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APPLICATIONS OF LAPLACE-ADOMIAN DECOMPOSITION METHOD FOR SOLVING TIME-FRACTIONAL ADVECTION DISPERSION EQUATION

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Abstract: In this paper, a space-time fractional partial differential equation, obtained from the standard partial differential equation by replacing the second order space-derivative by a fractional derivative of order $\beta > 0$ and the first order time-derivative by a fractional derivative of order $\alpha > 0$ has been recently treated by a number of authors. A time fractional advection-dispersion equation is obtained from the standard advection-dispersion equation by replacing the first order derivative in time by a fractional derivative in time of order α ($0 < \alpha \leq 1$). In the present paper, the solution of the analytical dispersion equation is derived using Laplace-Adomian Decomposition Method (LADM). This method has higher convergences as the solutions both of fractional order and integral are obtained in the form of series. In this method the Caputo derivatives are used to define fractional order derivatives. To confirm validity of this method illustrative examples are given.

Keywords: space-time fractional partial differential equation; advection-dispersion equation; Laplace-Adomian decomposition method.

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1. INTRODUCTION

Most of the models in physics, biology and other physical processes can be expressed by fractional differential equations. There are various methods of solving these equations which do not have analytical solution. The Fractional Advection Dispersion equations (FADE) are special type of fractional partial differential equations and have been applied to many other problems. FADE are used to model transportation of passive tracer in porous medium whenever fluid flows through it. Many problems have been applied to this method [1-9], such as fractional iterative method [10], homotopy analysis method [11], homtopy perturbation method [12], Green function [13], and least square method [14].

If we consider one dimensional Advection Dispersion equation as in [9] given as

$$\frac{D\partial^2 u(x,t)}{\partial x^2} - \frac{V\partial u(x,t)}{\partial x} - \lambda RC = \frac{R\partial c(x,t)}{\partial t}$$

Where D is dispersion coefficient and R is Retardation fraction. The solution of this problem has been given in [15] as it is of integer order. But in order to derive the solution of FADE then we have to know the concepts of fractional order derivatives that are given as under.

2. FRACTIONAL CALCULUS

Some definitions, lemmas and theorems related to fractional calculus is given as under

Definition 1: Reimann –Liouville Fractional integral is given by

$$I_x^\alpha g(x) = \begin{cases} g(x) & if\ \alpha = 0 \\ \frac{1}{\Gamma(\alpha)} \int_0^x (x-v)^{\alpha-1}\ g(v)dv & if\ \alpha > 0 \end{cases}$$

Where Γ denotes the gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-x}\, x^{z-1} dx, \quad z \in C$$

Definition 2: The Caputo operator of order α for a fractional derivative is given by following mathematical expression foe n ϵN, $x > 0$, g ϵC_t, $t \geq -1$

$$D^\alpha g(x) = \frac{\partial^\alpha g(x)}{\partial t^\alpha} = \begin{cases} I^{n-\alpha}\ \frac{\partial^\alpha g(x)}{\partial t^\alpha} & n-1 < \alpha < n,\ n \in N \\ \frac{\partial^\alpha g(x)}{\partial t^\alpha} & \end{cases}$$

Hence, we require the related properties which are given in the next Lemma.

Lemma 1:- If $n-1 < \alpha < n,\ n \epsilon N$ and g ϵC_t with t ≥ -1 , then

$I^{\alpha}I^{a}$g(x) = $I^{\alpha+a}g(x)a$, $\alpha \geq 0$

$I^{\alpha}x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)}x^{\alpha+\lambda}\alpha > 0, \lambda > -1 , x > 0$

$I^{\alpha}D^{\alpha}g(x) = g(x) - \sum_{x=0}^{n-1} g^{k}(0^{+})\frac{x^{k}}{k!}$, $x > 0,\ n-1 < \alpha < n$

Due to some disadvantages in other fractional operators, the Caputo operator is given preference.

Definition 3:- The Laplace transform of z (t), t > 0 is defined by [16]

Z(s) = Ł [z (t)] = $\int_0^{\infty} e^{-st} z(t)dt$

Definition 4. The Laplace transform in term of convolution is given by

Ł $[z_1 * z_2]$ = *Ł* $[z_1(t)]$ $*$*Ł* $[z_2(t)]$

Here $z_1 * z_2$ is convolution between z_1 and z_2

$(z_1 * z_2)\ t = \int_0^{\gamma} z_1(x)z_2(t-\gamma)dt$

Laplace transform of fractional derivative of a function z(t) is given by

Ł $[D_t^{\alpha} z(t)] = s^{\alpha}Z(s) - \sum_{k=0}^{n-1} s^{\alpha-1-1}z^{k}(0),\ n-1 < \alpha < n$

Where Z(s) is the Laplace transform of z(t).

Definition 5:- The Mittag–Leffler function, Fα (p) for $\alpha > 0$ is given as

$F_{\alpha}\ (p) = \sum_{n=0}^{\infty} \frac{p^n}{\alpha^n+1}\ , \alpha > 0\ , p\ \epsilon C.$

3. Fractional Laplace-Adomian Decomposition Method

To start with we apply Laplace-Adomian Decomposition Method (FADM) to Fractional Advection-Dispersion Equation (FADE) with initial conditions;

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = k\frac{\partial^2 u(x,t)}{\partial x^2} + v\frac{\partial u(x,t)}{\partial x} + g(x,t),\ x > 0, t > 0, m-1 < \alpha < m \quad (1)$$

Where $g(x,t)$, is a source function, subject to the conditions:

$$u(x,0) = h_1 \quad (2)$$

$$u'(x,0) = h_2 x$$

$u''(x,0) = h_3 x$ …..

$$u^n(x,0) = h_{n+1} x \tag{3}$$

Applying Laplace transform on both sides of Eq. 1, we get

$$Ł\,[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}] = Ł\,[k\frac{\partial^2 u(x,t)}{\partial x^2} + v\frac{\partial u(x,t)}{\partial x} + g(x,t)\ ] \tag{4}$$

$$s^\alpha u(x,t) - s^{\alpha-1}u(x,0) - s^{\alpha-2}u'(x,0) - \cdots - s^{\alpha-1-(n-1)}u^{\alpha-1}(x,t)$$

$$= Ł\,[k\frac{\partial^2 u(x,t)}{\partial x^2} + v\frac{\partial u(x,t)}{\partial x} + g(x,t)\,],$$

$$Łu(x,t) = \frac{h_1(x)}{s} + \frac{h_2(x)}{s^2} + \cdots + \frac{h_{n-1}(x)}{s^{-1-(n-1)}} + \frac{1}{s^\alpha}Ł\,[k\frac{\partial^2 u(x,t)}{\partial x^2} + v\frac{\partial u(x,t)}{\partial x} + g(x,t)\ ], \tag{5}$$

The LADM solution of $u(x,t)$ is represented by infinite series

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t) \tag{6}$$

And other nonlinear terms are expressed by infinite series of adomian polynomial

$$N\,u(x,t) = \sum_{j=0}^{\infty} B_j$$

$$B_j = \frac{1}{\Gamma(j+1)}\,[\frac{d^j}{d\lambda^j}\,[N\,\sum_{j=0}^{\infty}\lambda^j u_j]]_{\lambda=0},\ j = 0,1,2,\ldots$$

Substituting Eq. 5 and Eq. 6 in Eq. 4

$$Ł\,[\sum_{j=0}^{\infty} u_j(x,t)] = \frac{h_1(x)}{s} + \ldots + \frac{h_{n-1}(x)}{s^{-1-(n-1)}} + \frac{1}{s^\alpha}Ł\,[k\frac{\partial^2 u_j(x,t)}{\partial x^2} + v\frac{\partial u_j(x,t)}{\partial x}] + \frac{1}{s^\alpha}Ł[g(x,t)\ ]$$

$$Ł\,[u_{j+1}(x,t)] = \frac{1}{s^\alpha}Ł\,[k\frac{\partial^2 u_j(x,t)}{\partial x^2} + v\frac{\partial u_j(x,t)}{\partial x}] + \frac{1}{s^\alpha}Ł[g(x,t)\ ]$$

Now applying inverse Laplace transform, we have

$$u_{j+1}(x,t) = Ł^{-1}[\frac{1}{s^\alpha}Ł\,(k\frac{\partial^2 u_j(x,t)}{\partial x^2} + v\frac{\partial u_j(x,t)}{\partial x})] + Ł^{-1}[\frac{1}{s^\alpha}Ł[g(x,t)\ ]]$$

4. Results

Example 1:

Consider the FADE $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k\frac{\partial^2 u(x,t)}{\partial x^2} - v\frac{\partial u(x,t)}{\partial x}$, $t > 0, 0 < \alpha < 1,$ (7)

With the initial conditions $u(x,0) = e^{-x}$

Taking Laplace transform of Eq. 7

$Ł\,[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}] = Ł\,[k\frac{\partial^2 u(x,t)}{\partial x^2} \quad - v\frac{\partial u(x,t)}{\partial x}]$

$s^\alpha\ \ Ł\ \ [u(x,t)]\text{-}\ s^{\alpha-1}u(x,0)\ = Ł\,[k\frac{\partial^2 u(x,t)}{\partial x^2} \quad - v\frac{\partial u(x,t)}{\partial x}]$

$Ł\ \ [u(x,t)]= \frac{1}{s}\ [u(x,0)] + \frac{1}{s^\alpha}Ł\,[k\frac{\partial^2 u(x,t)}{\partial x^2} \quad - v\frac{\partial u(x,t)}{\partial x}]$

Taking inverse Laplace transform on both sides

$u(x,t)=\ e^{-x}\ + Ł^{-1}[\frac{1}{s^\alpha}Ł\,(k\frac{\partial^2 u(x,t)}{\partial x^2} \quad - v\frac{\partial u(x,t)}{\partial x})]$

Using ADM approach, we have

$\sum_{j=0}^{\infty} u_j(x,t) = =\ e^{-x}\ + Ł^{-1}[\frac{1}{s^\alpha}Ł\,(k\frac{\partial^2 u_j(x,t)}{\partial x^2} \quad - v\frac{\partial u_j(x,t)}{\partial x})]$

$u_{j+1}(x,t)\ = Ł^{-1}[\frac{1}{s^\alpha}Ł\,(k\frac{\partial^2 u_j(x,t)}{\partial x^2} \quad - v\frac{\partial u_j(x,t)}{\partial x})]$

Now for j = 0,1,2 … we have

$u_1(x,t)\ = Ł^{-1}[\frac{1}{s^\alpha}Ł\,(k\frac{\partial^2 u_0(x,t)}{\partial x^2} \quad - v\frac{\partial u_0(x,t)}{\partial x})]$

$u_1(x,t)= \quad \frac{e^{-x}\,(k+v)t^\alpha}{\Gamma(\alpha+1)}$

$u_2(x,t) = Ł^{-1}[\frac{1}{s^\alpha}Ł\,(k\frac{\partial^2 u_1(x,t)}{\partial x^2} \quad - v\frac{\partial u_1(x,t)}{\partial x})]$

$u_2(x,t) = Ł^{-1}[\frac{1}{s^{2\alpha+1}}[e^{-x}(k^2+v^2)]$

$u_2(x,t)= \frac{e^{-x}(k^2+v^2)t^{2\alpha}}{\Gamma(2\alpha+1)}$

The LADM solution is given by

$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \ldots,$

$u(x,t)=\ e^{-x}\ + \frac{e^{-x}\,(k+v)t^\alpha}{\Gamma(\alpha+1)} + \frac{e^{-x}(k^2+v^2)t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots$

The solution obtained here is same as the solution obtained in [15], where the solutions are expressed as Mittag-Leffler function when taken n up to infinity; same can be obtained in this example as well.

Example 2:-

Consider another FADE $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \omega\frac{\partial^2 u(x,t)}{\partial x^2} \quad +\beta\ \frac{\partial u(x,t)}{\partial x} = 0, t > 0, \quad 1 < \alpha < 2$ (8)

With the initial conditions $u(x,0) = sinx\,, \quad \frac{du(x,0)}{dx} = \ e^{-x}$

Taking Laplace transform on both sides

$$Ł\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right] = -Ł\left[\omega \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial u(x,t)}{\partial x}\right]$$

$$s^\alpha \; Ł\;[u(x,t)] - s^{\alpha-1}[(x,0)] - s^{\alpha-2}\left[\frac{\partial u(x,0)}{\partial x}\right] = -Ł\left[\omega \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial u(x,t)}{\partial x}\right]$$

$$Ł\;[u(x,t)] = \frac{\sin x}{s} + \frac{e^{-x}}{s^2} - \frac{1}{s^\alpha} Ł\left[\omega \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial u(x,t)}{\partial x}\right]$$

Taking inverse on both sides

$$u(x,t) = Ł^{-1}\left[\frac{\sin x}{s} + \frac{e^{-x}}{s^2}\right] - Ł^{-1}\left[\frac{1}{s^\alpha} Ł\left[\omega \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial u(x,t)}{\partial x}\right]\right]$$

$$u(x,t) = \sin x + t\,e^{-x} - Ł^{-1}\left[\frac{1}{s^\alpha} Ł\left[\omega \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial u(x,t)}{\partial x}\right]\right]$$

Applying adomian decomposition method we have

$$\sum_{j=0}^{\infty} u_j(x,t) = \sin x + t\,e^{-x} - Ł^{-1}\left[\frac{1}{s^\alpha} Ł\left[\omega \frac{\partial^2 u_j(x,t)}{\partial x^2} + \beta \frac{\partial u_j(x,t)}{\partial x}\right]\right]$$

$$u_0(x,t) = \sin x + t\,e^{-x}$$

$$u_1(x,t) = -Ł^{-1}\left[\frac{1}{s^\alpha} Ł\left[\omega \frac{\partial^2 u_0(x,t)}{\partial x^2} + \beta \frac{\partial u_0(x,t)}{\partial x}\right]\right]$$

$$u_1(x,t) = -Ł^{-1}\left[\frac{1}{s^\alpha}\left[-\omega\, sinx + \beta cosx + (\omega-\beta)t\,e^{-x}\right]\right.$$

$$u_1(x,t) = \left[\omega\, sinx - \beta cosx - (\omega-\beta)t\,e^{-x}\right]\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_1(x,t) = \left[\omega\, t^\alpha \sin x - \beta\, t^\alpha\, cosx - (\omega-\beta)t^{\alpha+1}e^{-x}\right]\frac{1}{\Gamma(\alpha+1)}$$

$$u_2(x,t) = -Ł^{-1}\left[\frac{1}{s^\alpha} Ł\left[\omega \frac{\partial^2 u_1(x,t)}{\partial x^2} + \beta \frac{\partial u_1(x,t)}{\partial x}\right]\right]$$

$$u_2(x,t) = \left[sinx\, t^{3\alpha}(\beta^2+\omega^2) + 2\,\omega\beta\, t^{\,3\alpha}\, cosx - \; t^{\,3\alpha+1}e^{-x}(\beta+\omega)^2\right]\frac{1}{\Gamma(2\alpha+1)}$$

$$u_3(x,t) = -Ł^{-1}\left[\frac{1}{s^\alpha} Ł\left[\omega \frac{\partial^2 u_2(x,t)}{\partial x^2} + \beta \frac{\partial u_2(x,t)}{\partial x}\right]\right];$$

$$u_3(x,t) = \left[sinx\, t^{6\alpha}(\beta^3+3\omega\beta^2) + cosx\, t^{6\alpha}(\beta^3+3\beta\omega^2) + \; t^{\,6\alpha+1}e^{-x}(\beta+\omega)^3\right]\frac{1}{\Gamma(3\alpha+1)}$$

The LADM solution is given by

$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \ldots,$

$$u(x,t) = sin\,x + t\,e^{-x} + \left[\omega\, t^\alpha \sin x - \beta\, t^\alpha\, cosx - (\omega-\beta)t^{\alpha+1}e^{-x}\right]\frac{1}{\Gamma(\alpha+1)}$$

$$+\left[\; sinx\, t^{3\alpha}(\beta^2+\omega^2) + 2\,\omega\beta\, t^{\,3\alpha}\, cosx - \; t^{\,3\alpha+1}e^{-x}(\beta+\omega)^2\right]\frac{1}{\Gamma(2\alpha+1)}$$

$$+ \quad [\ sinx\, t^{6\alpha}(\beta^3 + 3\omega\beta^2) + cosx\, t^{6\alpha}(\beta^3 + 3\beta\omega^2) + \ t^{\,6\alpha+1}e^{-x}(\beta+\omega)^3]\ \frac{1}{\Gamma(3\alpha+1)} \ldots..$$

The solution obtained is in the form of adomian polynomial.

5. CONCLUSION

The fundamental aim of this paper was to obtain efficient results and much more convenient. Previously the results were obtained fractional iterative method in which domain was confined to only $0 < \alpha < 1$, but by this method we can generalize to higher fractional order. This method is more fully appropriate to physical problem.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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