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OPTIMAL CONTROL AND DISCRETE-TIME MODELLING OF ALCOHOL MODEL WITH PHYSICAL AND PSYCHOLOGICAL COMPLICATIONS

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Abstract. In this paper, we propose a discrete mathematical model that describes the interaction between the classes of drinkers, namely, potential drinkers (P), moderate drinkers (M), heavy drinkers (H), heavy drinkers with physical complications (C_1), the individuals with psychological complications (C_2) and recovered and quitters of drinking (Q). We also focus on the importance of treatment of heavy drinkers with and without complications to aiming to find the optimal strategies to minimize the number of drinkers and maximize the number of the individuals who recovered and quitters of drinking. We use three controls which represent awareness programs through media and education for the potential drinkers, efforts treatment for the heavy drinkers and treatment for the heavy drinkers with physical complications. We use Pontryagin's maximum principle in discrete time to characterize these optimal controls. The resulting optimality system is solved numerically by Matlab. Consequently, the obtained results confirm the performance of the optimization strategy.

Keywords: mathematical model; alcohol drinking; physical complications; psychological complications; optimal control.

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1. INTRODUCTION

Alcoholism is a social and global phenomenon that is harmful to all social groups due to some physiological, genetic or even neurological factors that cause this disease, which need to drink isolated and uncontrollable alcohol. This is why an alcoholic will continue to drink despite serious family, social, professional, and legal problems arising from his addiction.

Harmful use of alcohol can harm others such as family members and bankruptcy may occur, bankruptcy in the relationship (divorce, separation, isolation), economic (loss of employment, financial difficulties), legal bankruptcy (loss of driver's license, arrest, imprisonment) Psychological (loss of self-esteem, dignity, freedom, autonomy, quality of life and well-being, physical and neurological complications, anxiety, depression, etc)[15]. According to the World Health Organization's 2016 report, alcohol drinking increases the risk of heart disease, high blood pressure and stroke. Risk increases in younger people due to party drinking, which can lead to accidents or violence, and there are 3 million deaths a year, which accounts for 6% of all deaths due to drinking alcohol [15]Alcoholism is a serious problem that causes many physical and psychological complications. Concerning physical complications, people who drink too much alcohol are at risk of harmful, potentially life-threatening liver problems. When you drink a lot, your liver breaks down alcohol and removes it from your blood. Drinking too much alcohol in a short period of time can lead to fatty liver and my father is considered a chronic condition that involves accumulation of bad fat in the liver. Which leads to the problem of obesity, liver failure and type 2 diabetes. Other serious liver complications are associated with prolonged and excessive alcohol consumption. There are other problems which are caused by drinking a lot of alcohol which are alcoholic hepatitis, brosis and cirrhosis. In order too treat these health problems, They require a proper medical diagnosis and intensive treatment plan. According to the World Health Organization's 2016 report, deaths caused by liver cirrhosis disease, age-standardized death rates (15+), per 100,000 population in 2016 in different countrys. in Egypt over there 200.4(Male) and 6.6 (Female), Russian Federation 40.3(Male) and 6.9 (Female) and in Brazil over there 26.7(Male), 6.4 (Female)[15], Cardiovascular (CV) health: Drinking alcohol affects cardiovascular health. Alcohol-related CV diseases caused an estimated 593.000

deaths globally consuming too much alcohol in 2016 is linked to high blood pressure, irregular heartbeat, trouble pumping blood through the body, blood clots, stroke, cardiomyopathy, or heart attack Excessive alcohol use, both directly and through malnutrition, can also lead to anemia. For example, it affects the circulatory system, veins and arteries negatively, leading to high blood pressure, as well as atherosclerosis and heart attacks, which may lead to death in some cases. It also causes ulcers and gastrointestinal problems, where pancreatitis, central nervous system, respiratory infections, immune system dysfunction, brain damage, malnutrition and vitamin deficiency, osteoporosis, Accidents and injuries.

In addition to the physical complications caused by alcoholism, the frequent use of alcohol leads to serious psychological complications. It may lead to personality changes: alcohol use can cause major changes to your personality. Attributes of the natural personality disappear during intoxication and are replaced by angry, selfish and aggressive behavior, mood swings are very common, and a general deterioration of morals also occurs. Alcohol slows the tangles in the brain and changes your body chemically by affecting levels of serotonin, the chemical responsible for transmitting mood signals to your mind. These physical changes make your emotions out of control and cause you to say and do things that you wouldn't normally do. Concomitant disorders: often there are psychological and mood disorders common among alcoholics. It can be caused by mental effects of light drinking, or it can be caused by heavy drinking in the first place. Often, a person with a common disorder such as depression or anxiety becomes addicted to alcohol after trying to treat his symptoms on his own. For example: depression, partial depression disorder, anxiety, compulsive behavior, deprivation, alcohol-induced depression disorder, alcohol-induced bipolar disorder, alcohol-induced sleep disorder, severe depression and some anxiety disorders, schizophrenia, bipolar disorder, etc., And mental disorder caused by alcohol. It changes the sleep pattern and personality, as it has cognitive effects such as short attention and reduced coordination problems. Several mathematicians did a lot of work in order to understand the dynamics of drinking and reduce its harm on the drinker and society as well as minimizing the number of addicted drinkers. For example[1, 4, 6, 7, 14]: H. F. Huo and N. Song[6] divided heavy drinkers in thier study into two types: those who confess drinking and those who do not and they proposed a two-stage model

for binge drinking problem taking into consideration the transition of drinkers from the class of susceptible individuals towards the class of admitting drinkers. H. F. Huo, and Q. Wang [7] developed a non-linear mathematical model with the effect of awareness programs on the binge drinking where they show that awareness programs are an effective measure in reducing alcohol problems. H. F. Huo et al [6] proposed a new social epidemic model to depict alcoholism with media coverage which was proven to be an effective way in pushing people to quit drinking. S. H. Ma et al [14] modeled alcoholism as a contagious disease and used an optimal control to study their mathematical model with awareness programs and time delay. Wang et al [4] proposed and analyzed a non-linear alcoholism model and used optimal control for the purpose of hindering interaction between susceptible individuals and infected individuals. I. k. Adu et al [1] used a non-linear *SHTR* mathematical model to study the dynamics of drinking epidemic, they divided their population into four classes: non-drinkers (S), heavy drinkers (H), drinkers receiving treatment (T) and recovered drinkers (R). They discussed the existence and stability of drinking-free and endemic equilibrium. Other mathematical models has also been widely used to study this phenomenon (For example, [10,12,13.....]). In addition, most of these previous researches have focused on continuous-time modeling. In this research, we will adopt the discrete-time modeling as the statistical data are collected at discrete time (day, week, month and year) as well as the treatment and vaccination of some patients are done in discrete-time. So, it is more direct, more convenient, and more accurate to describe the phenomena by using the discrete-time modeling than the continuous-time modeling and the use of discrete time models avoids some mathematical complications such as choosing a space, function, and regularity of the solution. Hence, difference equations appear as a more natural way to describe the epidemic models and discrete problems. Moreover, numerical solutions of differential equations use discretization and this encourages us to employ difference equations directly. The numerical exploration of discrete-time models is rather straightforward and therefore can be easily implemented by non mathematicians. Besides these works, we will study the dynamics of a mathematical alcohol model $PMHC_1Q C_2$ which contains the following additions:

- A discrete-time mathematical modeling.

- A compartment C_1 that represents the number of the heavy drinkers with physical complications.
- A compartment C_2 that represents the number of individuals with psychological complications.
- The death rate induced by the heavy drinkers δ_1 .
- The death rate induced by the heavy drinkers with physical complications δ_2 .

The drinkers classes of this model are divided into six compartments: Potential drinkers (P), Moderate drinkers (M), Heavy drinkers (H), The heavy drinkers with physical complications (C_1), The individuals with psychological complications (C_2) and recovered and quitters of drinking (Q). Throughout this research, we seek to find the optimal strategies to minimize the number of heavy drinkers with and without complications and maximize the number of recovered and quitters of drinking.

In order to achieve this purpose, we use optimal control strategies associated with three types of controls: the first represents awareness programs for potential drinkers, the second is the treatment for heavy drinkers and the third is the treatment for heavy drinkers with physical complications (C_1).

The paper is organized as follows. In Section 2, we present our $PMHC_1QC_2$ discrete mathematical model that describes the dynamics of the classes of drinkers. In Section 3 and 4, we present the optimal control problem for the proposed model where we give some results concerning the existence of the optimal controls and we characterize these optimal controls using Pontryagin's Maximum Principle in discrete time. Numerical simulations are given in Section 5. Finally, we conclude the paper in Section 6.

2. MODEL FORMULATION

We propose a discrete model $PMHC_1QC_2$ to describe the dynamics of population and the transmission of drinking. The population is divided into six compartments denoted by P , M , H , C_1 , Q and C_2 .

The graphical representation of the proposed model is shown in Figure 1.

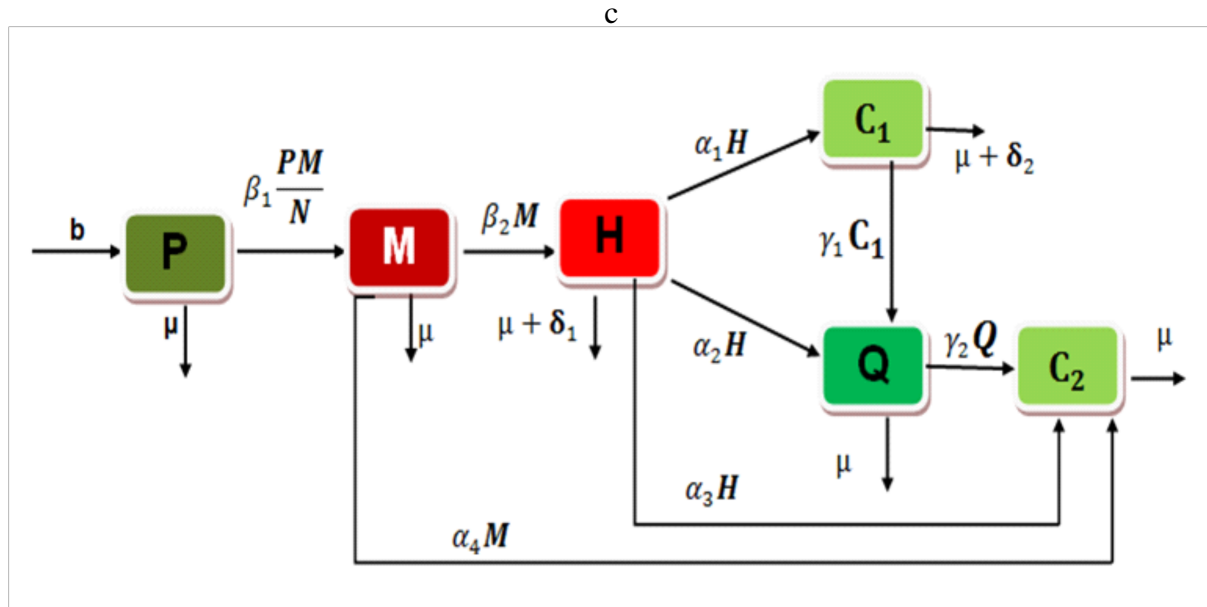


FIGURE 1. Schematic diagram of the six drinking classes in the model

The mathematical representation of the model consists of a system of non-linear difference equations:

$$(1) \quad \begin{cases} P_{k+1} = b - \beta_1 \frac{P_k M_k}{N_k} + (1 - \mu) P_k \\ M_{k+1} = \beta_1 \frac{P_k M_k}{N_k} - \alpha_4 M_k + (1 - \mu - \beta_2) M_k \\ H_{k+1} = \beta_2 M_k + (1 - \mu - \delta_1 - \alpha_1 - \alpha_2 - \alpha_3) H_k \\ C_{1,k+1} = \alpha_1 H_k + (1 - \mu - \gamma_1 - \delta_2) C_{1,k} \\ Q_{k+1} = \alpha_1 H_k + \gamma_1 C_{1,k} + (1 - \mu - \gamma_2) Q_k \\ C_{2,k+1} = \gamma_2 Q_k + \alpha_3 H_k + \alpha_4 M_k + (1 - \mu) C_{2,k} \end{cases}$$

where $P_0 \geq 0, M_0 \geq 0, H_0 \geq 0, C_{1,0} \geq 0, Q_0 \geq 0$ and $C_{2,0} \geq 0$.

The compartment P: contains the potential drinkers who represent individuals whose age is over adolescence and adulthood and may become drinkers. This compartment is increased by the recruitment rate denoted by b and decreased by an effective contact with the moderate drinkers at a rate β_1 and natural death μ . It is assumed that potential drinkers can acquire drinking behavior and can become moderate drinkers through effective contact with moderate drinkers in some social occasions like weddings, celebrating graduation ceremonies, week-end

parties and end of the year celebration. In other words, it is assumed that the acquisition of a drinking behavior is analogous to acquiring disease infection.

The compartment M : is composed of the moderate drinkers who drink alcohol in a controlled manner during some events and occasions or in a way that is unapparent to their social environment. It is increased by potential drinkers who turn to be moderate drinkers at a rate β_1 . This compartment is decreased when moderate drinkers become heavy drinkers at a rate β_2 , and those who quit drinking with psychological complication (C_2) at a rate α_4 and also by natural death at rate μ .

The compartment H : comprises the heavy drinkers. This compartment becomes larger as the number of heavy drinkers increases by the rate β_2 and decreases when some of them give up drinking who quit drinking at a rate α_2 and outhter becomes heavy drinkers with physical complications (C_1) at rate α_1 (α_1 is a rate of heavy drinkers becomes heavy drinkers with physical complications) and some becomes heavy drinkers with psychological complications (C_2) at rate α_3 (α_3 is a rate of heavy drinkers becomes individuals with psychological complications). In addition, this compartment decreases by natural death μ and due to deaths caused by diseases resulted from excessive alcohol intake at a rate δ_1 .

The compartment C_1 : represents the number of heavy drinkers with physical complications (C_1). This compartment is increased by the rate α_1 and decreased by the rates γ_1 and $(\mu + \delta_2)$, where γ_1 is the individuals heavy drinkers with physical complications who quit drinking and treated and δ_2 is the death rate induced by the heavy drinkers with physical complications.

The compartment Q : contains the number of individuals who recovered from physical complications and quitters of drinking. This compartment is increased by the rates α_2 and γ_1 and decreased by the rates γ_2 (γ_2 is a rate of the individuals who recovered from physical complications and quitters of drinking becomes individuals with psychological complications) and μ .

The compartment C_2 : encompasses the individuals who recovered from physical complications and quitters of drinking and becomes individuals with psychological complications for alcohol-induced depressive disorder, alcohol-induced bipolar disorder, alcohol-induced sleep disorder, alcohol-induced psychotic disorder, and more. These disorders are temporary and can occur after significant intoxication and/or withdrawal. mental illnesses which can develop

separately from the disorder and may even predate it, such as major depression, some anxiety disorders, schizophrenia, bipolar disorder, and others. It is increased with the recruitment of individuals who have been treated at rate γ_2 and α_4 (α_4 is a rate of the moderate drinkers who becomes to individuals with psychological complications). It also increases at the rate α_3 of those who quit and recovered alcohol and decreases at the rate μ due to natural deaths.

The total population size at time k is denoted by N_k with $N_k = P_k + M_k + H_k + C_{1,k} + Q_k + C_{2,k}$ and it is supposed as constant.

3. THE OPTIMAL CONTROL PROBLEM

The objective of the proposed control strategy is to minimize the number of heavy drinkers with and without complications and maximize the number of the recovered and quitters of drinking (Q) in the community during the time step $k = 0$ to $k = T$. The cost spent in the awareness programs and treatment are also to be minimized.

In order to achieve these objectives, we introduce three control variables. The first control u_1 represents the effort of the awareness programs to protect the potential drinkers not to be drinkers. The second control u_2 measures the treatment of the heavy drinkers. We note that the control function εu_2 represents the fraction of the heavy drinkers who will be treated and become moderate drinkers and the fraction $(1 - \varepsilon)u_2$ of those leaving the heavy drinkers class who will receive treatment and quit drinking and u_3 measures the treatment of the heavy drinkers with physical complications.

So, the controlled mathematical system is given by the following system of difference equations:

$$(2) \quad \begin{cases} P_{k+1} = b - \beta_1 \frac{P_k M_k}{N_k} + (1 - \mu)P_k - u_{1,k}P_k \\ M_{k+1} = \beta_1 \frac{P_k M_k}{N_k} - \alpha_4 M_k + (1 - \mu - \beta_2)M_k + \varepsilon u_{2,k}H_k \\ H_{k+1} = \beta_2 M_k + (1 - \mu - \delta_1 - \alpha_1 - \alpha_2 - \alpha_3)H_k - u_{2,k}H_k \\ C_{1,k+1} = \alpha_1 H_k + (1 - \mu - \gamma_1 - \delta_2)C_{1,k} - u_{3,k}C_{1,k} \\ Q_{k+1} = \alpha_2 H_k + \gamma_1 C_{1,k} + (1 - \mu - \gamma_2)Q_k + u_{1,k}P_k + (1 - \varepsilon)u_{2,k}H_k + u_{3,k}C_{1,k} \\ C_{2,k+1} = \gamma_2 Q_k + \alpha_3 H_k + \alpha_4 M_k + (1 - \mu)C_{2,k} \end{cases}$$

where $P_0 \geq 0, M_0 \geq 0, H_0 \geq 0, C_{1,0} \geq 0, Q_0 \geq 0$ and $C_{2,k} \geq 0$.

The optimal control problem to minimize the objective functional is given by:

$$(3) \quad \begin{aligned} J(u_1, u_2, u_3) &= H_T + C_{1,T} + C_{2,T} - Q_T \\ &+ \sum_{k=0}^{T-1} \left(H_k + C_{1,k} + C_{2,k} - Q_k + \frac{A_{1,k}u_{1,k}^2}{2} + \frac{A_{2,k}u_{2,k}^2}{2} + \frac{A_{3,k}u_{3,k}^2}{2} \right) \end{aligned}$$

Where the parameters $A_{1,k} > 0, A_{2,k} > 0$ and $A_{3,k} > 0$ are selected to weigh the relative importance of the cost of awareness programs and treatment respectively.

The aim is to find an optimal control u_1^*, u_2^* and u_3^* such that :

$$(4) \quad J(u_1^*, u_2^*, u_3^*) = \min_{(u_1, u_2, u_3) \in U_{ad}^3} J(u_1, u_2, u_3)$$

Where U_{ad} is the set of admissible controls defined by

$$(5) \quad U_{ad} = \{ (u_{1,k}, u_{2,k}, u_{3,k}) / 0 \leq u_{j \min} \leq u_{j,k} \leq u_{j \max} \leq 1; j = 1, 2, 3 \text{ and } k = 0, 1, 2, \dots, T-1 \}$$

The sufficient condition for the existence of an optimal control (u_1^*, u_2^*, u_3^*) for problem (2) and (3) comes from the following theorem.

Theorem 1. *There exists the optimal control (u_1^*, u_2^*, u_3^*) such that*

$$(6) \quad J(u_1^*, u_2^*, u_3^*) = \min_{(u_1, u_2, u_3) \in U_{ad}^3} J(u_1, u_2, u_3)$$

subject to the control system (2) with initial conditions.

Proof. Since the coefficients of the state equations are bounded and there are a finite number of time steps, $P = (P_0, P_1, P_2, \dots, P_T)$, $M = (M_0, M_1, M_2, \dots, M_T)$, $H = (H_0, H_1, H_2, \dots, H_T)$, $C_1 = (C_{1,0}, C_{1,1}, C_{1,2}, \dots, C_{1,T})$, $Q = (Q_0, Q_1, Q_2, \dots, Q_T)$ and $C_2 = (C_{2,0}, C_{2,1}, C_{2,2}, \dots, C_{2,T})$ are uniformly bounded for all (u_1, u_2, u_3) in the control set U_{ad} ; thus $J(u_1, u_2, u_3)$ is bounded for all $(u_1, u_2, u_3) \in U_{ad}^3$. Since $J(u_1, u_2, u_3)$ is bounded, $\inf_{(u_1, u_2, u_3) \in U_{ad}^3} J(u_1, u_2, u_3)$ is finite, and there exists a sequence $(u_1^j, u_2^j, u_3^j) \in U_{ad}^3$ such that $\lim_{j \rightarrow +\infty} (u_1^j, u_2^j, u_3^j) = \inf_{(u_1, u_2, u_3) \in U_{ad}^3} J(u_1, u_2, u_3)$ and corresponding sequences of states $P^j, M^j, H^j, C_1^j, Q^j$ and C_2^j . Since there is a finite number of

uniformly bounded sequences, there exist $(u_1^*, u_2^*, u_3^*) \in U_{ad}^3$ and $P^*, M^*, H^*, C_1^*, Q^*$ and $C_2^* \in \mathbb{R}^{T+1}$ such that, on a subsequence, $\lim_{j \rightarrow +\infty} (u_1^j, u_2^j, u_3^j) = (u_1^*, u_2^*, u_3^*)$, $\lim_{j \rightarrow +\infty} P^j = P^*$, $\lim_{j \rightarrow +\infty} M^j = M^*$, $\lim_{j \rightarrow +\infty} H^j = H^*$, $\lim_{j \rightarrow +\infty} C_1^j = C_1^*$, $\lim_{j \rightarrow +\infty} Q^j = Q^*$ and $\lim_{j \rightarrow +\infty} C_2^j = C_2^*$. Finally, due to the finite dimensional structure of system (2) and the objective function $J(u_1, u_2, u_3)$, (u_1^*, u_2^*, u_3^*) is an optimal control with corresponding states $P^*, M^*, H^*, C_1^*, Q^*$ and C_2^* .

Therefore $\inf_{(u_1, u_2, u_3) \in U_{ad}^3} J(u_1, u_2, u_3)$ is achieved. □

4. CHARACTERIZATION OF THE OPTIMAL CONTROLS

We apply the discrete version of Pontryagin’s Maximum Principle [2,3,8,11,16]. The key idea is introducing the adjoint function to attach the system of difference equations to the objective functional resulting in the formation of a function called the Hamiltonian. This principle converts the problem of finding the control to optimize the objective functional subject to the state difference equation with initial condition to find the control to optimize Hamiltonian pointwise (with respect to the control).

Now we have the Hamiltonian \hat{H} at time step k , defined by:

$$(7) \quad \hat{H}_k = H_k + C_{1,k} - Q_k + C_{2,k} + \frac{A_{1,k}u_{1,k}^2}{2} + \frac{A_{2,k}u_{2,k}^2}{2} + \frac{A_{3,k}u_{3,k}^2}{2} + \sum_{i=1}^6 \lambda_{i,k+1} f_{i,k+1}$$

where $f_{i,k+1}$ the right-hand side of the system of difference equations (2) of the i^{th} state variable at time step $k + 1$.

Theorem 2. *Given an optimal control $(u_1^*, u_2^*, u_3^*) \in U_{ad}^3$ and solutions $P_k^*, M_k^*, H_k^*, C_{1,k}^*, Q_k^*$ and $C_{2,k}^*$ of corresponding state system (2), there exist adjoint functions $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 satisfying the equations:*

$$\begin{aligned} \lambda_{1,k} &= \frac{\partial \hat{H}_k}{\partial P_k} = (\lambda_{2,k+1} - \lambda_{1,k+1})\beta_1 \frac{M_k}{N_k} + (\lambda_{5,k+1} - \lambda_{1,k+1})u_{1,k} + \lambda_{1,k+1}(1 - \mu). \\ \lambda_{2,k} &= \frac{\partial \hat{H}_k}{\partial M_k} = (\lambda_{2,k+1} - \lambda_{1,k+1})\beta_1 \frac{P_k}{N_k} + \beta_2 \lambda_{3,k+1} + \lambda_{2,k+1}(1 - \mu - \beta_2 - \alpha_4) \\ \lambda_{3,k} &= \frac{\partial \hat{H}_k}{\partial H_k} = 1 + \varepsilon u_{2,k} \lambda_{2,k+1} + \lambda_{3,k+1}(1 - \mu - \delta_1 - \alpha_1 - \alpha_2 - \alpha_3 - u_{2,k}) + \\ &\quad \alpha_1 \lambda_{4,k+1} + \alpha_2 \lambda_{5,k+1} + [(1 - \varepsilon)u_{2,k} + \alpha_3] \lambda_{6,k+1}. \end{aligned}$$

$$\begin{aligned}
(8) \lambda_{4,k} &= \frac{\partial \hat{H}_k}{\partial C_{1,k}} = 1 + \gamma_1 (\lambda_{5,k+1} - \lambda_{4,k+1}) + (1 - \mu - \delta_2) \lambda_{4,k+1} + u_{3,k} (\lambda_{5,k+1} - \lambda_{4,k+1}). \\
\lambda_{5,k} &= \frac{\partial \hat{H}_k}{\partial Q_k} = -1 + \gamma_2 \lambda_{6,k+1} + \lambda_{5,k+1} (1 - \mu - \gamma_2) \\
\lambda_{6,k} &= \frac{\partial \hat{H}_k}{\partial C_{2,k}} = 1 + \lambda_{6,k+1} (1 - \mu).
\end{aligned}$$

with the transversality conditions at time T

$$(9) \quad \lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 1, \lambda_4(T) = 1, \lambda_5(T) = -1 \text{ and } \lambda_6(T) = 1$$

Furthermore, for $i = 0, 1, \dots, T - 1$ we obtain the optimal control $(u_{1,k}^*, u_{2,k}^*, u_{3,k}^*)$ as

$$\begin{aligned}
u_{1,k}^* &= \min \left\{ \max \left(u_{1 \min}, \frac{(\lambda_{1,k+1} - \lambda_{5,k+1}) P_k}{A_{1,k}} \right), u_{1 \max} \right\} \\
u_{2,k}^* &= \min \left\{ \max \left(u_{2 \min}, \frac{(\lambda_{3,k+1} - \lambda_{5,k+1}) H_k + (\lambda_{5,k+1} - \lambda_{2,k+1}) \varepsilon H_k}{A_{2,k}} \right), u_{2 \max} \right\} \\
(10) \quad u_{3,k}^* &= \min \left\{ \max \left(u_{3 \min}, \frac{(\lambda_{4,k+1} - \lambda_{5,k+1}) C_{1,k}}{A_{3,k}} \right), u_{3 \max} \right\}
\end{aligned}$$

Proof. The Hamiltonian \hat{H}_k at time step k is given by

$$\begin{aligned}
(11) \quad \hat{H}_k &= H_k - C_{1,k} - Q_k - C_{2,k} + \frac{A_{1,k} u_{1,k}^2}{2} + \frac{A_{2,k} u_{2,k}^2}{2} + \frac{A_{3,k} u_{3,k}^2}{2} \\
&+ \lambda_{1,k+1} \left[b - \beta_1 \frac{P_k M_k}{N_k} + (1 - \mu) P_k - u_{1,k} P_k \right] \\
&+ \lambda_{2,k+1} \left[\beta_1 \frac{P_k M_k}{N_k} - \alpha_4 M_k + (1 - \mu - \beta_2) M_k + \varepsilon u_{2,k} H_k \right] \\
&+ \lambda_{3,k+1} \left[\beta_2 M_k + (1 - \mu - \delta_1 - \alpha_1 - \alpha_2 - \alpha_3) H_k - u_{2,k} H_k \right] \\
&+ \lambda_{4,k+1} \left[\alpha_1 H_k + (1 - \mu - \gamma_1 - \delta_2) C_{1,k} - u_{3,k} C_{1,k} \right] \\
&+ \lambda_{5,k+1} \left[\alpha_2 H_k + \gamma_1 C_{1,k} + (1 - \mu - \gamma_2) Q_k + u_{1,k} P_k + (1 - \varepsilon) u_{2,k} H_k + u_{3,k} C_{1,k} \right] \\
&+ \lambda_{6,k+1} \left[\gamma_2 Q_k + \alpha_3 H_k + \alpha_4 M_k + (1 - \mu) C_{2,k} \right].
\end{aligned}$$

For $k = 0, 1, \dots, T - 1$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's Maximum Principle, in discrete time, given in [2,3,8,11,16] such that

$$\begin{aligned}
 \lambda_{1,k} &= \frac{\partial \hat{H}_k}{\partial P_k}, \quad \lambda_1(T) = 0 \\
 \lambda_{2,k} &= \frac{\partial \hat{H}_k}{\partial M_k}, \quad \lambda_2(T) = 0 \\
 \lambda_{3,k} &= \frac{\partial \hat{H}_k}{\partial H_k}, \quad \lambda_3(T) = 1 \\
 \lambda_{4,k} &= \frac{\partial \hat{H}_k}{\partial C_{1,k}}, \quad \lambda_4(T) = 1 \\
 \lambda_{5,k} &= \frac{\partial \hat{H}_k}{\partial Q_k}, \quad \lambda_5(T) = -1 \\
 \lambda_{6,k} &= \frac{\partial \hat{H}_k}{\partial C_{2,k}}, \quad \lambda_6(T) = 1
 \end{aligned}
 \tag{12}$$

For $k = 0, 1, \dots, T - 1$, the optimal controls $u_{1,k}^*$, $u_{2,k}^*$ and $u_{3,k}^*$ can be solved from the optimality condition

$$\begin{aligned}
 \frac{\partial \hat{H}_k}{\partial u_{1,k}} &= A_k u_{1,k} - \lambda_{1,k+1} P_k + \lambda_{6,k+1} P_k = 0 \\
 \frac{\partial \hat{H}_k}{\partial u_{2,k}} &= B_k u_{2,k} - \lambda_{3,k+1} H_k + \lambda_{4,k+1} \varepsilon H_k + \lambda_{5,k+1} (1 - \varepsilon) H_k = 0 \\
 \frac{\partial \hat{H}_k}{\partial u_{3,k}} &= C_k u_{3,k} - \lambda_{2,k+1} \theta Q_k + \lambda_{6,k+1} \theta Q_k = 0
 \end{aligned}
 \tag{13}$$

So, we obtain:

$$\begin{aligned}
 u_{1,k} &= \frac{(\lambda_{1,k+1} - \lambda_{5,k+1}) P_k}{A_{1,k}} \\
 u_{2,k} &= \frac{(\lambda_{3,k+1} - \lambda_{5,k+1}) H_k + (\lambda_{5,k+1} - \lambda_{2,k+1}) \varepsilon H_k}{A_{2,k}} \\
 u_{3,k} &= \frac{(\lambda_{4,k+1} - \lambda_{5,k+1}) C_{1,k}}{A_{3,k}}
 \end{aligned}
 \tag{14}$$

By the bounds in U_{ad} of the controls, it is easy to obtain $u_{1,k}^*$, $u_{2,k}^*$ and $u_{3,k}^*$ in the form of (10). □

5. NUMERICAL SIMULATION

In this section, we shall solve numerically the optimal control problem for our $PMHC_1QC_2$ model. Here, we obtain the optimality system from the state and adjoint equations. The proposed optimal control strategy is obtained by solving the optimal system which consists of six difference equations and boundary conditions. The optimality system can be solved by using an iterative method. Using an initial guess for the control variables, $u_{1,k}$, $u_{2,k}$ and $u_{3,k}$, the state variables, P, M, H, C_1, Q and C_2 are solved forward and the adjoint variables λ_i for $i = 1, 2, 3, 4, 5, 6$ are solved backwards at times step $k = 0$ and $k = T$. If the new values of the state and adjoint variables differ from the previous values, the new values are used to update $u_{1,k}$, $u_{2,k}$ and $u_{3,k}$, and the process is repeated until the system converges.

The numerical solution of model (1) is executed using Matlab with the following parameter values and initial values of state variable: $P_0 = 600$, $M_0 = 200$, $H_0 = 100$, $C_{1,0} = 50$, $Q_0 = 20$, $C_{2,0} = 30$, $b = 300$, $N = 1000$, $\mu = 0.035$, $\beta_1 = 0.75$, $\beta_2 = 0.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.001$, $\alpha_3 = 0.001$, $\alpha_4 = 0.001$, $\gamma_1 = 0.001$, $\gamma_2 = 0.0001$, $\delta_1 = 0.002$, $\delta_2 = 0.002$.

We begin by presenting the solution evolution of our model (1) with and without controls that are represented in Figures 2.

The proposed control strategy in this work helps to achieve several objectives:

First objective: Protecting and preventing potential drinkers from falling into alcohol addiction

To realize this objective, we apply only the control u_1 i.e. the implementation of awareness, information and educational programs on potential drinkers and to make them know the risks of this phenomenon and the resulting health and social damages. Figure 2(a) shows that the number of moderate drinkers decreases from 2159 (without control u_1) to 1488 (with control u_1) at the end of the proposed program and the number of heavy drinkers decreases from 636 (without control u_1) to 438 (with control u_1) at the end of the proposed control strategy (see Figure 2(b)). Figure 2(c) shows that the number of heavy drinkers with physical complications (C_1) decreases from 4310 (without control u_1) to 2828 (with control u_1) at the end of the proposed control strategy. Also, we observe in Figure 2(d) that the number of individuals with

psychological complications decreases and has reached the value 53.6 (with control u_1) compared to the situation when there is no control 71 at the end of the proposed strategy. So, our objective has been achieved.

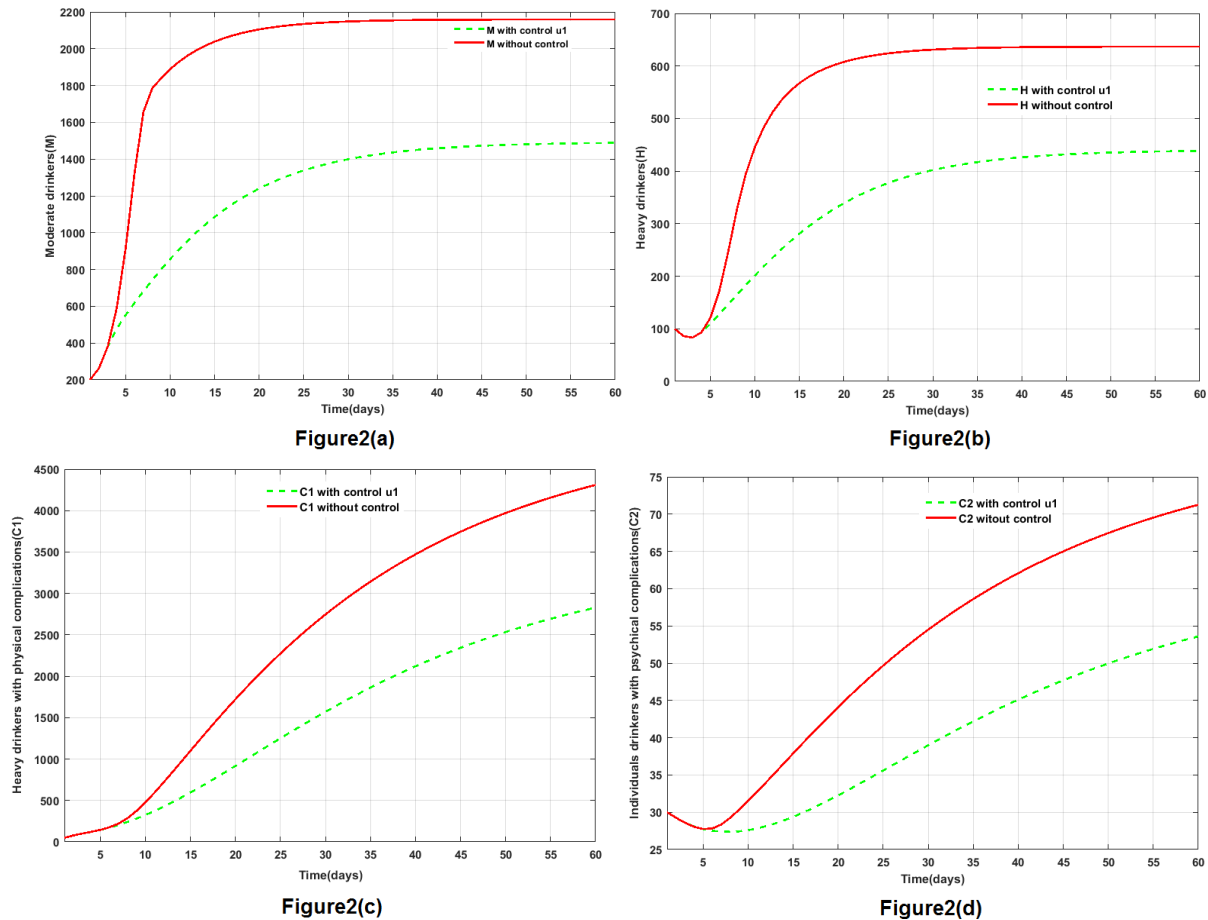


FIGURE 2. represents the drinkers class with and without control u_1

Second objective: Treatment for the addicted individuals.

To achieve this objective, we only use the control u_1 and u_2 i.e. treatment of the heavy drinkers. In Figure3(a), it is observed that there is a significant decrease in the number of moderate drinkers with control compared to a situation when there is no control where the decrease reaches 41.20% at the end of the proposed control strategy. Figure 3(b) shows that the number of the heavy drinkers decreased from 636.88 (without control) to 338.88 (with control) at the end of the proposed control. Figure 3(c) shows that the number of the heavy drinkers with physical complications C_1 decreased from 4310 (without control) to 2200 (with control) at

the end of the proposed control. Figure 3(d) shows that the number of the individuals with psychological complications decreased from 71.25 (without control) to 52.72 (with control) at the end of the proposed control. Figure 3(e) shows that the number of people who recovered and quitters of drinking without control increases and approaches a value of 97.39. It's increase appears with control u_1 and u_2 .

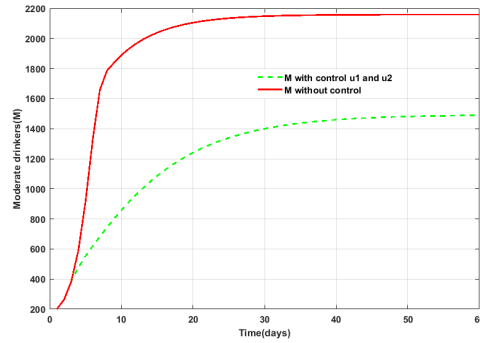


Figure3(a)

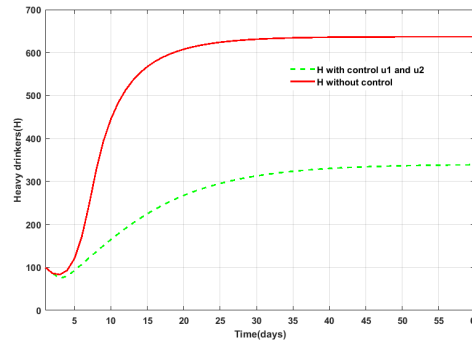


Figure3(b)

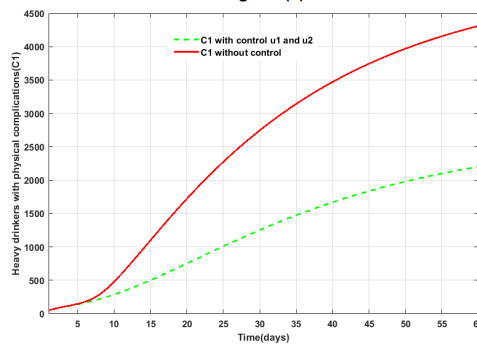


Figure3(c)

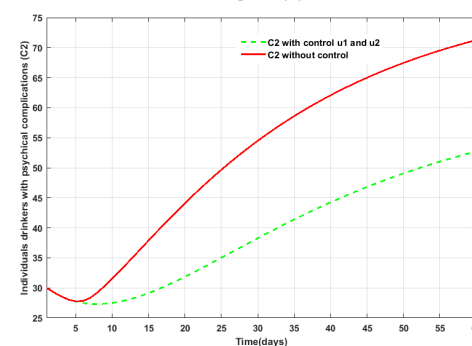


Figure3(d)

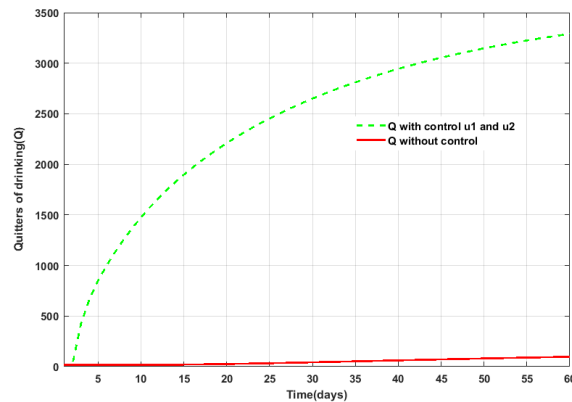


Figure3(e)

FIGURE 3. represents the drinkers class with and without control u_1 and u_2

Third objective: Prevention and treatment for the addicted individuals with and without complications.

To meet this objective, we use the controls u_1 , u_2 and u_3 i.e. awareness programs for the potential drinkers, treatment for heavy drinkers and treatment for heavy drinkers with physical complications (C_1). Figure 4(a) shows that the number of the heavy drinkers increases starting from the early days a value 636.88 (without controls) to 425.96 (with controls). Also, Figure 4(b) shows that the number of the heavy drinkers with physical complications (C_1) decreases from 4310 (without controls) to 291.42 (with controls). The number of the individuals with psychological complications decreases by 14% (See Figure 4(c)). Figure 4(d) depicts clearly an increase in the number of the recovered and quitters of drinking from 97.39 (without controls) to 5169 (with controls). As a result, the objective set before has been achieved.

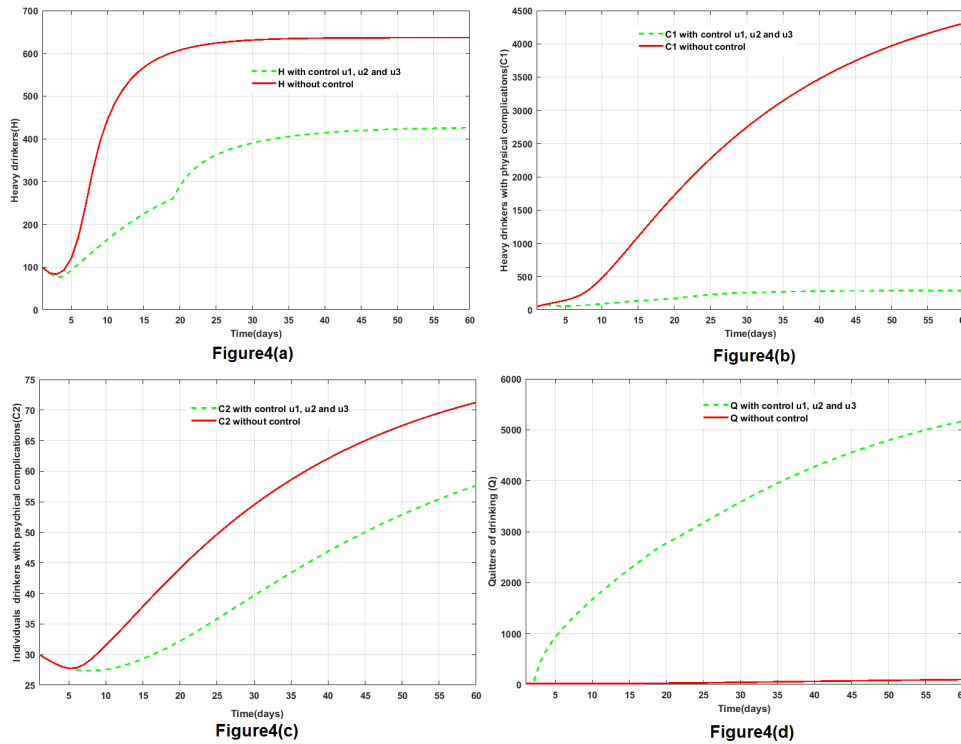


FIGURE 4. represents the drinkers class with and without optimal control u_1 , u_2 and u_3 .

6. CONCLUSION

In this research paper, we introduced a discrete modeling of drinking with and without complications for the purpose of minimizing the number of drinkers with and without complications and maximizing the number of recovered and quitters of drinking. Unlike some other previous models, we have taken into account the impact the treatment for physical complications on alcoholics. The results showed that treatment have substantial influence on the dynamics of alcoholism and can greatly impact the spread of the drinking, thus, it is crucial to urge people to know complications to recovered and quitters of drinking. We also presented three controls which represent awareness programs and treatment respectively. We applied the results of the control theory and we managed to obtain the characterizations of the optimal controls. The numerical simulation of the obtained results showed the effectiveness of the proposed control strategies.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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