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SOLUTION METHODS FOR INTEGRAL EQUATIONS - A SURVEY

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Abstract: The theory of integral equations has been an active field of research for many years and is inextricably related with other areas of Mathematics such as complex and mathematical analysis, function theory, integral transforms and functional analysis. Integral Equations arise naturally in applications, in many areas of Mathematics, Engineering, Science and Technology and have been studied extensively both at the theoretical and practical level. It is significant to note that a MathSciNet keyword search on Integral Equations returns more than eleven thousand items. In this paper, we do a brief survey of the existing literature on methods of solving integral equations of Volterra and Fredholm type of the first, second and third kind, Cauchy type singular integral equations and integral equations over an infinite interval. The objective is to classify the selected methods and evaluate their applicability while discussing challenges faced by individual researchers in this field. We also provide a rather extensive bibliography for the reader who would be interested in learning more about various theoretical and computational aspects of Integral Equations.

Keywords: fredholm; volterra; integral equations; linear; non-linear.
2010 AMS Subject Classification: 45B05.

1. INTRODUCTION

Integral equations have been one of the essential tools for various areas of applied mathematics and they occur naturally in many fields of science and engineering [152]. Integral equations are encountered in a variety of applications which include continuum mechanics, potential theory,

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geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, fracture mechanics, and radiative heat transfer problems. In fact, very recently, due to increasing usage of stochastic integral equations in applicable problems, the need to extend the numerical solution for this type of equation has been felt. In particular, stochastic integral equations characterized by fractional Brownian motion have been studied ([163], [164]).

An integral equation is a functional equation in which the unknown function appears under one or several integral signs. In an integral equation of Volterra type for instance, the integrals containing the unknown function are characterized by a variable upper limit of integration. To be more precise, an integral equation of the form

$$\lambda\int_a^x k(x,s)y(s)ds = f(x) \tag{1}$$

is called a linear Volterra integral equation of the first kind and that of the form

$$y(x)-\lambda\int_a^x k(x,s)y(s)ds = f(x) \tag{2}$$

is called a linear Volterra integral equation of the second kind. Here, x, s and a are real numbers, λ is a parameter, $y(s)$ is an unknown function, while $f(x)$ and $k(x,s)$ are given functions which are square integrable on $[a,b]$ and in the domain $a \le x \le b, a \le s \le x$, respectively. The function $f(x)$ is called the free term, while the function $k(x,s)$ is called the kernel.

Volterra equations may be regarded as a special case of Fredholm equations with the Kernel $k(x,s)$ defined on the square $a \le x \le b, a \le s \le b$ and vanishing on the triangle $a \le x < s \le b$.

For example

$$\int_a^b k(x,s)y(s)ds = f(x) \tag{3}$$

is called a linear Fredholm integral equation of first kind or

$$y(x)-\int_a^b k(x,s)y(s)ds = f(x) \tag{4}$$

is called a linear Fredholm integral equation of the second kind ([120], [91]). If $f(x)=0$ in equation (2), then the equation is called homogenous, otherwise it is called non homogenous. For homogenous equations, λ is an eigenvalue, because in such cases the integral equations present eigenvalue problems in which the objective is to determine those values of λ called eigenvalues for which the integral equations possess non trivial solutions called eigen-functions.

If the kernel $k(x,s)$ is continuous, then the integral equation is said to be non-singular. If the range of integration is infinite, or if the kernel violates the above conditions, then the equation is said to be singular.

The solution of an integral equation of any type is to find the unknown function $y(s)$ satisfying that equation. In many cases however, the determination of the solution of an integral equation by analytical techniques is out of question, and a straight forward numerical approach, restrictive though to a class of well-posed integral equations, is to replace the integral equation by a set of linear algebraic equations solved by any matrix methods. On the other hand, an attempt to solve Volterra integral equations of the first kind using the above numerical approach may not be fruitful after all. This is so, because Volterra integral equations of the first kind, in a sense appear to be situated mid-way between Volterra integral equations of the second kind and those of Fredholm integral equations of the first kind. Precisely, if a Volterra integral equation of the second kind is well-posed and can effectively be solved by any classical means, a Fredholm integral equation of the first kind is ill-posed, given any preconceived functional space solvable only by special approximate methods, and Volterra/Fredholm integral equation of the first kind, may, either be well-posed or ill-posed depending upon the choice of the solution space and the nature of the technique used ([96],[90]). It is worthy to note that Fredholm integral equations of the first kind are often ill-posed problems that may have no solution, or if a solution exists, it is not unique and may not depend continuously on the data $f(x)$ ([143],[144],[108]). Also, first kind Volterra integral equations are not ill-posed problems since they can be easily converted to second kind Volterra integral equations which always have unique solutions.

The ill-posedness [142] in most equations of the form (1) is the origin of frequent difficulties when dealing with methods for solving equation (1) numerically. The trouble with classical solution is that a discretization process transforms equation (1) into another equation (often a system of linear equations which may be solved by well-known singular value decomposition) which can lead to solutions that may deviate strongly from the (minimum norm) solution of equation (1).

The conditionalities for well-posed problems include the following

(i) Existence of solution;

(ii) The uniqueness of such a solution and

(iii) The stability of the solution.

A number of methods have been analyzed thoroughly based on these conditionalities, but much were done under the assumption that the kernel $k(x,s)$ and the right-hand side or free function $f(x)$ are known without error and that the approximating equation can be exact. When, as is often the case, equation (1) arises in the analysis of experimental data, these assumptions may not manifest. As was pointed out by [84], appreciable perturbations in $f(x)$ can make the standard numerical methods useless. We also note that, with the exception of the work on simple Abel's equation

[82], the problem of solving equation (1) in the presence of data error has received some unprecedented attention. This is attested to in the following monographs: ([34], [53], [147], [86]).

2. METHODS OF SOLVING VOLTERRA INTEGRAL EQUATIONS

It is known that if the kernel $k(x,s)$ is a continuous function in the domain $R\{a \le s \le x \le b\}$, and *f(x)* is continuous in the interval *[a,b]*, then the integral equation (4) has a unique solution for any λ [82]. The methods of solving equation (4) use quadrature rules some of which include repeated trapezoidal rule, Newton-Cotes, Clenshaw-Curtis, and Simpson's rule. Recently, other methods for solving equation (4) have been introduced. These include the power series method and Monte Carlo method for system of linear Volterra integral equations of the second kind. For non-linear Volterra integral equations,the methods of solution include the quadrature rule, Adomian decomposition technique for Volterra-Fredholm integral equations and recently, an iterative scheme based on the homotopy analysis method (HAM) has been used to solve non-linear Volterra integral equations. These methods can as well be applied in solving both linear Volterra and Fredholm integral equations of the first and second kinds.

Linear Volterra integral equations of the first kind: The identification of various important problems in electrical engineering, in modeling of dynamic impulse systems, and in non-linear dynamic system identification can be treated in terms of equation (1) which does not have classical continuous solutions [8]. Let us now consider equation (1), the linear Volterra integral equation of the first kind. Efforts in solving equation (1) numerically date back to 1953, when Fox and Goodwin used finite difference methods, although equation (1) was not treated explicitly. However, [63] was concerned with the study of the trapezoidal rule for solving first kind Volterra integral equation with convolution kernels

$$\int_0^x k(x-s)\, g(s)\, ds = f(x), \tag{5}$$

and noted that the solutions obtained oscillated about the exact solution. [69] gave a convergence argument for the trapezoidal rule. However, [79][80] considered several finite difference methods and showed that the mid-point rule was convergent and that high order Newton-Gregory formulae were not. Linz's work was important, because of the catalytic effect it had on other researchers. However, [153] produced high accuracy block-by-block methods which proved convergent. This was followed by other researchers, notably, [48] who displayed six interpolatory quadrature rules which yielded convergent schemes up to order six. However, [46] developed a new class of quadrature methods for solving equation (5) based on the following assumptions:

C 2.1 *f* and *k* are continuously differentiable to sufficiently high order based on their arguments on $S_1 = \{x | 0 \le s \le x \le a\}$, $(a < \infty)$ and $S_2 = \{(x, S) | 0 \le s \le x \le a\}$, respectively.

C 2.2 $f(0) = 0$; and

C 2.3 $k(x,x) \neq 0$ for all $x \in S$.

Holyhead, Mckee and Taylor (1975)[56] propounded a general concept of stability with an associated root condition. In this study, a general theorem demonstrates that consistency plus stability imply convergence. These results although aimed at cyclic interpolatory-type methods are really quite general and essentially subsume the results obtained by [57]. Furthermore, [56] introduced the concept of weak stability and tackled the convergence problem using generating functions as the essential tools. Taylor [135], in this interesting paper derived stable methods by "inverting" backward differentiation formulae. [67] derived a semi-explicit third order method (which may be viewed as a Rung-Kutta method) while Andrade, [6] derived a fourth and sixth order stability method for linear Volterra integral equations, the latter having all its zeros of its associated polynomial at the origin. [94] unified the two papers by [56] and [58] under the assumption that consistency could be expressed as an asymptotic expansion and in a further study [6] considered the problem of solving first kind Volterra integral equation (5) directly when $k(s,s) \equiv 0$. Again, [156] considered reducible quadrature methods also based on the above assumptions. [47] constructed families of methods depending on free parameters for the solution of equation (5). These parameters are restricted to certain regions so that a certain polynomial satisfies both stability and a consistency condition. This is an optimal choice if the free parameters were outlined in order that the $L_2 - norm$ of the roots of the polynomial was minimized.

Recently, [129] studied equation (5) in terms of generalized functions. A generalized solution is the basis of mathematical models formulated in terms of impulses theory [162]. Various well known electrical engineering problems can be formulated in terms of impulse theory [29]. The solutions consist of singular and regular components which can be constructed separately. The singular component is constructed as solutions of the special linear algebraic system while the regular component is constructed as solutions of special Volterra integral equation of the third kind. Most recently, [1] proposed a new method called the homotopy analysis method (HAM) for solving the first and second kinds linear Volterra integral equations. This method was applied to solve different test problems with known exact solutions and the numerical solutions obtained confirm the validity of the numerical method and suggest that it is an interesting and viable alternative to existing numerical methods for solving the problem under consideration. The homotopy analysis (HAM) was first introduced by Liao ([76]; [77]). In this method, the solution is considered to be the summation of an infinite series, which usually converge rapidly to the exact solution. The HAM is based on homotopy, a fundamental concept in topology and differential geometry.

We shall consider another version of equation (5) expressed in the form:

$$\int_a^x \frac{k(x-s)}{(x-s)^\alpha} y(s)\, ds = f(x) \;, 0 < \alpha < 1,\; f(\alpha) = 0 \tag{8}$$

called Abel's integral equation of the first kind. Observe that equation (8) is equivalent to equation (5) with $\alpha = 0$. The application of product integration methods for solution of Volterra integral equations was carried out by [160] although he did not explicitly advocate them for first kind equations. [81] was probably the first researcher to suggest the use of such methods. He presented some high order methods and outlined the convergence arguments. The product mid-point and the trapezoidal rule were first theoretically justified by ([154], [153]). However, Weiss was only able to prove convergence for $\alpha \in [0.1292,\ 1]$, while [30] indeed showed that the product trapezoidal rule was convergent for all $\alpha \in (0,\ 1)$. [14] proved the convergence of the product mid-point and trapezoidal rule for the equation with the kernel $k(x,s)(x-s)^{-\alpha}(x+s)^{-\beta}, 0<\alpha,\ \beta<1.$ [10] considered and proved the convergence of the product trapezoidal rule for the equation with the kernel $k(x,s)(x^2-s^2)^{-\frac{1}{2}}$. However, [57] showed that his general analysis of equation (5) could be extended to equation (8) and that his concepts of stability and weak stability were relevant. Furthermore, [19] considered two families of methods, the so called implicit and explicit backward difference product integration methods (IBDPIM'S and EBDPIM'S). From a simple sufficient condition he was able to determine theoretically the precise range of α for which the IBDPIM's are convergent.

Kosarev (1973)[72] considered the case of $\alpha = \frac{1}{2}$ and $k(x,s) \equiv 1$, in which equation (8) is reduced to the form

$$\int_0^a \frac{y(s)}{\sqrt{(x-s)}}\,ds = f(x),\ 0 \le x \le a\,. \tag{9}$$

He presented a method for the calculation of the unknown function *y(x)* which takes into account both the statistical properties of the function *f(x)*, connected with the errors of measurement, and also the analytic properties of Abel's transformation, expressed by equation (9). This method is based on the expansion of the unknown function *y(x)* in eigen-functions of the integral operators

$$Ay = \frac{1}{2\sqrt{x}}\int_0^x \frac{y(s)}{\sqrt{x-s}}\,ds, \tag{10}$$

which are the power functions

$$y_n(x) = x^n, \quad n = 0,1,2,\cdots\ . \tag{11}$$

The corresponding eigenvalues were calculated using the recurrence formula

$$\lambda_0 = 1, \quad \lambda_n = \frac{\lambda_{n-1}}{\left(1+\frac{1}{2n}\right)}, \quad n = 1,2,\cdots\ . \tag{12}$$

Recently, [159] have constructed high accuracy mechanical quadrature rule for solving equation (8). In order to avoid the ill-posed nature of the problem, the first kind Abel integral equation was transformed into the second kind Volterra integral equation with a continuous kernel and a smooth right-hand term expressed by weakly singular integrals. From the periodization method and modified trapezoidal integration rule, not only high accuracy approximation of the kernel and the right-hand side term could be easily computed, but also two quadrature algorithms for solving first kind Abel integration equation were proposed, which have the high accuracy $0\left(h^2\right)$ and asymptotic expansion of the errors. From the Richardson extrapolation, an approximation with higher accuracy order $0\left(h^3\right)$ was obtained. In addition, an aposteriori error estimate for the algorithms was derived.

Linear Volterra integral equations of the second kind:

We shall consider the second kind Volterra integral equations of the form

$$y(x) = f(x) + \int_a^x k(x,s)\, y(s)\, ds, \qquad a \le x \le b, \tag{13}$$

where the function *f(x)* and the regular kernel $k(x,s)$ are given, and *y(s)* is the unknown function to be determined. This integral equation is a mathematical model of many evolutionary problems with memory from biology, chemistry, engineering ([83]; [27]).

In recent years, there are several numerical techniques using quadrature rules such as repeated trapezoidal rule, Newton-Cotes, Clenshaw-Curtis, and Simpsons rule. Other methods include the power series method and Monte Carlo method.

However, the convection dominated problem always encountered in approximation of equation (13) is the spontaneous formation of non-smooth micro scale features which pose a challenge for high resolution computations. To overcome this problem, [98] introduced a modified method based on the Simpson’s quadrature formula. The idea is to approximate the solution of equation (13) in even number of equally spaced points (or a given mesh).

[134] presented a numerical method for the solution of equation (13) based on power series method. The proposed method provides the Taylor expansion of the exact solution of the integral equation using simple computation with quite acceptable approximate solution. For equations with polynomial solutions, the proposed method gave exactly the same solutions as the analytical method.

It is, in general, very difficult to find a useful solution of a linear Volterra integral equation of the second kind if the solution depends on several variables or if the equation is coupled with other integral equations. Suppose that equation (13) is a system of linear Volterra integral equations of the second kind where,

$y(x)=\left(y_1(x),\cdots,y_m(x)\right)^T$,

$f(x)=\left(f_1(x),\cdots,f_m(x)\right)^T$,

$k(x,s)=\left[k_{ij}(x,s)\right],\quad i=1,2,\cdots,m,\quad j=1,2,\cdots,m,$

and $f(x),k(x,s)$ are known functions and $y(x)$ is to be determined. Well known methods of solution are mostly ineffective because the amount of computation involved is too great, even for the latest machines. In many cases, however, especially in particular transport problems, a statistical procedure - the Monte-Carlo method can be used to find a solution which is sufficiently accurate for practical purposes [54]. The application of a well known Monte-Carlo method for the solution of an equation with non-negative kernel has been illustrated by ([74], [124], [54]). The idea by these authors was further developed by [117] and applied to the solutions of systems of equation (13). The result obtained showed the effectiveness of the method. The Monte-Carlo method is developed as iteration technique where the approximation is over a finite interval for the unknown function. At each step, there is a new partition for the interval and such partition comes from the generation of random numbers on the interval.

Linear Volterra integral equations of the third kind: It is worthy of note that the Volterra integral equation of the first kind can be transformed into Volterra integral equation of the third kind when the function $k(x,s)$ has zero outside the interval $[0,a]$. This case has been investigated by only a few researchers [142] based on the general Volterra integral equation of the third kind of the form

$$g(x)\,y(x)-\lambda\int_0^x k(x,s)\,y(s)\,ds=f(x),\ g(0)=0\ , \tag{14}$$

where λ is a real parameter. [121] demonstrated the existence of a one parameter family of solution of equation (14) in the case when $g(x)=x,\lambda=-1$. These results were generalized to a system of such equations. The most interesting results were obtained by [105], when questions of the solvability of equation (14), and consequently that of equation (5) in a specially introduced Banach space $C_\beta(w)$ for an arbitrary function $g(x)\geq 0$ were considered. However, the structure of the space $C_\beta(w)$, constructed only for the function $g(x)$, and the method of proof, led to the isolation of the family of solutions. This was due to the fact that the dependence of the behaviour of the solutions of equation (14) on the behaviour of the function $k(x,s)$ in the neighborhood of zero was not completely taken into account. [87] constructed the Banach space B_y^{α} and $M_{q,y}^{x,\alpha}$, a special form multiparameter families of solutions of equation (5) directly, and for a wide class of equations of the form (14), obtained similar results as corollary. Other references for works on Linear Volterra integral equations of the 3rd kind are ([122], [38], [41], [42], [107], [18]).

Nonlinear Volterra integral equations of the second kind: Consider the non-linear Volterra integral equations of the second kind,

$$y(x) = f(x) + \int_0^x k(x,s,y(s))\,ds, \qquad x \in I := [a,b], \tag{15}$$

where the kernel *k* is at least continuous on $s\ x\ R^n,\ S := \{(x,s) : a \le s \le x \le b\}$, and that the solution *y* exists uniquely and is continuous on *I*. The general Volterra-Rung-Kutta methods have been employed to solve equation (15).

In order to introduce the discretization of equation (15) by (implicit or explicit) Rung-Kutta methods, let $x_n = a + nb, \quad n = 0,1,2,\cdots,N$, with $h = \frac{b-a}{N}(N \ge 1)$, and denote by y_n any approximation to $y(x_n)$. Furthermore, define

$$F_n(x) := f(x) + \int_a^{x_n} K(x,s,y(s))\,ds,\ x \ge x_n \quad (n = 0,1,\cdots,N-1) \tag{16}$$

and let $\tilde{F}_n(x)$ be an approximation to $F_n(x)$. An *m*-stage (implicit) Volterra Rung-Kutta method for equation (16) has been given by Aparo (1959) to be

$$\begin{cases} Y_i^{(n)} = \tilde{F}_n(x_n + \theta_i h) + h\sum_{i=1}^{m} a_{ij} k\left(x_n + d_{ij} h_j x_n + c_j h, Y_j^{(n)}\right) \\ y_{n+1} = Y_{m\neq 1}^{(n)} = \tilde{F}_n(x_n + h) + h\sum_{i=1}^{m} b_i k\left(x_n + e_i h_i x_n + c_i h, Y_j^{(n)}\right) \\ i = 1,\cdots m, \end{cases} \tag{17}$$

where

$$c_i = \sum_{j=1}^{m} a_{ij} \qquad (i = 1,\cdots,m). \tag{18}$$

The method for equation (17) is completely characterized by the parameters $a_{ij}, d_{ij}, b_i, e_i, \theta_i$. The two terms on the right-hand side of equation (17) are called the lag term and the Runge-Kutta part of the Rung-Kutta method. Let us consider two special subclasses of this method:

(a) The Pouzet-type method (PRK-method) which was proposed by [111].

The method is as follows:

If $d_{ij} = c_i \quad (i, j = 1, \cdots, m), \quad e_i = 1, \quad \theta_i = c_i \left(i = 1, \cdots, m\right)$, we obtain

$$\begin{cases} Y_i^{(n)} = \tilde{F}_n\left(x_n + c_i h\right) + h\sum_{i=1}^{m} a_{ij} k\left(x_n + c_{in}, x_n + c_j h, Y_j^{(n)}\right) \\ y_{n+1} = Y_{m\neq 1}^{(n)} = \tilde{F}_n\left(x_n + h\right) + h\sum_{i=1}^{m} b_i k\left(x_n + h, x_n + c_i h, Y_j^{(n)}\right) \\ i = 1, \cdots m \end{cases} \tag{19}$$

This is the implicit version of Pouzet's Runge-Kutta method for equation (15); in the explicit case the upper limit of the summation is replaced by I=1 in the first formula of equation (19). In order that the argument of k in equation (19) lies in $S \times R^n$. It is expected that

$$c_i \geq c_j \quad if \quad a_{ij} \neq 0 . \tag{20}$$

For explicit methods this condition is satisfied if $c_1 \leq c_2 \leq \cdots \leq c_m \leq 1$. Equation (20) is referred to as the kernel condition.

(b) The Bel'tyukov-Type method (BRK-method)which was introduced by [13]. It is as follows:

If $d_{iij} = e_j \left(i, j = 1, \cdots, m\right), \quad \theta_i = c_i \left(i = 1, \cdots, m\right)$, then

$$\begin{cases} Y_i^{(n)} = \tilde{F}_n\left(x_n + c_i h\right) + h\sum_{i=1}^{m} a_{ij} k\left(x_n + e_{\backslash j} h, x_n + c_j h, Y_j^{(n)}\right) \\ y_{n+1} = Y_{m\neq 1}^{(n)} = \tilde{F}_n\left(x_n + h\right) + h\sum_{i=1}^{m} b_i k\left(x_n + e_i h, x_n + c_j h, Y_i^{(n)}\right) \end{cases} \tag{21}$$

For this type of method, the kernel condition is

$$e_i \geq c_i, \quad i = 1, \cdots, m . \tag{22}$$

It is worthy of note that the Runge-Kutta theory for Volterra integral equations of the second kind was given a solid foundation by [17] when the order conditions were derived from the theory of *V*-series. The results were then applied to the subclasses introduced by Pouzet and Bel'tyukov as earlier mentioned.

Let us consider a non-linear Volterra integral equation of the form

$$y\left(x\right) = \int_0^x K\left(x, s, y\left(s\right)\right) ds + g\left(x\right), \qquad \left(x > 0\right) . \tag{23}$$

Weiss (1972) derived two schemes called implicit block by block method based on interpolatory quadrature rules such that

$$\int_0^{x_j} \phi(x)\, dx \approx \sum_{k=0}^{n} w_k^j \phi(u_k) \tag{24}$$

and

$$\int_0^{y} \phi(x)\, dx \approx \sum_{k=0}^{p} w_k^j \phi(u_k), \tag{25}$$

where

$$w_k^j = \int_0^{u_j} L_k(x)\, dx \tag{26}$$

$$w_k = w_k^p = \int_0^{1} L_k(x)\, dx, \tag{27}$$

and

$$L_k^{(x)} = \prod_{\substack{j=0 \\ j\neq k}}^{p} \frac{(x-u_j)}{(u_k-u_j)} \tag{28}$$

for the solution of equation (23) and proved the convergence of the method. [88] extended this method to the solution of the nonlinear Volterra integro-differential equation of the form

$$y'(x) = G\left(x, y(x), \int_0^x k(x,t,y(t)\, dt) \right), \ x \geq 0. \tag{29}$$

The convergence of this method was proved and the rate of convergence was also found. The convergence results obtained were analogous to those obtained by [153].

Let us now consider the nonlinear Volterra-Fredolm integral equations from the modeling of many applications

$$y(x) = f(x) + \lambda_1 \int_a^x k_1(x,s)\, g_1(y(s))ds + \lambda \int_a^b k_2(x,s)\, g_2(y(s))ds \tag{30}$$

where $k_1(x,s)$ and $k_2(x,s)$ are the kernels, g_1 and g_2 are nonlinear functions of y_1 and *f(x)*, a given function. In recent times [2], Adomian polynomial algorithm has been proposed for the solution of equation (30). The algorithm (a decomposition method) assumes a series solution for the unknown quantity. It has been shown by [22] that the series converge fast, and with only few terms, this series approximates the exact solution with a fairly reasonable error. [26] adapted the algorithm and a modification version of the algorithm by [151] to the solution of equation (30).

The scheme was shown to be highly accurate, and only few terms were required to obtain accurate computable solutions.

Most recently, [1] has developed an iterative scheme based on the homotopy analysis method (HAM) for nonlinear Volterra integral equations of the form

$$y(x) = g(x) + \int_a^x k(x,s)\, f(y(s))\, ds \,. \tag{31}$$

This scheme was applied to the solution of different test problems with known exact solutions and the numerical solutions obtained confirmed the validity of the numerical method and suggest that it is an interesting and viable alternative to existing numerical methods for solution of the problem under consideration. Convergence was also observed.

3. METHODS OF SOLVING FREDHOLM INTEGRAL EQUATIONS

We shall now consider linear Fredholm integral equation of the first kind

$$\int_a^b k(x,s)\, y(s)\, ds = f(x) \tag{32}$$

and of the second kind

$$y(x) - \lambda \int_a^b k(x,s)\, y(s)\, ds = f(x), \tag{33}$$

where the functions *f(x)* and the kernel $k(x,s)$ are known and *y(s)* is the unknown function.

For equations in the form of equation (32), their ill-posedness nature is the origin of the frequently occurring difficulties when dealing with methods for solving them numerically. The trouble with the collocation methods is that a discretization process transforms equation (32) into another problem (often a system of linear equations probably solvable by the well-known singular value decomposition) which can lead to solutions and may or may not deviate strongly from the (minimum norm) solution of equation (32). On the other hand, regularization methods (e.g. Tikhonov's method) may suffer from the fact that approximate solutions obtained by these methods are dependent on the chosen regularization parameter α [141]. There is also the regularized collocation method for the solution of equation (32).

Methods of solution of equation (33) can be subdivided into two classes. The first class includes the quadrature methods. For example, Nystrom methods and product integration methods otherwise called multi-step methods. The second class constitutes projection methods. Examples of such methods are collocation methods and Galarkin methods [116]. A method using cubic spline approximation for the numerical solution of equation (33) has also been used. Recently, other methods such as the cubic spline collocation [113], adaptive solution and homotopy analysis method (HAM) have also been applied to the numerical solution of equation (33).

Linear Fredholm integral equations of the first kind:

Let us now consider equation (32). For notational convenience, equation (32) can be phrased abstractly as

$$Ky = f \tag{34}$$

where *K* is a compact linear operator from a Hilbert space H_1 into a Hilbert space H_2. We do not assume that equation (34) has a unique solution; by the value of *y*, we mean the solution with minimum norm, that is, the unique solution which is orthogonal to the null space of *K*. It is well known that for non-degenerate kernels equation (34) is ill-posed, that is, the minimum norm solution *y* does not depend continuously on the right-hand side *f*. Recognizing the inherent computational difficulties when *f* is not precisely known, [143] proposed what has become known as the regularization method for the solution of equation (34). The method involves taking as an approximation to *y* the minimizer y^α of the so called Tikhonov functional [95].

$$F_\alpha(z) = \|Kz - f\|^2 + \alpha\|Z\|^2, \tag{35}$$

where α is a positive parameter, the regularization parameter of y^α, being the minimize of F_α over an infinite dimensional space is not effectively computable. ([92], [100], [93]) obtained a computable approximation y_m^α by minimizing F_α over a finite dimensional subspace V_m of H_1. However, [52] related α to m in such a way that y_m^α converges to *y* as $m \to \infty$ and also studied the asymptotic order of convergence of $\|y_m^\alpha - y\|$. It is important to define algorithms giving α in a way that does not crucially depend on the intuition of the user of a regularization method. However, some a priori knowledge of the shape or smoothness of the minimum norm solution of equation (34) will probably always be needed if ill-posed problems are to be solved. [93] showed that Sobolev spaces may serve as a tool for a partial regularization of equations in the form of equation (34), where the Sobolev space $H^m(0,1)$ for magnetic integers *m* and on the real open interval *(0,1)* is defined as the completion of the set $C^m(0,1)$ of bounded continuous and *m-times* bounded continuously differentiable real functions on (0, 1) based on the Sobolev norm given by

$$\|f\|_m = \left(\sum_{i=0}^{m} \int_0^1 f^{(i)}(t)^2 \, dt \right)^{\frac{1}{2}}, \; f \in C^m(0,1). \tag{36}$$

Recently, [99] have shown that, if the priori information mentioned above is not available, then a combination of collocation with Tikhonov regularization can be the method of choice. They analyzed the regularized collocation in a rather general setting, when a solution smoothness is given as a source condition with an operator monotone index function. This setting covered all types of smoothness studied in the theory of Tikhonov regularization. They also discussed on a posteriori choice of the regularization parameter, which allows an optimal order of accuracy for deterministic noise model to be reached without any knowledge of solution smoothness.

Linear Fredholm integral equations of the second kind:

Generally speaking, Fredholm integral equations of the second kind are the most studied among all integral equations. For thesolution of such equations, various numerical methods have been constructed and their classification has been carried out. The first thing to mention here is the work by [74] where a general theory of approximate algorithms for well-posed problems in operator form was constructed.

Again, consider the Fredholm integral equation of the second kind

$$y(x) + \int_{-1}^{1} k(x,s)\, y(s)\, ds = 1, \tag{37}$$

where

$$k(x,s) = \frac{1}{\pi}\left[\frac{d}{d^2 + (x-s)^2} \right], \tag{38}$$

and d is a positive real number occurring in the problem of determining the capacity of a circular plate condenser as was considered by Love (1949). He showed, by analytical methods, that there exists a unique, continuous, real and even solution, and that it can be expressed as a convergent series of the form:

$$y(x) = 1 + \sum_{i=1}^{\infty} (-1)^n \int_{-1}^{n} k_n(x,s)\, ds, \tag{39}$$

where the iterated kernels $k_n(x,s)$ are given by

$$\left.\begin{aligned} k_1(x,s) &= \frac{d}{\pi\left[d^2 + \left(x - s^2\right) \right]} \\ k_n(x,s) &= \int_{-1}^{1} k_{n-1}(x,t)\, k_1(t,s) dt \end{aligned}\right\}. \tag{40}$$

This method of solution is somewhat laborious and numerical solutions to this problem were found by several researchers ([35]; [160]; [32]; [155]; [31]). [109] investigated the problem only for the case *d = 1.0*. He has discussed the problem even for smaller values of *d* which it was more interesting. [119] studied equation (37) by the trapezoiodal and the Chebyshev series method and the result obtained showed that application of these methods is easy only for the case *d = 1.0*. In search of an efficient method, [119] also solved equation (37) using cubic spline method and compared his result with those obtained using trapezoidal and Chebyshev series method. It was found that the method is unsuitable for finding the solution for larger values of d as the convergence is rather slow. The numerical result also showed that the cubic spline method is potentially useful.

[60] analyzed and applied an interpolation scheme based on piecewise cubic to the solution of equation (33). An experimental comparison of Nystrom and collocation methods showed that the collocation method is faster than that of Nystrom for problems with smooth solutions and non-smooth kernels. [101] discussed the adaptive solution of

$$y(x)=\int_0^1 k(x,s)\,y(s)\,ds+g(x), \qquad x\in[0,1], \tag{41}$$

where $k(x,s)$ is a regular kernel. The method is based on the trapezoidal rule for obtaining the numerical solution of equation (41). The idea is to start with a given number of equally spaced points (or a given mesh). The solution at this stage is obtained by the solution of a linear system of algebraic equations. The program then decides if the mesh should be refined and where. This is done in such a way that both change in the approximate solution and its gradient is equi-distributed.

[23] proposed a quadrature scheme called Clenshaw-Curtis quadrature scheme for the approximate solution of the equation

$$y(x)+\int_{-1}^1 k(x,s)\,y(s)\,ds=f(x), \tag{42}$$

where the kernel $k(x,s)$ is smooth. This method is a variation of the Chebyshev series method. The method provides the solution as a Chebyshev expansion by [32]. It was found that Clenshaw-Curtis quadrature scheme gives better accuracy than the Chenyshev series method. [66] also developed a new highly accurate numerical approximation scheme based on a Gauss-type Clenshaw-Curtis quadrature for the solution of equation (42) in the range *[a,b]*. They considered a case where the kernel $k(x,s)$ is either discontinuous or not smooth along the main diagonal. They discovered that the scheme is of spectral accuracy when $k(x,s)$ is infinitely differentiable away from the diagonal $x = t$. They related the result to singular value decomposition and also applied it to the solution of integro-differential Schrodinger equations with non-local potentials.

Recently, [104] applied cubic spline collocation to the solution of integral equations in the form of equation (33). Three cubic spline collocation methods are proposed namely, Chebyshev, orthogonal and equally spaced cubic spline collocation methods. They were applied to some Fredholm integral equations after the integrals had been evaluated. Numerical computations were carried out in order to compare the three methods on the basis of computational costs, efficiency and accuracy. The three techniques produced good numerical solutions to the integral equation but a comparison of the three methods reveals that Chebyshev cubic spline collocation method gives the best result with minimum error.

The following monographs are also devoted to the study of numerical methods for solving Fredholm integral equations of the second kind: ([73], [50], [123]).

Linear Fredholm integral equations of the third kind: The theory of linear Fredholm integral equations of third kind

$$\alpha(x)u(x)+\int_a^b K(x,s)u(s)ds = f(x), \quad x\in[a,b],$$

where $\alpha(x)$ is continuous and vanishes at some but not all points in $[a,b]$, K is a continuous function, $u(x)\in W_2^1[a,b]$, and $f(x)\in W_2^1[a,b]$, with $W_2^1[a,b]$ being defined in the work by [45]. Such integral equations contain a variable coefficient, multiplying the identity operator, and vanishing at a number of points in the domain of definition of the equation. Fredholm Integral equations of the third kind are widely investigated in theory and used in applications. A number of important problems in elasticity, neutron transport, particle scattering lead to such equations. The third kind Fredholm integral equations of the form above arises in the theories of singular integral equations with degenerate symbol and boundary value problems for mixed type partial differential equations. Therefore, the investigations in this area are of great interest. Integral equations of the third kind were the object of special investigations by Bateman, Picard, Fubini, and Platrier. [36] performed, in the Hilbert space, spectral analysis of the operator corresponding to the above equation under the assumption that $\alpha(x) = x$. [12] investigated the solvability of the equation in the class of generalized functions. [126] discussed the solvability of the equation in the class of Holder functions assuming that $\alpha(x)$ has a simple zero. However, as we know, there are a few valid methods for solving Fredholm integral equations of the third kind. [39] studied the equations using a new direct method and a special collocation method. [127;125] investigated the equations basing on the ideas of the theory of spectral expansions.

Nonlinear Fredholm integral equations of the second kind: Let us consider a nonlinear operator equation

$$y = Ky, \tag{43}$$

where K is a completely continuous operator defined on a Banach space Y and y is the solution to be determined. Equation (43) is the nonlinear integral equation

$$\begin{aligned} & y(x) = (Ky)(x) = \int_0^1 k(x, s, y(s))\,ds, x \in [0,1], \\ & y \in c[0,1] \end{aligned} \tag{44}$$

with $k(x,s,u)$ sufficiently smooth on $a \equiv [0,1]\times[0,1]\times R$ so that $Ky \in C[0,1]$. The discrete Galarkin scheme [9] for the integral equation (44) is given by

$$Z_n(*) = \sum_{i=1}^{n} b_{ni} U_{ni}(x),$$

$$\sum_{i=1}^{n} b_{ni} < U_{ni} \leq \sum_{k=1}^{mn} k\left(x, s_x, \sum_{i=1}^{n} b_{ni} U_{ni}(s_k)\right) w_k, U_{ni} > n, (45) \tag{45}$$

$$\tilde{Z}_n(t) = \sum_{i=1}^{mn} k(x, s_i, z_n(s_i))\, w_i. \tag{46}$$

Song (1992)[133] showed that the approximate $\tilde{Z}_n$ converges under suitable conditions to an exact solution x^* of equation (43) and also analyzed the rate of this convergence. He also showed that, under mild conditions, $\tilde{Z}_n$ has a higher order of convergence than the discrete Galarkin approximation z_n converging to x^*. This phenomenon is known as super convergence and was studied by [9].

We shall now consider a non-linear Fredholm integral equation of the form

$$y(x) = r(x) + \int_0^1 g(x,s) F(s, y(s))\, ds, 0 \leq x \leq 1, \tag{47}$$

where the kernel function $g(x,s)$ is continuous, but its partial derivatives have finite jump discontinuities across $x = s$. A quadrature method that has been used for such equation is one based on the trapezoidal rule that has a low accuracy ([11]; [27]). [128] added suitable correction terms to the trapezoidal rule derived by analysis of the corresponding Euler-Maclaurin expansion. They also proved an existence and uniqueness theorem for the quadrature method of solutions.

4. METHOD OF SOLVING SINGULAR INTEGRAL EQUATIONS

An integral equation is said to be singular if the range of integration is infinite or if the kernel becomes infinite in the range of integration and is said to be weakly singular if its kernel $k(x,s)$ has a singularity on the diagonal $s = x$ of order not higher than $|s-x|$, $0 < a < d$ (d being the dimension of the equation), including the kernels of the form $k(x,s) = log|x-s|$. In the case of a higher order singularity, the equation will be singular. ([70]; [131]; [89]; [110]). Examples of singular integral equations are the equations with Cauchy kernels, equations of Wiener-Hoff types and various dual integral equations arising in the solution of boundary value problems of mathematical physics.

A singular integral equation with wide application is the equation with Cauchy kernel, otherwise called Cauchy singular integral equations. These equations arise most naturally and directly from boundary value problems from elasticity, aerodynamics, hydrodynamics, electro-magnetic theory, etc ([97]; [150]; [162]). In the last three decades or more, several researchers have studied the numerical solution of these equations and among the methods used are the quadrature, Galerkin, collocation, spline and the trigonometric polynomial with Cauchy kernels that are non-linear, but there has been little or no literature in the area.

Several methods also exist for the solution of singular integral equations where the range of integration is infinite. These methods include the Wiener-Hopf technique, the preconditioned conjugate gradient (PCG) methods, Galarkin method and the quadrature methods. For equations with weakly singular kernels, Galarkin and iterated Galarkin methods are most widely used [75].

Cauchy-type singular integral equations: An equation of the form

$$a(x)y(x) + \frac{b(x)}{\pi}\int_{-1}^{1} k\frac{y(s)}{s-x}ds + \frac{1}{\pi}\int_{-1}^{1} k(x,s)y(s)ds = f(x), \tag{48}$$

where the functions $a(x)$, $b(x)$, $k(x,s)$ and $f(x)$ are assumed to be known, and $y(s)$ is the unknown function is called Cauchy-type singular integral equation. The first integral in equation (48) is termed the Cauchy principal value. The functions *a(x)* and *b(x)* are real, and *b(x)* is assumed not to be zero. If *a(x) = 0*, then equation (47) is said to be of the first kind and for $a(x)\neq 0$, $b(x)\neq 0$, an equation of the second kind. If $k(x,s)=0$, equation (48) is called a dominant equation, otherwise it is known as a complete equation.

Equation (48) can be solved numerically either directly through the use of an appropriate numerical integration rule and reduction to a system of linear equations, or indirectly by reduction of a singular integral equation to an equivalent Fredholm integral equation (FIE) of the second kind and solution of the resulting FIE by numerical techniques ([62]; [68]). Presently, methods of direct numerical solutions to singular integral equations of the form of equation (48) without transformation to Fredholm integral equations are preferred and are being intensively investigated and developed ([68]; [24]). It is well known that the unknown function *y(s)* in equation (48) possesses singularities at endpoints $s=\pm 1$ and is possibly unbounded at these points ([97]; [5]). Therefore, it is appropriate to express it as $y(s)=w(s)\phi(s)$ where $\phi(s)$ is a regular function and w(s) is the weight, $w(s)=(1-s)^{\alpha}(1+s)^{\beta}$, $\alpha,\beta>-1$.

The Jacobi numerical integration rule [71] well known for regular integrals is most widely applied to the solution of the Cauchy singular integral equation (47). [138] used Gauss-Jaocbi numerical integration rule together with the application of the resulting approximate equation at a certain number of properly selected points x_r of the integration interval *(-1,1)*, thereby reducing equation (48) to a system of linear equations of the form

$$\sum_{k=1}^{n} a_k \left\{ \frac{b(x_r)}{s_x - x_r} + k(s_k, x_r) \right\} \phi(s_k) = f(x_r), \quad r=1,2,\dots,m, \tag{49}$$

where, in most cases, *m = n-1, n* or *n + 1*. [136] solved equation (48) in the case of $\alpha=p=-\frac{1}{2}$ and $a(x)=0$. [137] also solved equation (48) in the case of $\alpha=\beta=0$. They also solved equation (48) in the case of $\alpha+\beta=-x$, where *x* is an integer number and *a(x)* and *b(x)* are constants. It was proved by [62] that the numerical results of the unknown function *y(s)* in equation (48) at the nodes using direct Gauss-Chebyshev quadrature method are the same with those obtained using the Gauss-Chebyshev method for the numerical solution of the corresponding Fredholm integral equation. The same was also proved by [138;139] for The Labatto-Chebyshev quadrature method. Furthermore, [141] introduced a natural interpolation formula based on the error term of the quadrature rule for the numerical solution of equation (48). Finally, [62] investigated the weighted Galerkin method for the numerical solution of equation (47). It was shown that the systems of linear algebraic equations and the numerical results obtained were similar on application of this method to equation (48) or its equivalent Fredholm integral equation. This permits the automatic transfer of the result obtained by the weighted Galerkin method when applied to Fredholm integral equation to the case of equation (48).

We remark that there are regularization methods for the solution of equation (48), which effectively transform the original integral equation into a new integral equation with compact operators involving double integrals ([62]; [44]). For the regularized equations obtained, several iterative methods are known to be readily applicable, although this regularization approach is expensive. It has been reported that the direct approach to the solution of singular integral equations is more efficient than first regularizing the equation and then solving the regularized equation, because the direct approach requires less numerical integration. Moreover, the solutions obtained from the direct and the regularized approach may be similar in the case of Cauchy singular integral equation (48) ([62], [21]). However, there has been little work done on direct iterative methods. [3] attempted to devise a modification of a two-grid method that yielded good results for the case of hyper-singular integral equations (HSIE's). Chen (1994) [21] considered the direct solution of non-compact integral operator equation by iterative methods. The result obtained was applicable to the direct solution of equation (48) and HSIE's. The idea is based on the introduction and identification of suitable splitting of singular integral operators into the most singular (but bounded) part and the compact part. [21] and [157] proposed the use of inverse of the bounded operator as a preconditioner for the equation. Numerical discretization revealed that the preconditioned equation can be solved efficiently and that iterative methods are applicable. The numerical experiments on equation (48) showed that the proposed method is very effective.

Let us consider theFredholm-type integral equation with logarithmic kernel

$$a\int_{-1}^{1} y(s)\,ds + \frac{b}{\pi}\int_{-1}^{1} y(s)\log|x-s|\,ds = f(x), -1 < x < 1. \tag{50}$$

The above integral equation can be transformed into a Cauchy singular integral equation of the form:

$$ah(x)+\frac{b}{\pi}\int_{-1}^{1}\frac{h(s)}{s-x}\,ds = f_0(x), \qquad -1<x<1, \tag{51}$$

where

$$f_0(x) = f(x) - h_0 G(x)$$

and

$$G(x) = a\frac{1+x}{2}-\frac{b}{\pi}+\frac{b}{2\pi}\left[(1-x)\log(1-x)+(1+x)\log(1+x)\right].$$

There exist many methods for the numerical solution of equation (51) ([64], [112]). [20] proposed a numerical method which consists of a consideration of the interpolation of the known function *f* and in the substitution of this in the expression of the solution *y*. Then, with the aid of the invariance properties of the orthogonal polynomials for the Cauchy integral equations, they obtained an approximate solution of the function *y*. They also gave weighted norm estimates for the error of this method. [65] studied the system of equation (48) in the case where *a* and *b* are given piecewise continuous functions using collocation method, based on the Chebyshev nodes of second kind as collocation points and on approximation of the solution by polynomials multiplied by the Chebyshev weight of second kind. They gave necessary and sufficient conditions for the stability of operator sequence $\{A_n\}$ belonging to a $C*-$ algebra, which is generated by the sequence of the collocation method for equation of type of equation (48).
Furthermore, [25] considered the case where $a(x)=0$ and $b(x)=1$ for equation (48) and solved the equation based on interpolation processes. The method was found to be stable and convergent. Error estimates and numerical test were also provided. Recently, [107] also solved equation (48) in the case where $a(x)=0$, $\alpha=\beta=-\frac{1}{2}$ and $\alpha=\beta=\frac{1}{2}$ based on Lagrange interpolation and Gauss-Jacobi quadrature, with the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ of degree n adopted as interpolation nodes.
It is worthy of mention at this juncture that, the application of the aforementioned methods of solution (in the case of real coefficients) of equations with complex coefficients is very difficult. This difficulty is as a result of the fact that the singular integral equations with complex coefficients which abound in many boundary-value problems of mathematical physics, have highly oscillatory solutions when the argument, *x*, approaches the boundary points [61]. [33]

first made an effort to develop an algorithm for the solution of equation (48), in which the coefficient in the dominant part of the equation is not necessarily restricted to be constant. However, [61] developed a principally new algorithm for the solution of equation (48) where the coefficients are purely imaginary.

It is necessary to further take note of the following works in singular integral equations: ([148], [118], [78], [115], [106], [132], [158]). Here, in particular, some effective versions of fully discrete projection and collocation methods for various singular equations were studied.

Integral equations over an infinite interval: In their analysis of the incompressible vision flow near the leading edge of a flate plate, [149] encountered the integral equation:

$$f(x) = (2\pi)^{-1} \int_0^\infty log|x-s| \, f(s) \, ds + x^{\frac{1}{2}} , \tag{52}$$

where the function $f(x)$ is related to the slip velocity on the plate. An exact solution of equation (52) presented by [16] was obtained by means of complex Fourier transforms and the Wiener-Hopf technique. The major problem encountered by Brown is that the Fourier transform of the kernel $log|x|$ does not exist. Therefore, Brown introduced a suitable convergence factor which, in fact, amounts to solution of the related integral equation

$$f(x) = (2\pi)^{-1} \int_0^\infty log|x-s| \, e^{-\varepsilon|x-s|} \, f(s) \, ds + x^{-\frac{1}{2}} e^{-\varepsilon x} \tag{53}$$

for $\varepsilon > 0$, and then taking the limit of the solution as $\varepsilon \to 0$.

In view of this difficulty, [15] solved equation (52) using a function-theoretic method developed by [55]. It is worthy of note, that, no attempts have so far been made in solving equation (52) numerically.

Let us now consider the Weiner-Hopf equations defined on the half-time $[0,\infty)$

$$y(x) + \int_0^\infty a(x-s) \, y(s) \, ds = f(x), 0 \le x \le \infty , \tag{54}$$

where $a(x) \in L_1(R)$ and $f(x) \in L_2[0,\infty)$ are given functions.

Wiener-Hopf equations have a variety of practical applications in mathematics and engineering, especially in the solutions of inverse problems. Typical examples are linear prediction problems, and scattering problems [51]. [49] solved equation (54) by the projection method, where the solution *y(x)* of equation (54) is approximated by the solution $y_T(x)$ of the finite-section equation

$$y_T(x) = \int_0^T a(x-s)\, y_T(s)\, ds = f(x),\ 0 \le x \le T. \tag{55}$$

They showed that

$$\lim_{T\to\infty} \|y_T - y\|_{L_p[0,T)} = 0, \qquad 1 \le p < \infty .$$

Finally, [37] solved equation (54) using high-order quadrature rules by preconditioned conjugate gradient (PCG) methods. They proposed the use of convolution operator as preconditioners for these equations. They also showed that with the proper choice of kernel functions for the preconditioners, the preconditioned equation would have clustered spectra and therefore can be solved by the PCG method with super-linear convergence rate. Moreover, the discretization of these equations by high-order quadrature rules leads to matrix systems that involve only Toeplitz or diagonal matrix-vector multiplications. Numerical results were given to illustrate the fast convergence of the method and the improvement on accuracy using higher order quadrature rule. Also considered was the performance of their preconditioners with the circulent integral operators.

Let us now consider an integral equation of the form

$$y(x) = f(x) + \int_0^\infty k(x,s)\, y(s)\, ds. \tag{56}$$

[130] gave convergence proof and error analysis of equation (56) for the Nystrom method. Two particular examples of the Nystrom method were discussed in detail, namely. that based on Gauss-Laguerre quadrature and that based on mapping the infinite interval to a finite interval using Gauss quadrature. For all of the methods considered, the rate of convergence was the same, apart from a constant factor, as that of the quadrature approximation to the integral equation. Recently, [102] presented the exact solution of equation (56) with degenerate kernel. Thereafter, they applied Galerkin method with Laguerre polynomial to get the approximate solution of equation (56). Numerical examples were given to show the validity of the method presented.

5. CONCLUSION

We have reviewed a number of methods for solving Volterra and Fredholm problems of the first, second and third kind, Cauchy type singular integral equations and integral equations over an infinite interval. Painstakingly, we have outlined various researchers' contributions in obtaining solutions to each category of integral equations, amidst the challenges and difficulties associated with their findings. In many cases, however, solving an integral equation by analytical techniques is out of question, and a straight forward numerical approach, restrictive though to a class of well-posed integral equations, is to replace the integral equation by a set of linear algebraic equations solved by any of the known matrix methods. We have seen regularization methods as involving the transformation of a first kind integral equation to second kind equation. These methods are observed to suffer from the fact that approximate solutions obtained by them are dependent on the chosen regularization parameter.

Interestingly, open problems (and conjectures) may arise in the discretization of Volterra integral equations, including equations with weakly singular kernels and delay arguments, by collocation methods in piecewise polynomial spaces. They focus on questions of stability versus accuracy; extrapolation on regular and graded meshes; and equations with certain variable delays.

CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest.

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