sim \neq [y]_\sim$. If $[z]_\sim$ is adjacent to both $[x]_\sim$ and $[y]_\sim$, then $d(x,y) = 2$. If $[z]_\sim$ is not adjacent to $[y]_\sim$ and it is adjacent to $[x]_\sim$ (or the other way round), then there is $t \in Z^*(R)$ such that $tz = 0$, $tx = 0$ and $ty \neq 0$ and we have two cases. First case if there is $r \in \operatorname{ann}_R(x)$ and $r \notin \operatorname{ann}_R(z)$ (since $[x]_\sim \neq [z]_\sim$) and since $ry \neq 0$, hence $[x]_\sim - [rz]_\sim - [ry]_\sim - [y]_\sim$ is a path between $[x]_\sim$ and $[y]_\sim$. Then $d(x,y) \leq 3$. Similarly, we can treat the second case when there is $r \in \operatorname{ann}_R(z)$ and $r \notin \operatorname{ann}_R(x)$. If $[z]_\sim$ is neither adjacent to $[x]_\sim$ nor $[y]_\sim$, then there is $t \in Z^*(R)$ such that $tz = 0$, $tx \neq 0$ and $ty \neq 0$. So $[x]_\sim - [tx]_\sim - [ty]_\sim - [y]_\sim$ is a path between $[x]_\sim$ and $[y]_\sim$. Thus $d(x,y) \leq 3$. Therefore, $IA(R)$ is connected and $\operatorname{diam}(IA(R)) \leq 3$. □

Remark 1.

- *If* $|Z(R/\sim)^*| = 2$, *then we have two cases. Firstly, if* $Z(R)$ *is not an ideal, then from theorem 2.1,* $IA(R)$ *is not a complete graph which means that* $IA(R)$ *is totally disconnected graph. Secondly, suppose that* $Z(R)$ *is an ideal and* $Z(R/\sim)^* = \{[a]_\sim, [b]_\sim\}$. *Then* $a+b \in Z(R)$. *If* $[a+b]_\sim = [0]_\sim$, *then* $a+b=0$. *Thus* $\mathrm{ann}_R(a) = \mathrm{ann}_R(b)$ *and this is a contradiction. Therefore,* $[a+b]_\sim = [a]_\sim$ *or* $[a+b]_\sim = [b]_\sim$. *If* $[a+b]_\sim = [a]_\sim$, *then for all* $r \in \mathrm{ann}_R(a) \setminus \{0\}$, $0 = ra = r(a+b) = ra + rb = rb$. *Thus* $r \in \mathrm{ann}_R(b)$. *In the same way, the case* $[a+b]_\sim = [b]_\sim$ *can be treated. Hence* $IA(R)$ *is complete.*
- *If* $|Z(R/\sim)^*| = 3$, *then the graph* $IA(R)$ *is connected. Therefore from corollary 2.9,* $\mathrm{gr}(IA(R)) = 3$. *It follows that* $IA(R)$ *is complete and* $\mathrm{diam}(IA(R)) = 1$.

The following theorem shows that $\Gamma_R(R)$ is complete if and only if $IA(R)$ is complete. Theorem 2.12 shows that $\Gamma_R(R)$ is connected if and only if $IA(R)$ is connected. Moreover, they have the same diameters.

Theorem 2.11. $\Gamma_R(R)$ *is complete if and only if* $IA(R)$ *is complete.*

Proof. Assume that $\Gamma_R(R)$ is complete. Let $[x]_\sim, [y]_\sim \in Z(R/\sim)^*$ be two distinct vertices. Then $x, y \in Z^*(R) = T(R)^*$ for any representative x, y. Hence there is $r \neq 0$ such that $r \in \mathrm{ann}_R(x) \cap \mathrm{ann}_R(y)$. Therefore $IA(R)$ is complete.

For the converse, let $x, y \in T(R)^* = Z^*(R)$. If $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, then $x - y$. If $\mathrm{ann}_R(x) \neq \mathrm{ann}_R(y)$, then $[x]_\sim$ and $[y]_\sim$ are two distinct vertices in $IA(R)$. Since $IA(R)$ is complete, then there is $r \in Z^*(R)$ such that $r \in \mathrm{ann}_R(x) \cap \mathrm{ann}_R(y)$. So $x - y$. Therefore $\Gamma_R(R)$ is complete. $\square$

Theorem 2.12. $\Gamma_R(R)$ *is connected if and only if* $IA(R)$ *is connected. Moreover, if* $|Z(R/\sim)^*| > 1$, *then* $\mathrm{diam}(\Gamma_R(R)) = \mathrm{diam}(IA(R))$.

Proof. Assume that $\Gamma_R(R)$ is connected. Let $[x]_\sim, [y]_\sim \in Z(R/\sim)^*$ be two distinct vertices in $IA(R)$. Then there is a path between x and y. Let $r_0 = x - r_1 - \dots. - r_{n-1} - y = r_n$ be the shortest path of length n between x and y, i.e. $d(x,y) = n$. If $\mathrm{ann}_R(r_i) = \mathrm{ann}_R(r_{i+1})$ for some $i \geq 1$, and since $\mathrm{ann}_R(r_{i-1}) \cap \mathrm{ann}_R(r_i) \neq \{0\}$, then $r_{i-1} - r_{i+1}$. We can collapse the path into $r_0 = x - r_1 - \dots - r_{i-1} - r_{i+1} - \dots - r_{n-1} - y = r_n$, i.e. $d(x,y) < n$, which is a contradiction. Therefore

$\mathrm{ann}_R(r_i) \neq \mathrm{ann}_R(r_{i+1})$ for all $0 \leq i \leq n-1$. Thus $[r_i]_\sim \neq [r_{i+1}]_\sim$ and $\mathrm{ann}_R(r_i) \cap \mathrm{ann}_R(r_{i+1}) \neq \{0\}$ for all $0 \leq i \leq n-1$. It follows that $[r_0]_\sim = [x]_\sim - [r_1]_\sim - .... - [r_{n-1}]_\sim - [y]_\sim = [r_n]_\sim$ is a path in $IA(R)$ between $[x]_\sim$ and $[y]_\sim$. Hence $IA(R)$ is connected and $d([x]_\sim, [y]_\sim) \leq n = d(x,y)$. So $\mathrm{diam}(IA(R)) \leq \mathrm{diam}(\Gamma_R(R))$.

For the converse, assume that $IA(R)$ is connected. Let $x, y \in Z^*(R)$. If $[x]_\sim = [y]_\sim$ i.e. $d([x]_\sim, [y]_\sim) = 0$, then $x - y$ is in the graph $\Gamma_R(R)$ which means that $d(x,y) = 1$. If $[x]_\sim \neq [y]_\sim$. Let $[s_0]_\sim = [x]_\sim - [s_1]_\sim - .... - [s_{m-1}]_\sim - [y]_\sim = [s_m]_\sim$ be the shortest path between $[x]_\sim$ and $[y]_\sim$ of length $m \geq 1$, i.e. $d([x]_\sim, [y]_\sim) = m$. It follows that $\mathrm{ann}_R(s_i) \neq \mathrm{ann}_R(s_{i+1})$ and $\mathrm{ann}_R(s_i) \cap \mathrm{ann}_R(s_{i+1}) \neq \{0\}$ for all $0 \leq i \leq m-1$. Therefore $s_0 = x - s_1 - ... - s_{m-1} - y = s_m$ is a path in $\Gamma_R(R)$ between x and y. Then $\Gamma_R(R)$ is connected and $d(x,y) \leq m = d([x]_\sim, [y]_\sim)$. It follows that $\mathrm{diam}(\Gamma_R(R)) \leq \mathrm{diam}(IA(R))$.

From the two parts of the proof, it follows that if $|Z(R/\sim)^*| > 1$, then $\mathrm{diam}(IA(R)) = \mathrm{diam}(\Gamma_R(R))$. □

2.3 Compressed intersection annihilator graph of $\mathbb{Z}_n$

Throughout this subsection, we assume that $R = \mathbb{Z}_n$ for some integer $n > 1$. The following two lemmas identify which elements in R are zero divisors, and when they have the same annihilators.

Lemma 2.13. *Let k and n be integers with $1 < k < n$. Then $k \in Z^*(R)$ if and only if $g.c.d(k,n) \neq 1$.*

Proof. Let $k \in Z^*(R)$. Then there is $l \in Z^*(R)$, $l < n$ such that $kl = 0\,(\mathrm{mod}\, n)$. Hence for some positive integer m, $kl = mn$. If $g.c.d(k,n) = 1$, so n/l. This is a contradiction with $l < n$.
For the converse, assume that $g.c.d(k,n) = r$, $r \neq 1$. Then there are integers $l \neq 0$, $m \neq 0$ such that $n = rl$ and $k = rm$. Hence $kl = rml = nm$, i.e. $kl = 0\,(\mathrm{mod}\, n)$. Therefore $k \in Z^*(R)$. □

Lemma 2.14. *Let $k \in R$ and $k \neq 0$. If $g.c.d(k,n) = l$ with $l \neq 1$, then* $\mathrm{ann}_R(k) = \mathrm{ann}_R(l)$.

Proof. Let $g.c.d(k,n) = l$, $l \neq 1$. Then there exist non-zero $r, s \in \mathbb{N}$ such that $k = rl$, $n = sl$ with $g.c.d(r,s) = 1$. It is clear that, $\mathrm{ann}_R(l) \subset \mathrm{ann}_R(k)$. Now, we want to show that $\mathrm{ann}_R(k) \subset$

$\mathrm{ann}_R(l)$. Let $h \in \mathrm{ann}_R(k)$, $h \neq 0$. Then $hk = qn$ for non-zero integer $q \in \mathbb{N}$. By substitution, $hrl = qsl$. By cancellation of $l \neq 0$, we have $hr = qs$ but $g.c.d(r,s) = 1$. Therefore s must divide h, then sl must divide hl, which means that hl is a multiple of n. Thus $hl = 0 \,(\mathrm{mod}\, n)$ and therefore $h \in \mathrm{ann}_R(l)$, and $\mathrm{ann}_R(k) \subset \mathrm{ann}_R(l)$. Thus $\mathrm{ann}_R(k) = \mathrm{ann}_R(l)$. □

Corollary 2.15. *Let $[k]_\sim \in Z(R/\sim)^*$. Then for some representative l, $[k]_\sim = [l]_\sim$ and l/n.*

Proof. From lemma 2.13, $k \in Z^*(R)$ implies that $g.c.d(k,n) \neq 1$ say, $g.c.d(k,n) = l$. Thus, from lemma 2.14, $\mathrm{ann}_R(k) = \mathrm{ann}_R(l)$. This means that $[k]_\sim = [l]_\sim$ and l/n as required. □

In the previous corollary, we identify the vertices $Z(R/\sim)^*$ of $IA(R)$ in view of properties of R, then we use it in proving the following two lemmas. In the next lemma, we determine when the vertices of the graph $IA(R)$ are adjacent.

Lemma 2.16. *Let $[k]_\sim, [l]_\sim \in Z(R/\sim)^*$ such that $[k]_\sim \neq [l]_\sim$. Then $g.c.d(k,l) \neq 1$ if and only if $[k]_\sim$ is adjacent to $[l]_\sim$.*

Proof. Let $[k]_\sim, [l]_\sim \in Z(R/\sim)^*$, where $[k]_\sim \neq [l]_\sim$. Without any loss of generality, we assume k/n and l/n.

Assume that $g.c.d(k,l) = r$ with $r \neq 1$. Then there are non-zero q, $p \in \mathbb{N}$ such that $k = qr$, and $l = pr$. Since k/n, and l/n, so that r/n, then there is a positive integer m such that $n = mr$. By substitution,

$$mk = mqr = qn = 0\,(\bmod n)$$

$$ml = mpr = pn = 0\,(\bmod n)$$

i.e. $m \in \mathrm{ann}_R(k) \cap \mathrm{ann}_R(l)$. Therefore $[k]_\sim$ is adjacent to $[l]_\sim$.

For the converse, assume that $[k]_\sim$ is adjacent to $[l]_\sim$. Then there is $h \in Z^*(R)$ such that $h \in \mathrm{ann}_R(k) \cap \mathrm{ann}_R(l)$. Thus from corollary 2.15 there is $m \in Z^*(R)$ such that $[h]_\sim = [m]_\sim$ and m/n, which implies that $m \in \mathrm{ann}_R(k) \cap \mathrm{ann}_R(l)$. Hence $mk = 0\,(\mathrm{mod}\, n)$, and $ml = 0\,(\mathrm{mod}\, n)$. Then there are positive integers p, q such that $mk = pn$ and $ml = qn$. Since m/n, then there is a positive integer $r \neq 1$ such that $n = rm$. It follows that $mk = prm$, and $ml = qrm$. By cancellation of $m \neq 0$; $k = pr$, $l = qr$, and $r \neq 1$. Therefore $g.c.d(k,l) \neq 1$. □

In the next lemma, we show that if n is divisible by at least three primes, then the graph $IA(R)$ is always connected, and its diameter is less than or equal to two. Therefore it has a cycle of length three.

Lemma 2.17. *Let $n=\prod_{i=1}^{m} p_i$, where $m\geq 3$ and p_i's are prime numbers. Then $IA(R)$ is a connected graph with* $\operatorname{diam}(IA(R))\leq 2$ *and* $\operatorname{gr}(IA(R))=3$.

Proof. Let $[v_1]_\sim, [v_2]_\sim \in Z^*(R/\sim)$ such that $[v_1]_\sim \neq [v_2]_\sim$. Then by corollary 2.15, there exist $k,l\in Z^*(R)$ such that $[k]_\sim=[v_1]_\sim$ and $[l]_\sim=[v_2]_\sim$ with k/n and l/n. We have two cases: if $g.c.d(k,l)\neq 1$, then by lemma 2.16, $[k]_\sim$ is adjacent to $[l]_\sim$. It follows that $d(k,l)=d(v_1,v_2)=1$. If $g.c.d(k,l)=1$, let $k=\prod_{i=1}^{r} p_{t_i}$ and $l=\prod_{i=1}^{s} p_{j_i}$, where the set $\{p_{t_i}\}_{i=1}^{r}\subset\{p_i\}_{i=1}^{m}$ is distinct from the set $\{p_{j_i}\}_{i=1}^{s}\subset\{p_i\}_{i=1}^{m}$ and $r,s<m$. Let $u=p_{j_f}p_{t_g}$ for some $p_{j_f}\in\{p_{j_i}\}_{i=1}^{s}$ and $p_{t_g}\in\{p_{t_i}\}_{i=1}^{r}$. Then $[u]_\sim\neq[k]_\sim$, $[u]_\sim\neq[l]_\sim$ and $[u]_\sim\neq[0]_\sim$. By lemma 2.16, $[k]_\sim-[u]_\sim-[l]_\sim$. Thus $d(k,l)=d(v_1,v_2)=2$. It follows that $IA(R)$ is connected and $\operatorname{diam}(IA(R))\leq 2$. From corollary 2.9, $\operatorname{gr}(IA(R))=3$. $\square$

Example 2. *Let p, q and r be distinct prime numbers. Then we have the following three cases:*

(1) Let $R=\mathbb{Z}_{pq}$. From corollary 2.15, $Z(R/\sim)^*=\{[p]_\sim,[q]_\sim\}$. Since $g.c.d(p,q)=1$, then from Lemma 2.16, the graph $IA(R)$ is a totally disconnected graph as shown in figure 3.

FIGURE 3. $IA(R), R=\mathbb{Z}_{pq}$

(2) Let $R=\mathbb{Z}_{pqr}$. From corollary 2.15, the set of vertices is

$$Z(R/\sim)^*=\{[p]_\sim,[q]_\sim,[r]_\sim,[pq]_\sim,[pr]_\sim,[qr]_\sim\}.$$

Then the graph is connected, $\operatorname{diam}(IA(R))=2$ *and* $\operatorname{gr}(IA(R))=3$ *as shown in figure 4.*

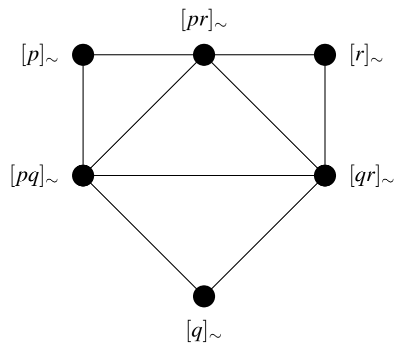

FIGURE 4. $IA(R)$, $R = \mathbb{Z}_{pqr}$

(3) Let $R = \mathbb{Z}_{p^m}$ for a positive integer m. In that case, $Z(R/\sim)^* = \{[p^i]_\sim | 1 \leq i < m\}$, and all vertices are adjacent. So, it is a complete graph with $(m-1)$-vertices. This means that $IA(R) = K_{m-1}$.

Remark 2.

- *Under the condition of lemma 2.17, if there is at least one distinct prime, then the graph is connected and* $\operatorname{diam}(IA(R)) = 2$. *If there is no distinct primes, then this is the case in part 3 of example 2.*
- *From 3 in example 2, all complete graphs of m-vertices may be realized as $IA(R) = K_m$. For instance, we can take $R = \mathbb{Z}_{p^{m+1}}$.*

2.4 Compressed intersection annihilator graph of a finite product of rings

In this subsection, we study the properties of the graph $IA(R)$ when the ring R is one of the following:

- The finite direct product of two or more integral domains with non-zero identities.
- The finite product of Artinian local rings.
- The direct product of two rings such that one of them is not an integral domain.

Now, we investigate the case when R is the finite direct product of two or more integral domains with non-zero identities. We notice that the graph of an integral domain is empty.

Theorem 2.18. *Let $R = A \times B$ where A and B are integral domains with non-zero identities. Then $IA(R)$ is a totally disconnected graph.*

Proof. We need to determine the set of vertices $Z(R/\sim)^*$. Let $(x,y) \in Z^*(R)$, then there is $(h,k) \in Z^*(R)$ such that $(h,k)(x,y) = (0_A, 0_B)$. Hence $hx = 0_A$ and $ky = 0_B$ and since A, B are integral domains, then $x = 0_A$ or $h = 0_A$ and $y = 0_B$ or $k = 0_B$. Therefore, the set of zero divisors without the zero element $(0_A, 0_B)$ may be partitioned into two disjoint sets, $V_A = \{(a, 0_B) \in R \,|\, a \in A \setminus \{0_A\}\}$ and $V_B = \{(0_A, b) \in R \,|\, b \in B \setminus \{0_B\}\}$. Thus $Z^*(R) = V_A \cup V_B$. But we can easily show that $\mathrm{ann}_R(u) = V_B$ for all $u \in V_A$ and similarly, $\mathrm{ann}_R(v) = V_A$ for all $v \in V_B$. Therefore $V_A \subseteq [(1_A, 0_B)]_\sim$ and $V_B \subseteq [(0_A, 1_B)]_\sim$. Thus $Z(R/\sim)^* = \{[(1_A, 0_B)]_\sim, [(0_A, 1_B)]_\sim\}$. Since $|Z(R/\sim)^*| = 2$ and $Z(R)$ is not an ideal, then from remark 1, $IA(R)$ is a totally disconnected graph with two vertices. □

Example 3. *Let $R = \mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are any two primes. Then $Z(R/\sim)^* = \{[(1,0)]_\sim, [(0,1)]_\sim\}$ and the graph $IA(R)$ is a disconnected graph as shown in figure 5.*

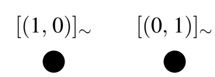

FIGURE 5. $IA(R), R = \mathbb{Z}_p \times \mathbb{Z}_q$

Theorem 2.19. *Let $\{A_i\}_{i=1}^n$ be a set of integral domains with non-zero identities and $n > 2$. If $R = \prod_{i=1}^n A_i$, then $IA(R)$ is connected with* $\mathrm{diam}(IA(R)) = 2$ *and* $\mathrm{gr}(IA(R)) = 3$.

Proof. Let $X, Y \in Z^*(R)$, $X = (a_1, a_2, ..., a_n)$ and $Y = (b_1, b_2, ..., b_n)$, where $a_i, b_i \in A_i$ with $a_k = 0_{A_k}$ and $b_l = 0_{A_l}$ for some $1 \le k, l \le n$. Let $[X]_\sim \neq [Y]_\sim$ and $\mathrm{ann}_R(X) \cap \mathrm{ann}_R(Y) = \{0\}$. Let $Z = (c_1, c_2, ..., c_n) \in Z^*(R)$ with $c_k = 0_{A_k}$ and $c_l = 0_{A_l}$. Define for any $1 \le j \le n$, $I_{A_j} = (u_1, u_2, ..., u_n)$ with $u_j = 1_{A_j}$ and $u_i = 0_{A_i}$ for all $1 \le i \le n$ and $i \neq j$. Then we can easily check that $I_{A_k}X = 0$, $I_{A_k}Z = 0$, $I_{A_l}Y = 0$ and $I_{A_l}Z = 0$. So that, there is an edge between $[X]_\sim$ and $[Z]_\sim$ and an edge between $[Z]_\sim$ and $[Y]_\sim$. Therefore there is a path between $[X]_\sim$ and $[Y]_\sim$. Thus $IA(R)$ is connected. Moreover, $\mathrm{diam}(IA(R)) \le 2$ and there are vertices which are not adjacent. For, $W = (1_{A_1}, 1_{A_2}, ... 1_{A_{n-1}}, 0_{A_n}) \in R$ and $V = (0_{A_1}, 1_{A_2}, ..., 1_{A_n}) \in R$ to find an element to annihilate V and W together it must be $0 = (0_{A_1}, ..., 0_{A_n}) \in R$. It follows that $\mathrm{diam}(IA(R)) = 2$. From corollary 2.9, $\mathrm{gr}(IA(R)) = 3$. □

Remark 3.

(1) One can easily check that for the ring R, defined in theorem 2.19, $|Z^*(R/\sim)| = 2^n - 2$.

(2) In the proofs of theorems 2.18 and 2.19 we used only that $Z(A_i) = \{0\}$, *for all* $1 \leq i \leq n$. *In fact the compressed intersection annihilator graph for a finite direct product of n integral domains is isomorphic to the compressed intersection annihilator graph of a finite direct product of n fields which are Artinian rings. Namely, different graphs constructed in that way, for a given n, would be isomorphic. Theorem 2.20 below is, in some sense, a generalization of theorem 2.19.*

Example 4. *Graph 6 below represents the following two very different cases:*

(1) Let p, q, r *be three prime numbers and* $R = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$. *Then*

$$Z(R/\sim)^* = \{[(0,0,1)]_\sim, [(0,1,0)]_\sim, [(1,0,0)]_\sim, [(0,1,1)]_\sim, [(1,0,1)]_\sim, [(1,1,0)]_\sim\}.$$

(2) Let $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. *Then the set of vertices is*

$$Z(R/\sim)^* = \{[(0,0,1)]_\sim, [(0,1,0)]_\sim, [(1,0,0)]_\sim, [(0,1,1)]_\sim, [(1,0,1)]_\sim, [(1,1,0)]_\sim\}.$$

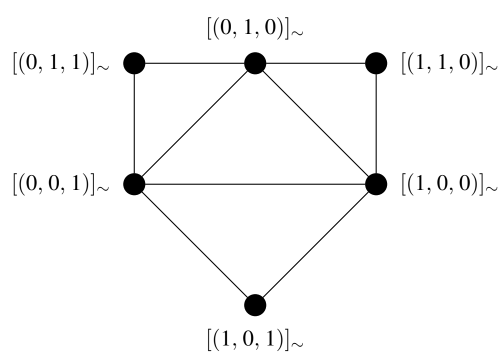

FIGURE 6. $IA(R)$, $R = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ & $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

Remark 4.

- *From the two parts in example 4, we notice that we may have isomorphic graphs for non-isomorphic rings and from part 2, we may have a finite graph for an infinite ring.*
- *Let p, q and r be three distinct primes. The graph* $IA(R)$ *when* $R = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ *is isomorphic to the graph* $IA(R)$, *when* $R = \mathbb{Z}_{pqr}$ *as shown in figures 4 and 6. The condition that p, q, and r are distinct is essential, since for example,* $IA(\mathbb{Z}_2 \times \mathbb{Z}_2)$ *is not isomorphic to* $IA(\mathbb{Z}_4)$.

Now, we investigate the properties of the compressed intersection annihilator graph $IA(R)$, when the ring R is a finite direct product of Artinian local rings with non-zero identities. Notice that the graph of an Artinian local ring is complete since the set of zero divisors is an annihilator ideal. In the next theorem, we show that the graph $IA(R)$ is connected and compute its diameter.

Theorem 2.20. *Let $\{R_i\}_{i=1}^n$ be a set of Artinian local rings with non-zero identities and $n \geq 2$. Assume that $a_i \in \operatorname{ann}_R(M_i)$ for some non-zero nilpotent element a_i, where $M_i = Z(R_i)$ is the unique maximal ideal of R_i for each $1 \leq i \leq n$ and let $R = \prod_{i=1}^n R_i$. Then $IA(R)$ is a connected graph with* $\operatorname{diam}(IA(R)) = 2$.

Proof. Let $X, Y \in R$ be two non-zero zero-divisors such that $[X]_\sim \neq [Y]_\sim$. Then for each i, $1 \leq i \leq n$, there exist $x_i, y_i \in R_i$, such that $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$ with $x_k \in M_k$ and $y_l \in M_l$ for some $1 \leq k, l \leq n$. Let $Z = (z_1, ..., z_n)$ with $z_k = a_k$, $z_l = a_l$ and $z_i \in R_i$ for all $i \neq k$ and $i \neq l$. Define $I_{R_j} = (u_1, ..., u_n)$ where $u_j = a_j$ and $u_i = 0_{R_i}$ for all $i \neq j$. By properties of nilpotent elements and $a_i \in \operatorname{ann}_R(M_i)$, we can easily show that $I_{R_k}X = 0$, $I_{R_k}Z = 0$, $I_{R_l}Z = 0$ and $I_{R_l}Y = 0$. It follows that, there is an edge between $[X]_\sim$ and $[Z]_\sim$ and thus, there is an edge between $[Z]_\sim$ and $[Y]_\sim$. Therefore, there is a path between $[X]_\sim$ and $[Y]_\sim$. This means that $IA(R)$ is connected and also $\operatorname{diam}(IA(R)) \leq 2$. However, there are vertices which are not adjacent. For instance, let $W = (1_{R_1}, 1_{R_2}, ... 1_{R_{n-1}}, 0_{R_n}) \in R$ and $V = (0_{R_1}, 1_{R_2}, ..., 1_{R_n}) \in R$. To find an element that annihilates V and W together it must be $0 = (0_{R_1}, ..., 0_{R_n}) \in R$. So $\operatorname{diam}(IA(R)) = 2$. □

Example 5. *Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then the set of vertices is*

$$Z(R/\sim)^* = \{[(0, 1)]_\sim, [(1, 0)]_\sim, [(0, 2)]_\sim, [(2, 0)]_\sim, [(2, 2)]_\sim, [(2, 1)]_\sim, [(1, 2)]_\sim\}$$

and the graph $IA(R)$ is represented in figure 7.

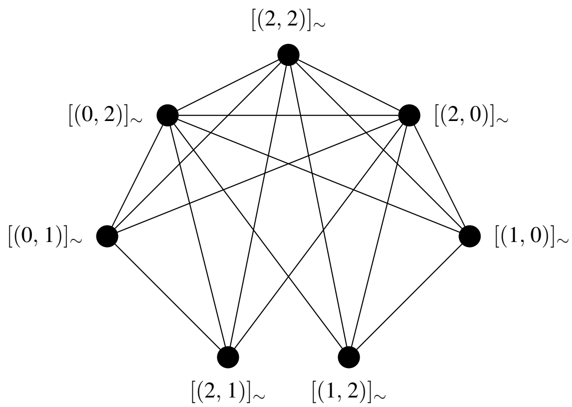

FIGURE 7. $IA(R), R = \mathbb{Z}_4 \times \mathbb{Z}_4$

In the next theorem, we show that if R is a finite direct product of two rings such that one of them is not an integral domain, then $IA(R)$ is connected, but not a complete graph with $\operatorname{diam}(IA(R)) \leq 3$ and $\operatorname{gr}(IA(R)) = 3$

Theorem 2.21. *Let R_1 and R_2 be two commutative rings with non-zero identities such that $Z^*(R_1) \neq \phi$ or $Z^*(R_2) \neq \phi$ and $R = R_1 \times R_2$. Then $IA(R)$ is connected, but not a complete graph with* $\operatorname{diam}(IA(R)) \leq 3$ *and* $\operatorname{gr}(IA(R)) = 3$.

Proof. Assume that $Z^*(R_1) \neq \phi$. Then there is $x \in Z^*(R_1)$ such that $rx = 0$ for some $r \in Z^*(R_1)$. We claim that we have three different vertices in $Z(R/\sim)^*$ namely, $[(1_{R_1}, 0_{R_2})]_\sim$, $[(0_{R_1}, 1_{R_2})]_\sim$ and $[(x, 0_{R_2})]_\sim$. It would follow by theorem 2.10, $IA(R)$ is connected and $\operatorname{diam}(IA(R)) \leq 3$ and by corollary 2.9, $\operatorname{gr}(IA(R)) = 3$. The vertices are distinct, since clearly $[(1_{R_1}, 0_{R_2})]_\sim \neq [(0_{R_1}, 1_{R_2})]_\sim$. On the other hand, $(r, 0_{R_2}) \in \operatorname{ann}(x, 0_{R_2})$ and $(r, 0_{R_2}) \notin \operatorname{ann}(1_{R_1}, 0_{R_2})$. Then $[(x, 0_{R_2})]_\sim \neq [(1_{R_1}, 0_{R_2})]_\sim$. Besides, $(0_{R_1}, 1_{R_2}) \in \operatorname{ann}(x, 0_{R_2})$ and $(0_{R_1}, 1_{R_2}) \notin \operatorname{ann}(0_{R_1}, 1_{R_2})$. Thus $[(x, 0_{R_2})]_\sim \neq [(0_{R_1}, 1_{R_2})]_\sim$. To show that $IA(R)$ is not a complete graph, it is easy to verify that $[(1_{R_1}, 0_{R_2})]_\sim$ and $[(0_{R_1}, 1_{R_2})]_\sim$ are not adjacent. Similarly, if $Z^*(R_2) \neq \phi$, then $IA(R)$ is connected, but not a complete graph with $\operatorname{diam}(IA(R)) \leq 3$ and $\operatorname{gr}(IA(R)) = 3$. $\square$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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