Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 5, 2025-2032
https://doi.org/10.28919/jmcs/4816
ISSN: 1927-5307

PROPERTIES OF BERNOULLI POLYNOMIALS

R. SIVARAMAN*

Department of Mathematics, D. G. Vaishnav College, Chennai, India

Copyright © 2020 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In 1713, Swiss mathematician Jacob Bernoulli published his work about "Bernoulli Numbers". In this paper, we try to generalize the Bernoulli numbers to class of polynomials called Bernoulli Polynomials. In particular, we see that the constant terms of Bernoulli polynomials are precisely the Bernoulli numbers. After introducing Bernoulli numbers and Bernoulli polynomials we discuss few interesting properties of Bernoulli polynomials.

Keywords: Taylor's series expansion; Bernoulli numbers; Bernoulli polynomials; generating functions; even and odd values.

2010 AMS Subject Classification: 11B68.

1. INTRODUCTION

Bernoulli numbers are special class of numbers which occur quite unexpectedly in several counting problems. It contains more applications and is considered to be one of important objects of research by mathematicians. The concept of Bernoulli polynomials was seen as a natural extension of Bernoulli numbers. This paper focuses about deriving few interesting properties of Bernoulli polynomials. We first look at the definitions of Bernoulli numbers and Bernoulli polynomials.

---

*Corresponding author
E-mail address: rsivaraman1729@yahoo.co.in
Received June 30, 2020

2. BERNOULLI NUMBERS

Bernoulli numbers are numbers which occur as coefficients of $\frac{x^n}{n!}$ in the Taylor's series expansion of the function $\frac{x}{e^x-1}$ about $x = 0$ (see **[1])**.

That is,

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!} \tag{2.1}$$

Since, $e^x-1=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ the constant term of $\frac{x}{e^x-1}$ is 1. Hence, from (2.1), it follows that $B_0=1$. We also know that $\sum_{k=0}^{n}\binom{n+1}{k}B_k=0$ (2.2)

In view of equation (2.2), we get set of equations given by

$$B_0+2B_1=0, B_0+3B_1+3B_2=0, B_0+4B_1+6B_2+4B_3=0, B_0+5B_1+10B_2+10B_3+5B_4=0,\cdots$$

Knowing $B_0=1$, from the equations mentioned above, we get the set of Bernoulli numbers as follows:

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42}, B_7=0, B_8=-\frac{1}{30},$$
$$B_9=0, B_{10}=\frac{5}{66}, B_{11}=0, B_{12}=-\frac{691}{2730}, B_{13}=0, B_{14}=\frac{7}{6}, B_{15}=0, B_{16}=-\frac{3617}{510},\cdots \tag{2.3}$$

The numbers listed in (2.3) are referred to as Bernoulli numbers. We notice that except B_1 all Bernoulli numbers of odd subscripts are zero.

3. BERNOULLI POLYNOMIALS

Bernoulli polynomials are polynomials of the form $B_n(t)$ which occur as coefficients of $\frac{x^n}{n!}$ in the Taylor's series expansion of $\frac{xe^{tx}}{e^x-1}$ about $x = 0$. We denote the nth Bernoulli Polynomial by $B_n(t)$. Hence we have $\frac{xe^{tx}}{e^x-1}=\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}$ (3.1)

First, we note that comparing equations (2.1) and (3.1), we have $B_n(0)=B_n$. That is, the Bernoulli numbers are values from Bernoulli polynomials evaluated at t = 0. In other words, Bernoulli numbers are the constant terms of the Bernoulli polynomials. In this sense, we can view Bernoulli polynomials as generalization of Bernoulli numbers.

3.1 Definition

The nth Bernoulli polynomial is given by $B_n(t)=\sum_{k=0}^{n}\binom{n}{k}B_k t^{n-k}$ (3.2) where B_k in the right hand side is the kth Bernoulli number.

3.2 Determining Bernoulli Polynomials

Using equation (3.2), we can find Bernoulli polynomials using the list of Bernoulli numbers obtained in equation (2.3)

First, we find for n = 0, $B_0(t)=B_0=1$ (3.3)

For n = 1, $B_1(t)=\sum_{k=0}^{1}\binom{1}{k}B_k t^{1-k}=B_0 t+B_1=t-\frac{1}{2}$ (3.4)

For n = 2, $B_2(t)=\sum_{k=0}^{2}\binom{2}{k}B_k t^{2-k}=B_0 t^2+2B_1 t+B_2=t^2-t+\frac{1}{6}$ (3.5)

For n = 3, $B_3(t)=\sum_{k=0}^{3}\binom{3}{k}B_k t^{3-k}=B_0 t^3+3B_1 t^2+3B_2 t+B_3=t^3-\frac{3}{2}t^2+\frac{1}{2}t$ (3.6)

Similarly for n = 4, 5 and 6 we get the following polynomials

$$B_4(t)=B_0 t^4+4B_1 t^3+6B_2 t^2+4B_3 t+B_4=t^4-2t^3+t^2-\frac{1}{30} \quad (3.7)$$

$$B_5(t)=B_0 t^5+5B_1 t^4+10B_2 t^3+10B_3 t^2+5B_4 t+B_5=t^5-\frac{5}{2}t^4+\frac{5}{3}t^3-\frac{1}{6}t \quad (3.8)$$

$$B_6(t)=B_0 t^6+6B_1 t^5+15B_2 t^4+20B_3 t^3+15B_4 t^2+6B_5 t+B_6=t^6-3t^5+\frac{5}{2}t^4-\frac{1}{2}t^2+\frac{1}{42} \quad (3.9)$$

We can find more Bernoulli polynomials by substituting successive natural number values of n. Equations (3.3) to (3.9) provide the first seven Bernoulli polynomials.

In view of equations (2.1) and (3.1), the functions $\frac{x}{e^x-1}$ and $\frac{xe^{tx}}{e^x-1}$ are called Generating Functions (GFs) of Bernoulli numbers and Bernoulli polynomials respectively.

4. MAIN RESULTS

We now try to prove some of the interesting properties through two theorems associated with Bernoulli polynomials.

4.1 Theorem 1

If $B_n(t)$ is the *n*th Bernoulli polynomial then

$$B_n(1-t) = (-1)^n B_n(t) \tag{4.1}$$

Proof: We make use of the generating function of Bernoulli polynomials to get

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n B_n(t)\frac{x^n}{n!} &= \sum_{n=0}^{\infty} B_n(t)\frac{(-x)^n}{n!} = \frac{-xe^{-tx}}{e^{-x}-1}\\ &= \frac{-xe^{-tx}}{e^{-x}-1}\times\left(\frac{-e^x}{-e^x}\right) = \frac{xe^{-tx}e^x}{e^x-1}\\ &= \frac{xe^{(1-t)x}}{e^x-1} = \sum_{n=0}^{\infty} B_n(1-t)\frac{x^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{x^n}{n!}$ on both sides we get $B_n(1-t) = (-1)^n B_n(t)$ as desired.

4.2 Theorem 2

For any whole number *n*, the Bernoulli polynomials satisfy the following properties

$$B_n(1) = (-1)^n B_n \tag{4.2}$$

$$B_n\left(\frac{1}{2}\right) = \left(2^{1-n}-1\right)B_n \tag{4.3}$$

$$B_{2n}\left(\frac{1}{3}\right) = -\frac{1}{2}\left(1-3^{1-2n}\right)B_{2n} \tag{4.4}$$

$$B_{2n}\left(\frac{1}{6}\right) = \frac{1}{2}\left(1-2^{1-2n}\right)\left(1-3^{1-2n}\right)B_{2n} \tag{4.5}$$

$$B_{2n}\left(\frac{1}{6}\right) = \left(2^{1-2n}-1\right)B_{2n}\left(\frac{1}{3}\right) \tag{4.6}$$

$$B_{2n-1}\left(\frac{1}{6}\right) = \left(2^{2-2n}+1\right)B_{2n-1}\left(\frac{1}{3}\right) \tag{4.7}$$

Proof: Equation (4.2) is an immediate consequence of (4.1) in Theorem 1, by taking $t = 0$. In doing so, we get $B_n(1-0) = (-1)^n B_n(0)$. But we know that $B_n(0) = B_n$. Hence we get $B_n(1) = (-1)^n B_n$ which is equation (4.2).

For proving equations (4.3) to (4.7), we use generating functions of Bernoulli numbers and polynomials as defined in equations (2.1) and (3.1). We prove these results by considering the generating functions on the right hand side and showing that they form generating function on the left hand side terms.

$$\sum_{n=0}^{\infty}\left(2^{1-n}-1\right)B_n\frac{x^n}{n!} = \sum_{n=0}^{\infty}2^{1-n}B_n\frac{x^n}{n!} - \sum_{n=0}^{\infty}B_n\frac{x^n}{n!} = 2\sum_{n=0}^{\infty}B_n\frac{\left(\frac{x}{2}\right)^n}{n!} - \sum_{n=0}^{\infty}B_n\frac{x^n}{n!}$$

$$= 2\frac{x/2}{e^{x/2}-1} - \frac{x}{e^x-1} = \frac{x\left(e^{x/2}+1\right)-x}{e^x-1} = \frac{xe^{x/2}}{e^x-1} = \sum_{n=0}^{\infty}B_n\left(\frac{1}{2}\right)\frac{x^n}{n!}$$

Comparing the coefficients of $\frac{x^n}{n!}$ on both sides we get $B_n\left(\frac{1}{2}\right) = \left(2^{1-n}-1\right)B_n$ which is equation

Now to prove (4.4), we consider the generating function of the right hand side and proceed as below:

$$\sum_{k=0}^{\infty} -\frac{1}{2}\left(1-3^{1-k}\right)B_k\frac{x^k}{k!} = -\frac{1}{2}\sum_{k=0}^{\infty}B_k\frac{x^k}{k!} + \frac{3}{2}\sum_{k=0}^{\infty}B_k\frac{\left(\frac{x}{3}\right)^k}{k!} = -\frac{1}{2}\frac{x}{e^x-1} + \frac{3}{2}\frac{x/3}{e^{x/3}-1}$$

$$= -\frac{1}{2}\frac{x}{e^x-1} + \frac{3}{2}\frac{x/3\left(e^{2x/3}+e^{x/3}+1\right)}{e^x-1} = \frac{1}{2}\frac{1}{e^x-1}\times\left[-x+xe^{2x/3}+xe^{x/3}+x\right]$$

$$= \frac{1}{2}\frac{xe^{2x/3}}{e^x-1} + \frac{1}{2}\frac{xe^{x/3}}{e^x-1} = \frac{1}{2}\left[\sum_{k=0}^{\infty}\left(B_k\left(\frac{2}{3}\right)+B_k\left(\frac{1}{3}\right)\right)\frac{x^k}{k!}\right] = \frac{1}{2}\left[\sum_{k=0}^{\infty}\left((-1)^k B_k\left(\frac{1}{3}\right)+B_k\left(\frac{1}{3}\right)\right)\frac{x^k}{k!}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty}\left[1+(-1)^k\right]B_k\left(\frac{1}{3}\right)\frac{x^k}{k!}\right]$$

Comparing the coefficients of $\frac{x^k}{k!}$ on both sides we get $-\frac{1}{2}\left(1-3^{1-k}\right)B_k = \frac{1}{2}\left[1+(-1)^k\right]B_k\left(\frac{1}{3}\right)$

If k is odd, then $1+(-1)^k = 0$ and so the right hand side is zero. Also, the left hand side term $B_k = 0$ for odd values of k except 1. When $k = 1$, $1-3^{1-k} = 0$ and so the left hand side is zero.

Hence for all odd positive integer values of k, both left and right hand sides of above equation becomes identically zero.

If k is even, then $1+(-1)^k=2$ and so the above equation becomes $-\frac{1}{2}\left(1-3^{1-k}\right)B_k=B_k\left(\frac{1}{3}\right)$.

Since, k is even we can take $k = 2n$ in the previous equation to get $B_{2n}\left(\frac{1}{3}\right)=-\frac{1}{2}\left(1-3^{1-2n}\right)B_{2n}$

which is precisely equation (4.4)

To prove (4.5), we proceed similarly by considering the generating function of the right hand side to get the following equations

$$\sum_{k=0}^{\infty}\left[\frac{1}{2}\left(1-2^{1-k}\right)\left(1-3^{1-k}\right)B_k\right]\frac{x^k}{k!}=\frac{1}{2}\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}-\sum_{k=0}^{\infty}B_k\frac{\left(\frac{x}{2}\right)^k}{k!}-\frac{3}{2}\sum_{k=0}^{\infty}B_k\frac{\left(\frac{x}{3}\right)^k}{k!}+3\sum_{k=0}^{\infty}B_k\frac{\left(\frac{x}{6}\right)^k}{k!}$$

$$=\frac{1}{2}\frac{x}{e^x-1}-\frac{x/2}{e^{x/2}-1}-\frac{3}{2}\frac{x/3}{e^{x/3}-1}+3\frac{x/6}{e^{x/6}-1}$$

$$=\frac{1}{2}\frac{x}{e^x-1}-\frac{1}{2}\frac{x\left(e^{x/2}+1\right)}{e^x-1}-\frac{1}{2}\frac{x\left(e^{2x/3}+e^{x/3}+1\right)}{e^x-1}+\frac{1}{2}\frac{x\left(e^{5x/6}+e^{2x/3}+e^{x/2}+e^{x/3}+e^{x/6}+1\right)}{e^x-1}$$

$$=\frac{1}{2}\frac{x}{e^x-1}\times\left[1-e^{x/2}-1-e^{2x/3}-e^{x/3}-1+e^{5x/6}+e^{2x/3}+e^{x/2}+e^{x/3}+e^{x/6}+1\right]$$

$$=\frac{1}{2}\frac{x}{e^x-1}\times\left[e^{x/6}+e^{5x/6}\right]=\frac{1}{2}\frac{xe^{x/6}}{e^x-1}+\frac{1}{2}\frac{xe^{5x/6}}{e^x-1}=\frac{1}{2}\sum_{k=0}^{\infty}\left[B_k\left(\frac{1}{6}\right)+B_k\left(\frac{5}{6}\right)\right]\frac{x^k}{k!}$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\left[1+(-1)^k\right]B_k\left(\frac{1}{6}\right)\frac{x^k}{k!}$$

Comparing the coefficients of $\frac{x^k}{k!}$ on both sides we get

$$\frac{1}{2}\left(1-2^{1-k}\right)\left(1-3^{1-k}\right)B_k=\frac{1}{2}\left[1+(-1)^k\right]B_k\left(\frac{1}{6}\right)$$

For all odd values of k, we see that both left and right hand side terms are zero. If k is even, then $1+(-1)^k=2$ and so the above equation becomes $\frac{1}{2}\left(1-2^{1-k}\right)\left(1-3^{1-k}\right)B_k=B_k\left(\frac{1}{6}\right)$. Since k is even, taking $k = 2n$, we have $B_{2n}\left(\frac{1}{6}\right)=\frac{1}{2}\left(1-2^{1-2n}\right)\left(1-3^{1-2n}\right)B_{2n}$ which is equation (4.5)

To prove the final two identities, first we consider the generating function of $\left(2^{1-k}-1\right)B_k\left(\frac{1}{3}\right)$ to get

$$\sum_{k=0}^{\infty}\left(2^{1-k}-1\right)B_k\left(\frac{1}{3}\right)\frac{x^k}{k!}=2\sum_{k=0}^{\infty}B_k\left(\frac{1}{3}\right)\frac{\left(\frac{x}{2}\right)^k}{k!}-\sum_{k=0}^{\infty}B_k\left(\frac{1}{3}\right)\frac{x^k}{k!}=2\frac{x/_2\, e^{x/_6}}{e^{x/_2}-1}-\frac{xe^{x/_3}}{e^x-1}$$

$$=\frac{xe^{x/_6}\left(e^{x/_2}+1\right)-xe^{x/_3}}{e^x-1}=\frac{xe^{2x/_3}+xe^{x/_6}-xe^{x/_3}}{e^x-1}=\sum_{k=0}^{\infty}\left[B_k\left(\frac{2}{3}\right)+B_k\left(\frac{1}{6}\right)-B_k\left(\frac{1}{3}\right)\right]\frac{x^k}{k!}$$

$$=\sum_{k=0}^{\infty}\left[(-1)^k B_k\left(\frac{1}{3}\right)+B_k\left(\frac{1}{6}\right)-B_k\left(\frac{1}{3}\right)\right]\frac{x^k}{k!}=\sum_{k=0}^{\infty}\left[B_k\left(\frac{1}{3}\right)\left((-1)^k-1\right)+B_k\left(\frac{1}{6}\right)\right]\frac{x^k}{k!}$$

Comparing the coefficients of $\frac{x^k}{k!}$ on both sides we get

$$\left(2^{1-k}-1\right)B_k\left(\frac{1}{3}\right)=B_k\left(\frac{1}{3}\right)\left((-1)^k-1\right)+B_k\left(\frac{1}{6}\right)$$

If k is even, then $(-1)^k-1=0$. Thus if $k = 2n$, then we get $B_{2n}\left(\frac{1}{6}\right)=\left(2^{1-2n}-1\right)B_{2n}\left(\frac{1}{3}\right)$ which is equation (4.6). We can also obtain this equation directly using equations (4.4) and (4.5).

If k is odd, then $(-1)^k-1=-2$. Thus if $k = 2n - 1$, then we get $B_{2n-1}\left(\frac{1}{6}\right)=\left(2^{2-2n}+1\right)B_{2n-1}\left(\frac{1}{3}\right)$ which is equation (4.7).

5. Conclusion

The concept of Bernoulli numbers and polynomials occur in variety of research areas in mathematics. In this paper, we have seen some new properties of Bernoulli polynomials listed in Theorem 2 of section 4. These new properties may add further understanding of the values of Bernoulli polynomials evaluated at certain values like $\frac{1}{2},\frac{1}{3},\frac{1}{6}$ through the corresponding Bernoulli numbers. Equations (4.6) and (4.7) provide the connection between Bernoulli polynomials for even and odd indices for the values $\frac{1}{6}$ and $\frac{1}{3}$ respectively. These new results may provide greater scope for future explorations in the understanding of Bernoulli polynomials, which in turn can be used in tackling various problems of Science and Technology.

CONFLICTS OF INTEREST

The authors declare that they have no con³icts of interest.

REFERENCES

[1] R. Sivaraman, Remembering Ramanujan, Adv. Math. 9 (1) (2020), 489–506.
[2] M. Açikgöz, D. Erdal, S. Araci, A New Approach to q-Bernoulli Numbers and q-Bernoulli Polynomials Related to q-Bernstein Polynomials, Adv. Differ. Equ. 2010 (2010), 951764.
[3] M. Momenzadeh, L.Y. Kakangi, Approximation Properties of q – Bernoulli Polynomials, Abstr. Appl. Anal. 2017 (2017), 9828065.
[4] M.-S. Kim, A note on Sums of Products of Bernoulli Numbers, Appl. Math. Lett. 24 (1) (2011), 55 – 61.
[5] R. Sivaraman, Sum of powers of natural numbers, Aut Aut Res. J. XI (IV) (2020), 353 – 359.
[6] H. BT Bahari, A Generalization of the Bernoulli Numbers, Master's Thesis, Universiti Putra Malaysia, 2006.