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OSCILLATIONS IN SECOND ORDER LINEAR NEUTRAL IMPULSIVE DIFFERENTIAL EQUATIONS WITH CONSTANT DELAYS

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Abstract: Sufficient conditions are established for the oscillation of all solutions of a class of second order neutral differential equations with constant delay arguments and constant impulsive jumps for the cases where the coefficient p is a constant and when p is a variable. Examples are provided for clarity.

Keywords: second-order; impulsive; neutral delay differential equation; oscillation.

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1. INTRODUCTION

It is ascertained that, notwithstanding the presence of broad writing in these fields, an excellent deal of interest continues to be targeted on the oscillations of ordinary and neutral delay differential equations [1-12]. Lately, the examination of the oscillatory/nonoscillatory properties of the attention-grabbing neutral impulsive differential equations, which is an area of impulsive

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differential equations, has again caught the consideration of many applied mathematicians just as different researchers around the globe [13-22].

However, in spite of the large number of investigations of impulsive differential equations, their oscillation theory has not yet been fully elaborated, unlike the case of oscillation theory for delay differential equations. The monographs by Erbe *etal*, Gyori and Ladas, and Ladde *etal* [1, 2-3] contain excellent surveys of known results for delay and neutral delay differential equations. Moreso, unlike oscillatory theory of first order neutral impulsive differential equations, not much has been done in the area of oscillations of second order neutral delay impulsive differential equations. This work, therefore, aims at establishing sufficient conditions for the oscillation of all solutions of a certain type of linear neutral impulsive differential equations of the second order with delay of the form

$$\left[y(t)+py(t-\tau)\right]''+q(t)y(t-\sigma)=0,\ \ t\neq t_k \tag{1}$$

$$\Delta\left[y(t_k)+py(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0,\ \ \forall t=t_k. \tag{2}$$

Second order differential equations in general, are most important in applications. Same also applies to neutral second order delay impulsive differential equations which have been developed to model impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth. The introduction of oscillation and non-oscillation theory has further boosted the concept and particularly helped in identifying more areas of applications both within and outside differential equations. In particular, we use second order differential equations with impulse to understand the mathematical model for collision of viscoelastic bodies (see for e.g. [26]) and in impact theory in which an impact is an interaction of bodies which happens in a short period of time and can be considered as an impulse. Giving importance to such type of application, an attempt is made here to study the oscillation properties of (1) and (2).

A neutral delay impulsive differential equation of the second order is a differential system comprising a second-order differential equation and its impulsive conditions in which the highest-order derivative of the unknown function appears in the differential equation both with and without delays.

Now, the above definition becomes more meaningful if we define other related terms and concepts that will continue to be useful as we progress through the article.

It is known that the solution $y(t)$ for $t\in[t_0,T)$ of a given impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k\in[t_0,T)$.

Let $r\in\mathbb{N}$, $D:=[T,+\infty)\subset R$ and let $S:=\{t_k\}_{k\in N}$ be fixed. Subsequently, we will assume that the elements of the sequence S are the impulse moments and fulfil the condition that If $\{t_k\}$ is defined for all $k\in\mathbb{N}$, then $0<t_1<t_2<\cdots$ and $\lim_{k\to\infty} t_k=+\infty$.

We denote by $PC(D,R)$ the set of all functions $\varphi:D\to R$ which is continuous for all $t\in D,\ t\notin S.$ They are continuous from the left and have discontinuity of the first kind at the points for $t\in S$. By $PC^r(D,R),$ we denote the set of functions $\varphi:D\to R$ having derivative $\frac{d^j\varphi}{dt^j}\in PC(D,R),\ 0\le j\le r$ [23, 24].

A function $y:[-\rho,+\infty)\to R$ is said to be a solution of equation (1) and (2) with initial function $\varphi\in C\left([-\rho,0],R\right)$ if

i) $y(t)=\varphi(t)$ for $t_0-\rho\le t\le t_0$, $y(t)$ is continuously differentiable for $t\ge t_0$ and $t\ne t_k, k\in N, t\in R_+$ and $y(t)\in PC(R_+,R)$ satisfies (1) and (2) for all sufficiently large $t\ge 0;$

ii) $z(t)=y(t)+py(t-\tau)$ and $z'(t)$ are continuously differentiable for $t\ge t_0,\ t\ne t_k,\ t\ne t_k+\tau,$ $t\ne t_k+\sigma,\ k\in N$ and satisfies equation (1);

iii) $y(t_k^+), y(t_k^-), y'(t_k^+)$ and $y'(t_k^-)$ exist, $y(t_k^-)=y(t_k),\ y'(t_k^-)=y'(t_k)$ and satisfy equation (2),

where $\rho=max\{\tau,\sigma\}$. Here, $PC\left(R_+,R\right)$ is the set of all functions $\varphi:R_+\to R$ which are continuous for $t\in R_+, t\ne t_k, k\in N$ and continuous from the left side for $t\in R_+$ with discontinuity of the first kind at the points $t_k\in R,\ k\in N,$ and $\Delta z'(t_k)=z'(t_k+0)-z'(t_k-0),\ y(t_k-0)=y(t_k)$ and $y(t_k-\tau-0)=y(t_k-\tau),$ where $k\in N$ and $z(t_k)=y(t_k)+py(t_k-\tau)$.

A solution $y(t)$ of (1) and (2) is said to be regular if it is defined in some half line $\left[T_y,+\infty\right)\subset\left[t_0,+\infty\right)$ and $sup\left\{|y(t)|:t\geq T\right\}>0$ for all $T\geq T_y$.

As is customary, a nontrivial solution $y(t)$ of an impulsive differential equation is said to be finally positive (finally negative) if there exist $T\geq 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t\geq T$; non-oscillatory if it is either finally positive or finally negative, and oscillatory if it is neither finally positive nor finally negative [23, 25].

An impulsive differential equation is said to be oscillatory if all its solutions are oscillatory.

2. STATEMENT OF THE PROBLEM

We now return to the linear neutral delay impulsive differential equation of the second order under investigation:

$$\begin{cases}\left[y(t)+py\left(t-\tau\right)\right]''+q\left(t\right)y\left(t-\sigma\right)=0, & t\notin S, t\geq t_0\\ \Delta\left[y(t_k)+py\left(t_k-\tau\right)\right]'+q_k y\left(t_k-\sigma\right)=0, & \forall t_k\in S,\end{cases} \tag{3}$$

where $\Delta y(t_k)=0,\ t, t_k$ and $q_k\in R\ \forall\ k\in N$, $\sigma, \tau\in R_+=(0,+\infty)$, $q(t)\in C(R_+,R_+)$ and $t_1,t_2,\cdots t_k$ are the moments of impulse effect. Our aim is to establish some new sufficient conditions for the oscillation of all solutions of equation (1). In this paper, if there is no special claim, we say an equation or inequality holds if it is satisfied for sufficiently large *t*. Without loss of generality, we can deal only with positive solutions of equation (3).

The following is a basic lemma that is essential in carrying out our investigation. It is an extension of Lemma 4.4.5 on page 239 of the monograph by Erbe *etal* [1].

Lemma 2.1: Assume that $p\geq 0,\ q_k\geq 0$ and $q(t)\in C\left(R_+,R_+\right)$. Let $y(t)$ be a finally positive solution of equation (3). Set

$$x\left(t\right)=y\left(t\right)+py\left(t-\tau\right)$$

and

$$z(t) = x(t) + p\,x(t-\tau). \tag{4}$$

Then $z(t)>0,\ \ z'(t), \Delta z(t_k)>0$ and $z''(t),\ \Delta z'(t_k) \le 0$ and

$$\begin{cases} z''(t) + \dfrac{q^*(t)}{1+p}\, z(t-\sigma) \le 0,\ \ t \notin S \\ \Delta z'(t_k) + \dfrac{q_k^*}{1+p}\, z(t_k-\sigma) \le 0,\ \ \forall\ t_k \in S \end{cases} \tag{5}$$

finally, where $q^*(t) = min\,\{q(t), q(t-\tau)\}$ and $q_k^* = min\,\{q(t_k), q(t_k-\tau)\}$.

Proof: Suppose $y(t)>0$ for $t \ge t_0$. Then $x(t)>0$ for $t \ge t_0+\tau$, $x''(t),\ \Delta x'(t_k)<0$ for $t \ge t_1 = t_0 + max\,\{\sigma,\tau\}$ and $k: t_k \ge t_1 = t_0 + max\,\{\sigma,\tau\}$. Therefore, $x'(t),\ \Delta x(t_k) > 0$ for $t \ge t_1$ and $k: t_k \ge t_1$. Then

$$z''(t) = x''(t) + px''(t-\tau) \le -q^*(t)\left[y(t-\sigma) + p\,y(t-\tau-\sigma)\right]$$

$$= -q^*(t)\,x(t-\sigma) \tag{6}$$

and

$$\Delta z'(t_k) = \Delta x'(t_k) + p\Delta x'(t_k-\tau) \le -q_k^*\left[y(t_k-\sigma) + py(t_k-\tau-\sigma)\right]$$

$$= -q_k^* x(t_k-\sigma). \tag{7}$$

Similar to the above, we have $z(t)>0,\ z'(t),\ \Delta z(t_k)>0$ and $z''(t),\ \Delta z'(t_k) \le 0$ for $t \ge t_2 \ge t_1,\ \forall\ k: t_k \ge t_2 \ge t_1$ and

$$z''(t) \le -q^*(t)\,x(t-\sigma) \le -\frac{q^*(t)}{1+p}\left[x(t-\sigma) + px(t-\sigma-\tau)\right] = -\frac{q^*(t)}{1+p}\,z(t-\sigma),$$

and

$$\Delta z'(t_k) \le -q_k^*\, x(t_k-\sigma) \le -\frac{q_k^*}{1+p}\left[x(t_k-\sigma) + px(t_k-\sigma-\tau)\right] = -\frac{q_k^*}{1+p}\,z(t_k-\sigma).$$

This completes the proof of Lemma 2.1.

3. MAIN RESULTS

Theorems 3.1 and 3.2 are extensions of the neutral delay versions of Theorem 4.4.3 and Theorem 4.4.4 found on pages 240 and 241, respectively, as identified in the monograph by Erbe *et al* [1].

Theorem 3.1: Let $p>0,\ q_k \geq 0$ and $q(t)\in PC(R_+,R_+)$. Assume that the second order impulsive ordinary differential equation

$$\begin{cases} x''(t) + \lambda q(t)\dfrac{t-\sigma}{t}x(t) = 0, & t \notin S \\ \Delta x'(t_k) + \lambda q_k \dfrac{t_k-\sigma}{t_k}x(t_k) = 0, & \forall\, t_k \in S \end{cases} \tag{8}$$

is oscillatory for some $\lambda \in (0,1)$. Then every solution of equation (3) is oscillatory.

Proof: Let us assume by contradiction that there exists a finally positive solution $y(t)$ of equation (3) and $z(t)$ is defined by equation (4). Then $z(t)$ satisfies all conditions of Lemma 2.1. Consequently, for every $\ell\in(0,1)$, there exists a $t_\ell \geq 0$ such that

$$\begin{cases} z(t-\sigma) \geq \ell\dfrac{t-\sigma}{t}z(t), & \text{for } t \geq t_\ell,\ t \notin S \\ z(t_k-\sigma) \geq \ell\dfrac{t_k-\sigma}{t_k}z(t_k), & \text{for } t_k \geq t_\ell,\ \forall\, t_k \in S. \end{cases} \tag{9}$$

By Lemma 2.1, inequality (5) is true. Combining inequalities (5) and (9), we obtain

$$\begin{cases} z''(t) + \dfrac{\ell(t-\sigma)\,q^*(t)}{(1+p)\,t}z(t) \leq 0, & \text{for } t \geq t_\ell,\ t \notin S \\ \Delta z'(t_k) + \dfrac{\ell(t_k-\sigma)\,q_k^*}{(1+p)\,t_k}z(t_k) \leq 0, & \text{for } t_k \geq t_\ell,\ \forall\, t_k \in S, \end{cases} \tag{10}$$

which implies that

$$\begin{cases} z''(t) + \dfrac{\ell(t-\sigma)\,q^*(t)}{(1+p)\,t}z(t) = 0, & t \notin S \\ \Delta z'(t_k) + \dfrac{\ell(t_k-\sigma)\,q_k^*}{(1+p)\,t_k}z(t_k) = 0, & \forall\, t_k \in S \end{cases} \tag{11}$$

has a non-oscillatory solution. This contradicts our initial assumption and, thus, completes the proof of Theorem 3.1.

We now consider the case of the linear equation (3) with variable coefficient p as follows:

$$\begin{cases} \left[y(t)+p(t)\,y(t-\tau)\right]'' + q(t)y(t-\sigma)=0, & t \ge t_0,\ t \notin S \\ \Delta\left[y(t_k)+p_k y(t_k-\tau)\right]' + q_k y(t_k-\sigma)=0, & t_k \ge t_0,\ \forall\, t_k \in S. \end{cases} \tag{12}$$

Theorem 3.2: Assume that

i) $\tau > 0,\ \sigma > 0,\ p_k > 0;$

ii) $q(t) \in PC(R_+, R_+)$ and there exist a constant q_0 such that $q(t) \ge q_0 > 0$ for $t \in [t_0, \infty)$;

iii) $p(t) \in PC^1(R_+, R)$ and there exist constants p_1 and p_2 such that $p_1 \le p(t) \le p_2$ for $t \in [t_0, \infty)$ and $p(t)$ is not finally negative.

Then every solution of equation (12) is oscillatory.

Proof: By contradiction, we assume that $y(t)$ is a finally positive solution of equation (12) and define

$$z(t) = y(t) - p(t)y(t-\tau)$$

We see that $z(t)$ takes on non-negative values finally. From equation (10), we have that

$$\begin{cases} z''(t) = -q(t)y(t-\sigma) \le -q_0 y(t-\sigma) < 0, & t \notin S \\ \Delta z'(t_k) = -q_k y(t_k-\sigma) \le -q_0 y(t_k-\sigma) < 0, & t_k \in S \end{cases} \tag{13}$$

which implies that $z'(t)$ is a strictly decreasing function of t and so $z(t)$ is a strictly monotone function. From the above observations it follows that either

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = -\infty$$

or

$$\lim_{t\to\infty} z'(t) = \ell \text{ is finite.} \tag{14}$$

Let us assume that condition (14) holds. Integrating both sides of inequality (13) from t_0 to t and letting $t \to \infty$, we obtain

$$\int_{t_0}^{\infty} q_0 y(s-\sigma)ds + \sum_{t_0 \le t_k < \infty} q_0 y(t_k - \sigma) \le z'(t_0) - \ell .$$

Hence,

$$\int_{t_0}^{\infty} q_0 y(s-\sigma)ds < \infty, \ \sum_{t_k \ge t_0} \Delta z'(t_k) < \infty,$$

which implies that $y(t) \in L_1[t_0, \infty)$ and so $z(t) \in L_1[t_0, \infty)$, where L_1 is the space of all Lebesgue integrable functions on $[t_0, \infty)$. Since $z(t)$ is monotone, it follows that

$$\lim_{t\to\infty} z'(t) = \lim_{t\to\infty} z(t) = 0 \tag{15}$$

and therefore $\ell = 0$. Finally, by conditions (14) and (15) with $\ell = 0$ and the decreasing nature of $z'(t)$, we conclude that $z(t) < 0$ and $z'(t) > 0$. In particular, $z(t) < 0$ finally, which contradicts condition (iii) of Theorem 3.2 and thus completes the proof.

We now examine the following illustrations:

Example 3.1: Consider the equation

$$\begin{cases} \left[y(t) + \left(\frac{1}{2} - \sin t\right) y(t-2\pi) \right]'' + \left(\frac{3}{2} - \sin t\right) y(t-4\pi) = 0, \ t \ge 0, \ \ t \notin S \\ \Delta \left[y(t_k) + \left(\frac{1}{2} - \sin t_k\right) y(t_k - 2\pi) \right]' + \left(\frac{3}{2} - \sin t_k\right) y(t_k - 4\pi) = 0, \ t_k \ge 0, \ \forall \ t_k \in S. \end{cases} \tag{16}$$

It is easy to see that the assumptions of Theorem 3.2 are satisfied here. Therefore, every solution of equation (16) oscillates. For instance, $y(t) = \frac{\sin t}{\frac{3}{2} + \sin t}$ is one of such solutions.

The following illustration shows that if the hypothesis (ii) of Theorem 3.2 is violated, the result may be wrong.

Example 3.2: Consider the equation

$$\begin{cases} \left[y(t) + (t-1)^{-\frac{1}{2}} y(t-1) \right]'' + \frac{1}{4} t^{-\frac{3}{2}} (t-2)^{-\frac{1}{2}} y(t-2) = 0, \ \ t \ge 2, \ t \notin S \\ \Delta \left[y(t_k) + (t_k - 1)^{-\frac{1}{2}} y(t_k - 1) \right]' + \frac{1}{4} t_k^{-\frac{3}{2}} (t_k - 2)^{-\frac{1}{2}} y(t_k - 2) = 0, \ t_k \ge 2, \ \forall \ t_k \in S. \end{cases}$$

All assumptions of Theorem 3.2, except (ii) are satisfied. Note, however, that $y(t)=t^{\frac{1}{2}}$ is a non-oscillatory solution.

The following corollary which is an extension of Corollary 4.4.3 on page 240 of the monograph by Erbe et al. [1] also holds true.

Corollary 3.1: Let $p>0,\ q_k\geq 0$ and $q\in PC(R_+,R_+)$. Then every solution of equation (3) is oscillatory if for some $\alpha\in(0,1)$,

$$\int_0^\infty t^\alpha q(t)\,dt+\sum_{0\leq t_k<\infty} t_k^\alpha q_k=\infty.$$

4. CONCLUSION

It has become imperative in recent times to determine the properties of the solutions of certain mathematical equations from the knowledge of associated equations. In this work, we have made an effort to study the oscillation properties of a class of neutral delay differential equations with impulse of the form (3) by establishing a comparison theorem which compares the neutral impulsive differential equation (3) with the impulsive ordinary differential equation (6), in the sense that every oscillation criterion for the impulsive ordinary differential equation (6) becomes an oscillation criterion for the neutral impulsive differential equation (3). The formulated comparison theorem essentially simplifies the examination of the oscillatory properties of equation (3) and enables us also to eliminate some conditions imposed on the given problem.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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