EXACT AND NUMERICAL SOLUTIONS OF NONLINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS USING EXP-FUNCTION METHOD AND HOMOTOPY ANALYSIS METHOD

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Abstract: In This study, we employed the Exp-function method to obtain the exact solutions of two systems of nonlinear partial differential equations the coupled Hirota-Satsuma-KdV equation and the Hirota–Satsuma coupled KdV and compare this exact solutions with corresponding numerical solution obtained by the homotopy analysis method (HAM).

Keywords: exp-function method; system of nonlinear partial differential equations; the coupled Hirota-Satsuma-KdV equation; Hirota–Satsuma coupled KdV system; homotopy analysis method.

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1. INTRODUCTION

The solutions of linear and non-linear differential equations are of great importance for scientific disciplines such as engineering, physics and other disciplines. There are many powerful methods

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that have been presented to obtain analytical solutions to differential equations for example, such as the tanh-function method [1-2] and sine–cosine method [3-4]. On the other hand there are methods that have been used to find numerical solutions for differential equations such as homotopy perturbation method [5-6], variational iteration method [7], [8] and Adomian decomposition method [9]–[11] , etc. The Exp-function method [12]–[14] is one of the most powerful methods used to find analytical solutions for differential equations, which are characterized by the ease of solving procedures and any of the mathematical package can be used to help make these procedures easier.

The present article is motivated by the desire to extend the Exp-function method to the coupled Hirota-Satsuma-KdV equation [15] and the Hirota–Satsuma coupled KdV System [15]–[19] and compare the exact solutions that we obtained with the corresponding numerical solution by homotopy analysis method then calculating the error.

2. EXP-FUNCTION METHOD

Consider the following nonlinear system of partial differential equations:

$$N_i(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{tx}, v_{tx}, u_{xx}, \dots) = 0, \qquad i = 1,2 \tag{1}$$

where N, is a nonlinear function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformations. Introducing a complex variation ϕ defined as:

$$u = u(\phi) \quad , \qquad v = v(\phi) \quad , \qquad \phi = kx + \omega t \tag{2}$$

where k, ω are constants to be determined later. Then Eq. (1) reduces to the ODE:

$$N_i(u, v, \omega u', ku', kv', \omega^2 u'', \omega^2 v'', \omega ku'', \omega kv'', k^2 u'', \dots) = 0 \ , \tag{3}$$

And then solution of u(ϕ) is a form of

$$u(\phi) = \frac{\sum_{f=-c}^{d} a_f \, e^{f\phi}}{\sum_{g=-p}^{q} b_g \, e^{g\phi}} \quad , v(\phi) = \frac{\sum_{h=-m}^{n} A_h \, e^{h\phi}}{\sum_{j=-l}^{r} B_j \, e^{j\phi}} \quad , \tag{4}$$

where c, m, p, l, d, n, r and q are arbitrary positive integer, the coefficients a_f, b_g, A_h and B_j are unknown constants that will be determined. By balancing the linear term with nonlinear

term of highest order in Eq. (3) we can determine the values of c, p, m and l. As well, by balancing the linear term with nonlinear term of lowest order in Eq. (3) we can calculate the values of d, q, n and r.

3. APPLICATION OF EXP-FUNCTION METHOD

3.1. The coupled Hirota-Satsuma-KdV equation

To illustrate the basic idea of the Exp-function method, we first consider the coupled Hirota-Satsuma-KdV equation [12] in the form

$$u_t - \frac{1}{4}u_{xxx} - 3uu_x + 6vv_x = 0, \tag{5}$$

$$v_t + \frac{1}{2}v_{xxx} + 3uv_x = 0, \tag{6}$$

By using Eq. (2), the system of Eq. (5) and (6) reduced to the following system of ordinary differential equation,

$$\omega u' - \frac{1}{4}k^3 u''' - 3kuu' + 6kvv' = 0, \tag{7}$$

$$\omega v' + \frac{1}{2}k^3 v''' + 3kuv' = 0, \tag{8}$$

where ($'$) denote derivative of u and v with respect to ϕ. From eq. (4) that the solutions of u and v takes the following forms:

$$u(\eta) = \frac{a_c e^{c\phi} + \cdots + a_{-d}e^{-d\phi}}{b_p e^{p\phi} + \cdots + b_{-q}e^{-q\phi}} \quad , \quad v(\eta) = \frac{A_m e^{m\phi} + \cdots + A_{-n}e^{-n\phi}}{B_l e^{l\phi} + \cdots + B_{-r}e^{-r\phi}} \tag{9}$$

where c, m, p, l, d, n, r and q are arbitrary positive integer, the coefficients a_f , b_g , A_h and B_j are unknown constants that will be determined. In order to evaluate the values of parameters c, m, p, l, d, n, r and q we balance the highest and lowest order of linear terms with the highest and lowest order of nonlinear terms in Eq. (7) and (8) respectively. From Eq. (8) and (9) we obtain:

$$u''' = \frac{c_3 e^{(c+7p+3l)\phi} + \cdots}{c_4 e^{(8p+3l)\phi} + \cdots}, uu' = \frac{d_3 e^{(2c+6p+3l)\phi} + \cdots}{d_4 e^{(8p+3l)\phi} + \cdots}, vv' = \frac{d_5 e^{(2m+l+8p)\phi} + \cdots}{d_6 e^{(8p+3l)\phi} + \cdots} \tag{10}$$

From eq. (10) by balancing the highest order of u''', uu' and vv' we get:

$c + 7p + 3l = 2c + 6p + 3l$ or $c + 7p + 3l = 2m + l + 8p$ which leads to $p = c$ or $m = l$.

In a similar way from eq. (8) we obtain:

$$v''' = \frac{c_3 e^{(m+7l+p)\phi} + \cdots}{c_4 e^{(8l+p)\phi} + \cdots}, uv' = \frac{d_3 e^{(c+m+7l)\phi} + \cdots}{d_4 e^{(8l+p)\phi} + \cdots}, uv' = \frac{d_3 e^{(c+m+7l)\phi} + \cdots}{d_4 e^{(8l+p)\phi} + \cdots} \qquad (11)$$

So, we have: $\qquad m + 7l + p = c + m + 7l$, $\qquad$ which leads to $\qquad p = c$

Likewise, determining the values of d, q, n and r. By equating the linear term of lowermost order

in Eq. (7) $\quad$ we get: $\qquad d = q \, or \, n = r.$

values of the constants c, d, m and n can be selected arbitrary. We consider the case that all

constants to be equal 1 i.e.

$$p = c = 1, d = q = 1, m = l = 1 \, and \, n = r = 1.$$

Consequently eq. (9) turn into:

$$u(\eta) = \frac{a_1 e^\phi + a_0 + a_{-1} e^{-\phi}}{b_1 e^\phi + b_0 + b_{-1} e^{-\phi}}, \qquad v(\eta) = \frac{A_1 e^\phi + A_0 + A_{-1} e^{-\phi}}{B_1 e^\phi + B_0 + B_{-1} e^{-\phi}} \qquad (12)$$

In the case $b_1 \neq 0$ and $B_1 \neq 0$ the system (12)can be simplified to

$$u(\eta) = \frac{a_1 e^\phi + a_0 + a_{-1} e^{-\phi}}{e^\phi + b_0 + b_{-1} e^{-d\phi}}, \qquad v(\eta) = \frac{A_1 e^\phi + A_0 + A_{-1} e^{-\phi}}{e^\phi + B_0 + B_{-1} e^{-\phi}} \qquad (13)$$

Substituting (13) into (7) and (8), we have

$$\frac{1}{G} [M_6 e^{6\phi} + M_5 e^{5\phi} + M_4 e^{4\phi} + M_3 e^{3\phi} + M_2 e^{2\phi} + M_1 e^\phi + M_0 + M_{-1} e^{-\phi} + M_{-2} e^{-2\phi}$$

$$+ M_{-3} e^{-3\phi} + M_{-4} e^{-4\phi} + M_{-5} e^{-5\phi} + M_{-6} e^{-6\phi}] = 0,$$

$$G = \left(e^\phi + B_0 + B_{-1} e^{-\phi}\right)^3 \left(e^\phi + b_0 + b_{-1} e^{-d\phi}\right)^4,$$

$$\frac{1}{H} [N_4 e^{4\phi} + N_3 e^{3\phi} + N_2 e^{2\phi} + N_1 e^\phi + N_0 + N_{-1} e^{-\phi} + N_{-2} e^{-2\phi} + N_{-3} e^{-3\phi} + N_{-4} e^{-4\phi}] = 0,$$

$$H = \left(e^\phi + B_0 + B_{-1} e^{-\phi}\right)^4 \left(e^\phi + b_0 + b_{-1} e^{-d\phi}\right)^1 \qquad (14)$$

$M_\sigma , (\sigma = -6, -5, \dots, 5,6) \, and \, N_\tau , (\tau = -4, -3, \dots, 3,4) \, and$ are constants to be obtained

By Calculating all coefficients of $e^{n\phi}$ and equals by zero we obtain system of algebraic

equations as follows:

$M_6 = 0, M_5 = 0, M_4 = 0, M_3 = 0, M_2 = 0, M_1 = 0, M_0 = 0, M_{-1} = 0, M_{-2} = 0, M_{-3} = 0,$

$M_{-4} = 0, M_{-5} = 0, M_{-6} = 0, N_4 = 0, N_3 = 0, N_2 = 0, N_1 = 0, N_0 = 0, N_{-1} = 0, N_{-2} = 0,$

$$N_{-3} = 0, N_{-4} = 0 \tag{15}$$

Solutions of system (15) will yield

Case 1. $\quad A_0 = -\dfrac{5k^2}{4} + 2, A_1 = 2 - \dfrac{7k^2}{4}, A_2 = \dfrac{1}{2} - \dfrac{7k^2}{16}, B_0 = 1, B_2 = \dfrac{1}{4}, a_0 = 1, a_1 = 1 - k^2,$

$$a_2 = \frac{1}{4} - \frac{1}{4}k^2, b_0 = 1, b_2 = \frac{1}{4}, \omega = \frac{5}{2}k^3 - 3k, \tag{16}$$

From eq. (16) and (13), we get the exact solution of u (x, t) and v (x, t) as follows:

$$v_1(x, t) = \frac{\left(\pm 2 \mp \dfrac{7k^2}{4}\right)e^{kx + \left(\frac{5}{2}k^3 - 3k\right)t} \mp \dfrac{5k^2}{4} \pm 2 + \left(\pm\dfrac{1}{2} \mp \dfrac{7k^2}{16}\right)e^{-kx - \left(\frac{5}{2}k^3 - 3k\right)t}}{e^{kx + \left(\frac{5}{2}k^3 - 3k\right)t} + 1 + \dfrac{1}{4}e^{-kx - \left(\frac{5}{2}k^3 - 3k\right)t}},$$

$$u_1(x, t) = \frac{(1 - k^2)e^{kx + \left(\frac{5}{2}k^3 - 3k\right)t} + 1 + \left(\dfrac{1}{4} - \dfrac{1}{4}k^2\right)e^{-kx - \left(\frac{5}{2}k^3 - 3k\right)t}}{e^{kx + \left(\frac{5}{2}k^3 - 3k\right)t} + 1 + \dfrac{1}{4}e^{-kx - \left(\frac{5}{2}k^3 - 3k\right)t}}. \tag{17}$$

Case 2.

$$A_0 = A_1 \mp \frac{k^2}{2}, A_1 = A_1, A_2 = \frac{1}{4}A_1, B_0 = 1, B_2 = \frac{1}{4}, a_0 = \mp\frac{1}{2}A_1 + \frac{7}{8}k^2, a_1 = \mp\frac{1}{2}A_1 - \frac{1}{8}k^2,$$

$$a_2 = \mp\frac{1}{8}A_1 - \frac{1}{32}k^2, b_0 = 1, b_2 = \frac{1}{4}, \omega = -\frac{1}{8}k^3 \pm \frac{3}{2}kA_1, \tag{18}$$

From eq. (18) and (13), we get the exact solution of u (x, t) and v (x, t) as follows:

$$v_2(x, t) = \frac{A_1 e^{kx + \left(-\frac{1}{8}k^3 \pm \frac{3}{2}kA_1\right)t} + \left(A_1 \mp \dfrac{k^2}{2}\right) + \left(\dfrac{1}{4}A_1\right)e^{-kx - \left(-\frac{1}{8}k^3 \pm \frac{3}{2}kA_1\right)t}}{e^{kx + \left(-\frac{1}{8}k^3 \pm \frac{3}{2}kA_1\right)t} + 1 + \dfrac{1}{4}e^{-kx - \left(-\frac{1}{8}k^3 \pm \frac{3}{2}kA_1\right)t}},$$

$$u_2(x, t) = \frac{\left(\mp\dfrac{1}{2}A_1 - \dfrac{1}{8}k^2\right)e^{kx + \left(-\frac{1}{8}k^3 \pm \frac{3}{2}kA_1\right)t} + \left(\mp\dfrac{1}{2}A_1 + \dfrac{7}{8}k^2\right) + \left(\mp\dfrac{1}{8}A_1 - \dfrac{1}{32}k^2\right)e^{-kx - \frac{1}{8}k^3 \pm \frac{3}{2}kA_1 t}}{e^{kx + \left(-\frac{1}{8}k^3 \pm \frac{3}{2}kA_1\right)t} + 1 + \dfrac{1}{4}e^{-kx - \left(\frac{1}{8}k^3 \pm \frac{3}{2}kA_1\right)\text{m} t}}. \tag{19}$$

Case 3.

$$A_0 = B_0 A_1 \mp \frac{k^2}{2}B_0, A_1 = A_1, A_2 = \frac{1}{4}A_1{B_0}^2, B_0 = B_0, B_2 = \frac{1}{4}{B_0}^2, a_0 = \mp\frac{1}{2}A_1 B_0 + \frac{7}{8}k^2 B_0,$$

$$a_1 = \mp\frac{1}{2}A_1 - \frac{1}{8}k^2, a_2 = \frac{1}{32}{B_0}^2(\mp 4A_1 - k^2), b_0 = B_0, b_2 = \frac{1}{4}{B_0}^2, \omega = -\frac{1}{8}k^3 \pm \frac{3}{2}kA_1, \tag{20}$$

From eq. (20) and (13), we get the exact solution of u (x, t) and v (x, t) as follows:

$$v_3(x,t) = \frac{A_1 e^{kx+\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t} + \left(B_0 A_1 \mp \frac{k^2}{2}B_0\right) + \left(\frac{1}{4}A_1 B_0{}^2\right)e^{-kx-\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t}}{e^{kx+\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t} + B_0 + \frac{1}{4}B_0{}^2 e^{-kx-\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t}},$$

$$u_3(x,t) = \frac{\left(\mp\frac{1}{2}A_1 - \frac{1}{8}k^2\right)e^{kx+\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t} \mp \frac{1}{2}A_1 B_0 + \frac{7}{8}k^2 B_0 + \frac{1}{32}B_0{}^2(\mp 4A_1 - k^2)e^{-kx-\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t}}{e^{kx+\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t} + B_0 + \frac{1}{4}B_0{}^2 e^{-kx-\left(-\frac{1}{8}k^3\pm\frac{3}{2}kA_1\right)t}} \tag{21}$$

Case 4.

$$A_0 = 0, A_1 = A_1, A_2 = -\frac{1}{4}A_1 B_0{}^2, B_0 = B_0, B_2 = \frac{1}{4}B_0{}^2, a_0 = -\frac{1}{8}\frac{B_0(-3k^4 + 16A_1{}^2)}{k^2}, b_0 = B_0,$$

$$a_1 = -\frac{1}{8}\frac{k^4 + 16A_1{}^2}{k^2}, a_2 = -\frac{1}{32}\frac{(k^4 + 16A_1{}^2)B_0{}^2}{k^2}, b_2 = \frac{1}{4}B_0{}^2, \omega = \frac{1}{8}\frac{-k^4 + 48A_1{}^2}{k} \tag{22}$$

From eq. (22) and (13), we get the exact solution of u(x, t) and v(x, t) as follows:

$$v_4(x,t) = \frac{A_1 e^{kx+\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t} + \left(-\frac{1}{4}A_1 B_0{}^2\right)e^{-kx-\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t}}{e^{kx+\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t} + B_0 + \frac{1}{4}B_0{}^2 e^{-kx-\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t}}, \tag{23}$$

$$u_4(x,t)$$

$$= \frac{\left(-\frac{k^4 + 16A_1{}^2}{8k^2}\right)e^{kx+\left(\frac{-k^4+48A_1{}^2}{8k}\right)t} - \frac{B_0(-3k^4 + 16A_1{}^2)}{8k^2} - \frac{1}{32}\frac{(k^4 + 16A_1{}^2)B_0{}^2}{k^2}e^{-kx-\left(\frac{-k^4+48A_1{}^2}{8k}\right)t}}{e^{kx+\left(\frac{-k^4+48A_1{}^2}{8k}\right)t} + B_0 + \frac{1}{4}B_0{}^2 e^{-kx-\left(\frac{-k^4+48A_1{}^2}{8k}\right)t}}$$

$$\tag{24}$$

Case 5.

$$A_0 = (B_0 - b_0)A_1, A_1 = A_1, A_2 = -\frac{1}{4}A_1 b_0(-b_0 + 2B_0), B_0 = B_0, B_2 = \frac{1}{4}b_0(-b_0 + 2B_0),$$

$$a_0 = -\frac{1}{8}\frac{b_0(-3k^4 + 16A_1{}^2)}{k^2}, a_1 = -\frac{1}{8}\frac{k^4 + 16A_1{}^2}{k^2}, a_2 = -\frac{1}{32}\frac{(k^4 + 16A_1{}^2)b_0{}^2}{k^2}, b_0 = b_0,$$

$$b_2 = \frac{1}{4}b_0{}^2, \omega = \frac{1}{8}\frac{-k^4 + 48A_1{}^2}{k} \tag{25}$$

From eq. (25) and (13), we get the exact solution of u(x ,t) and v(x, t) as follows:

$$v_5(x,t) = \frac{A_1 e^{kx+\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t} + (B_0 - b_0)A_1 - \frac{1}{4}A_1 b_0(-b_0 + 2B_0)e^{-kx-\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t}}{e^{kx+\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t} + B_0 + (\frac{1}{4}b_0(-b_0 + 2B_0))e^{-kx-\left(\frac{1-k^4+48A_1{}^2}{8}\frac{}{k}\right)t}},$$

$u_5(x, t)$

$$= \frac{\left(-\dfrac{1}{8}\dfrac{k^4 + 16A_1{}^2}{k^2}\right) e^{kx+\left(\frac{1-k^4+48A_1{}^2}{8}\cdot\frac{1}{k}\right)t} - \dfrac{b_0\left(-3k^4 + 16A_1{}^2\right)}{8k^2} - \dfrac{\left(k^4 + 16A_1{}^2\right)b_0{}^2}{32k^2} e^{-kx-\left(\frac{1-k^4+48A_1{}^2}{8}\cdot\frac{1}{k}\right)t}}{e^{kx+\left(\frac{1-k^4+48A_1{}^2}{8}\cdot\frac{1}{k}\right)t} + b_0 + \dfrac{1}{4}b_0{}^2 e^{-kx-\left(\frac{1-k^4+48A_1{}^2}{8}\cdot\frac{1}{k}\right)t}} . (26)$$
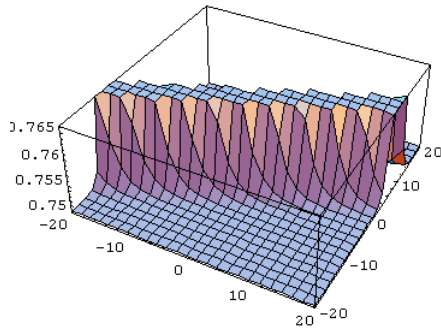
Fig.(1a) exact solution of $u_1(x, t)$ $\qquad (k = -\frac{1}{2})$ $\qquad$ Fig.(1b) exact solution of $v_1(x, t)$
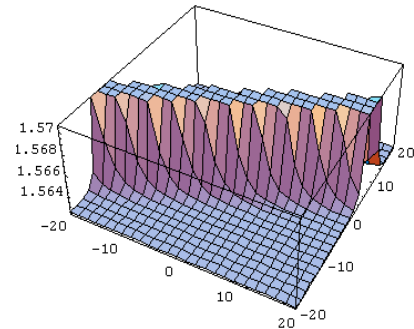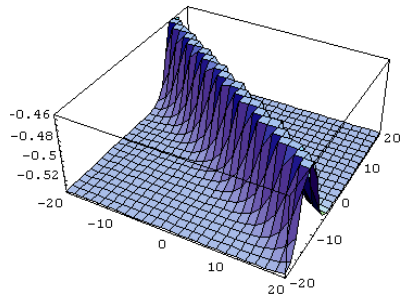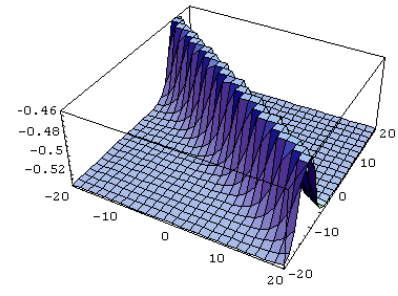

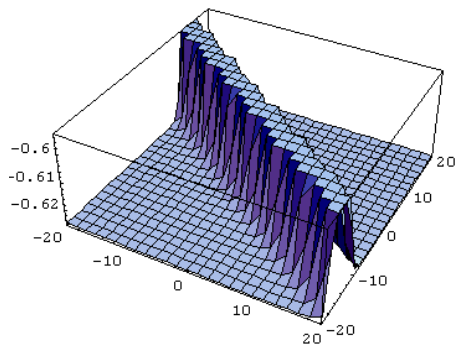Fig. (2a) exact solution of $u_2(x, t)$ $\qquad (k = -\frac{1}{2})$ $\qquad$ Fig.(2b) exact solution of $v_2(x, t)$

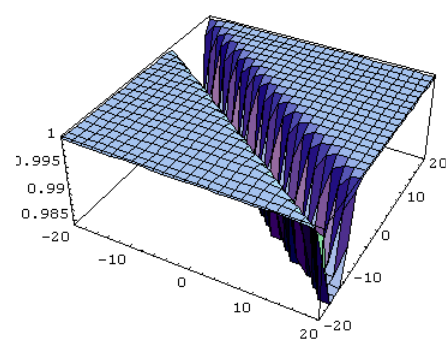Fig. (3a) exact solution of $u_3(x, t)$ $\qquad (B_0 = 1, A_1 = 1, k = 1)$ $\qquad$ Fig.(3b) exact solution of $v_3(x, t)$
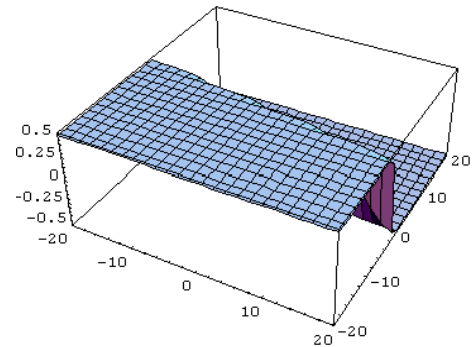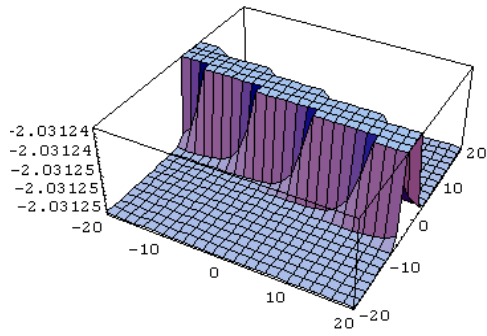
Fig.4 a exact solution of u₄(x, t) $(B_0 = \frac{1}{5}, A_1 = \frac{1}{2}, k = -\frac{1}{2},)$ Fig.(4b) exact solution of v₄(x ,t)
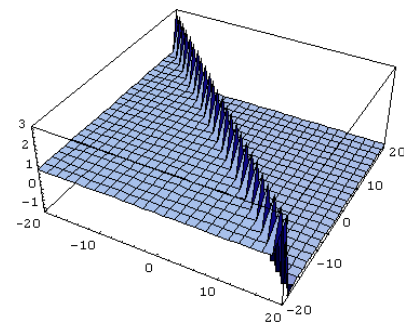

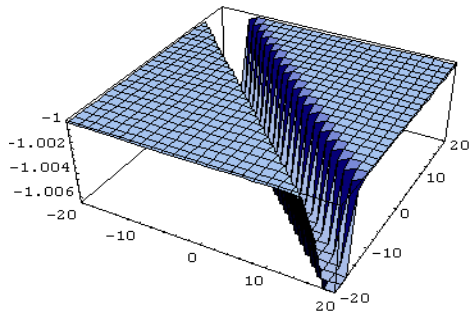Fig.(5a) exact solution of u₅(x, t) $(B_0 = 2; b_0 = -1; A_1 = 1; k = -2,)$ Fig.(5b) exact solution of v₅(x, t)

3.2. The Hirota–Satsuma coupled KdV System

We consider the Hirota–Satsuma coupled KdV System [13, 14] in the form

$$u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3(vw)_x = 0,$$

$$v_t + v_{xxx} - 3uv_x = 0,$$

$$w_t + w_{xxx} - 3uw_x = 0, \tag{27}$$

By using Eq. (2), the system of Eq. (27) reduced to the following system of ordinary differential equation,

$$\omega u' - \frac{1}{2}k^3 u''' + 3kuu' - 3kvw' - 3kwv' = 0,$$

$$\omega v' + k^3 v''' - 3kuv' = 0,$$

$$\omega w' + k^3 w''' - 3kuw' = 0, \tag{28}$$

where (') denote derivative of u, v and w with respect to ϕ. From eq. (4) that the solutions of u, v and w takes the following forms:

$$u(\eta) = \frac{a_c e^{c\phi} + \cdots + a_{-d} e^{-d\phi}}{b_p e^{p\phi} + \cdots + b_{-q} e^{-q\phi}}, \qquad v(\eta) = \frac{A_m e^{m\phi} + \cdots + A_{-n} e^{-n\phi}}{B_l e^{l\phi} + \cdots + B_{-r} e^{-r\phi}},$$

$$w(\eta) = \frac{F_g e^{g\phi} + \cdots + F_{-h} e^{-h\phi}}{Z_s e^{s\phi} + \cdots + Z_{-j} e^{-j\phi}}, \tag{29}$$

where $c, m, g, p, l, s, d, e, h, q, r$ and j are arbitrary positive integer, the coefficients a_f, b_g, A_h, B_j, F_o and Z_x are unknown constants that will be determined. In order to evaluate the values of parameters $c, m, g, p, l, s, d, e, h, q, r$ and j we balance the highest and lowest order of linear terms with the highest and lowest order of nonlinear terms respectively. From Eq. (28), we get:

$$u''' = \frac{c_3 e^{(c+7p+2l+s)\phi} + \cdots}{c_4 e^{(8p+2l+s)\phi} + \cdots}, \qquad uu' = \frac{d_3 e^{(2c+6p+3l)\phi} + \cdots}{d_4 e^{(8p+3l)\phi} + \cdots},$$

$$vw' = \frac{d_3 e^{(m+g+s+8p)\phi} + \cdots}{d_4 e^{(8p+l+2s)\phi} + \cdots}, \qquad wv' = \frac{d_5 e^{(m+g+l+8p)\phi} + \cdots}{d_6 e^{(s+2l+8p)\phi} + \cdots}, \tag{30}$$

$$v''' = \frac{c_3 e^{(m+7l+p)\phi} + \cdots}{c_4 e^{(8l+p)\eta} + \cdots}, \qquad uv' = \frac{d_3 e^{(c+m+7l)\phi} + \cdots}{d_4 e^{(8l+p)\eta} + \cdots},$$

$$w''' = \frac{c_3 e^{(g+7s+p)\phi} + \cdots}{c_4 e^{(8s+p)\phi} + \cdots}, \qquad uw' = \frac{d_3 e^{(c+g+7s)\phi} + \cdots}{d_4 e^{(8s+p)\phi} + \cdots}. \tag{31}$$

By equating the highest order of Exp-function in Eq. (28) using Eq. (30) and (31) we obtain:

$$p = c \text{ or } m + g = l + s \qquad \text{and } p = c$$

In the same way, we can obtain the values of d, q, n and r by equating the lowest order in Eq. (28) using Eq. (30) and (31) we get:

$$d = q \text{ or } n + h = r + j.$$

$c, m, g, p, l, s, d, e, h, q, r$ and j can be selected arbitrary. We consider the case that all constants to be equal 1 i.e.

$$p = c = 1, d = q = 1, m = l = 1, n = r = 1, g = s = 1 \text{ and } h = j = 1.$$

Thus, (43) - (45) becomes as:

$$u(\eta) = \frac{a_1 e^{\phi} + a_0 + a_{-1} e^{-\phi}}{b_1 e^{c\phi} + b_0 + b_{-1} e^{-d\phi}}, \qquad v(\eta) = \frac{A_1 e^{\phi} + A_0 + A_{-1} e^{-\phi}}{B_1 e^{\phi} + B_0 + B_{-1} e^{-\phi}},$$

$$w(\eta) = \frac{c_1 e^\phi + c_0 + c_{-1}e^{-\phi}}{y_1 e^\phi + y_0 + y_{-1}e^{-\phi}} \cdot \tag{32}$$

In the case $b_1 \neq 0$, $B_1 \neq 0$ *and* $y_1 \neq 0$ *the system* (32)*can be simplified to*

$$u(\eta) = \frac{a_1 e^\phi + a_0 + a_{-1}e^{-\phi}}{e^\phi + b_0 + b_{-1}e^{-d\phi}}, v(\eta) = \frac{A_1 e^\phi + A_0 + A_{-1}e^{-\phi}}{e^\phi + B_0 + B_{-1}e^{-\phi}}, w(\eta) = \frac{c_1 e^\phi + c_0 + c_{-1}e^{-\phi}}{e^\phi + y_0 + y_{-1}e^{-\phi}}$$

$$\tag{33}$$

Substituting Eq. (33) into Eq. (28), we obtain system of algebraic equations for a_1, a_0, a_{-1}, b_0,

b_{-1}, A_1, A_0, A_{-1}, B_0, B_{-1} , c_1, c_0, c_{-1}, y_0, y_{-1} by setting the coefficients of all powers of $\exp(n\ \phi)$ to

be zero. After solving this system, we get:

Case 1.

$$A_0 = -\frac{k^2 c_0 (4a_2 - k^2)}{(c_0 \mp 2c_2)^2}, A_1 = -\frac{k^2 c_2 (4a_2 - k^2)}{(c_0 \mp 2c_2)^2}, \omega = -k^3 + 3ka_2 ,$$

$$A_2 = -\frac{k^2(\pm k^2 c_2 \mp 4c_2 a_2 + 4c_0 a_2 - k^2 c_0)}{(c_0 - 2c_2)^2}, B_0 = \pm 2, B_2 = 1, a_0 = \mp 2k^2 \pm 2a_2, a_1 = a_2,$$

$$a_2 = a_2, b_0 = \pm 2, b_2 = 1, c_0 = c_0, c_1 = -c_2 \pm c_0, k = k, y_0 = 2, y_2 = 1, \tag{34}$$

From eq. (34) and (33), we get the exact solution of u (x, t) and v (x, t) as follows:

v_1

$$= \frac{-\frac{k^2 c_2(4a_2 - k^2)}{(c_0 \mp 2c_2)^2}e^{k(x+(-k^2+3a_2)t)} - \frac{k^2 c_0(4a_2 - k^2)}{(c_0 \mp 2c_2)^2} - \frac{k^2(\pm c_2(k^2 \mp 4a_2) + c_0(4a_2 - k^2))}{(c_0 - 2c_2)^2}e^{-kx-(-k^3+3ka_2)t}}{e^{k(x+(-k^2+3a_2)t)} \pm 2 + e^{-kx-(-k^3+3ka_2)t}}$$

$$u_1 = \frac{a_2 e^{kx+(-k^3+3ka_2)t} \mp 2k^2 \pm 2a_2 + a_2 e^{-kx-(-k^3+3ka_2)t}}{e^{kx+(-k^3+3ka_2)t} \pm 2 + e^{-kx-(-k^3+3ka_2)t}},$$

$$w_1 = \frac{(-c_2 \pm c_0)e^{kx+(-k^3+3ka_2)t} + c_0 + c_2 e^{-kx-(-k^3+3ka_2)t}}{e^{kx+(-k^3+3ka_2)t} \pm 2 + e^{-kx-(-k^3+3ka_2)t}}, \tag{35}$$

Case2.

$$A_0 = \frac{8k^2(\pm 5k^2 c_2 + 2a_0 c_2 \mp a_0 c_0 - 3k^2 c_0)}{(c_0 - 2c_2)^2}, A_1 = -\frac{2k^2(\pm 7k^2 c_0 - 12k^2 c_2 \mp 4a_0 c_2 + 2a_0 c_0)}{(c_0 - 2c_2)^2},$$

$$A_2 = -\frac{2k^2(\pm 7k^2 c_0 - 12k^2 c_2 \mp 4a_0 c_2 + 2a_0 c_0)}{(c_0 - 2c_2)^2}, B_0 = \pm 2, B_2 = 1, a_0 = a_0, a_1 = \pm \frac{1}{2}a_0 + 2k^2, \tag{36}$$

$$a_2 = \pm \frac{1}{2}a_0 + 2k^2, b_0 = \pm 2, b_2 = 1, c_0 = c_0, c_1 = c_2, c_2 = c_2, k = k, y_0 = \pm 2, y_2 = 1, \omega = 5k^3 \pm \frac{3}{2}ka_0$$

From eq. (36) and (33), we get the exact solution of u (x, t) and v (x, t) as follows:

v_2

$$= \frac{-\frac{2k^2(\pm 7k^2 c_0 - 12k^2 c_2 \mp 4a_0 c_2 + 2a_0 c_0)}{(c_0 - 2c_2)^2} e^{kx+\left(5k^3 \pm \frac{3}{2}ka_0\right)t} + \frac{8k^2(\pm 5k^2 c_2 + 2a_0 c_2 \mp a_0 c_0 - 3k^2 c_0)}{(c_0 - 2c_2)^2} - \frac{2k^2(\pm 7k^2 c_0 - 12k^2 c_2 \mp 4a_0 c_2 + 2a_0 c_0)}{(c_0 - 2c_2)^2} e^{-kx-\left(5k^3 \pm \frac{3}{2}ka_0\right)t}}{e^{kx+\left(5k^3 \pm \frac{3}{2}ka_0\right)t} \pm 2 + e^{-kx-\left(5k^3 \pm \frac{3}{2}ka_0\right)t}}$$

$$u_2 = \frac{\left(\pm \frac{1}{2}a_0 + 2k^2\right)e^{kx+\left(5k^3 \pm \frac{3}{2}ka_0\right)t} + a_0 + \left(\pm \frac{1}{2}a_0 + 2k^2\right)e^{-kx-\left(5k^3 \pm \frac{3}{2}ka_0\right)t}}{e^{kx+\left(5k^3 \pm \frac{3}{2}ka_0\right)t} \pm 2 + e^{-kx-\left(5k^3 \pm \frac{3}{2}ka_0\right)t}},$$

$$w_2 = \frac{c_2 e^{kx+\left(5k^3 \pm \frac{3}{2}ka_0\right)t} + c_0 + c_2 e^{-kx-\left(5k^3 \pm \frac{3}{2}ka_0\right)t}}{e^{kx+\left(5k^3 \pm \frac{3}{2}ka_0\right)t} \pm 2 + e^{-kx-\left(5k^3 \pm \frac{3}{2}ka_0\right)t}}, \tag{37}$$

Case 3.

$$A_0 = 0, A_1 = -\frac{48k^4 y_2^3}{(4y_2 + b_0^2)^2}, A_2 = \frac{3}{4}\frac{k^4 y_2 b_0^6}{(4y_2 + b_0^2)^2 c_2}, B_0 = \frac{1}{8}\frac{b_0(4y_2 + b_0^2)}{y_2}, B_2 = \frac{1}{16}\frac{b_0^4}{y_2},$$

$$a_0 = 0, a_1 = k^2, a_2 = \frac{1}{4}k^2 b_0^2, b_0 = b_0, b_2 = \frac{1}{4}b_0^2, c_0 = 0, c_1 = -\frac{1}{4}\frac{c_2 b_0^2}{y_2^2}, c_2 = c_2, k = k,$$

$$y_0 = \frac{1}{2}\frac{4y_2 + b_0^2}{b_0}, y_2 = y_2, \omega = 2k^3, \tag{38}$$

From eq. (36) and (33), we get the exact solution of u (x, t) and v (x, t) as follows:

$$v_3 = \frac{-\frac{48k^4 y_2^3}{(4y_2 + b_0^2)^2 c_2}e^{kx+2k^3 t} + \frac{3}{4}\frac{k^4 y_2 b_0^6}{(4y_2 + b_0^2)^2 c_2}e^{-kx-2k^3 t}}{e^{kx+2k^3 t} + \frac{1}{8}\frac{b_0(4y_2 + b_0^2)}{y_2} + \frac{1}{16}\frac{b_0^4}{y_2}e^{-kx-2k^3 t}},$$

$$u_3 = \frac{k^2 e^{kx+2k^3 t} + \frac{1}{4}k^2 b_0^2 e^{-kx-2k^3 t}}{e^{kx+2k^3 t} + b_0 + \frac{1}{4}b_0^2 e^{-kx-2k^3 t}}, \quad w_3 = \frac{-\frac{1}{4}\frac{c_2 b_0^2}{y_2^2}e^{kx+2k^3 t} + c_2 e^{-kx-2k^3 t}}{e^{kx+2k^3 t} + \frac{1}{2}\frac{4y_2 + b_0^2}{b_0} + y_2 e^{-kx-2k^3 t}}, \tag{39}$$

Case 4.

$$A_0 = 0, A_1 = A_1, A_2 = -A_1, B_0 = 2, B_2 = 1, a_0 = \frac{1}{2}\frac{-3k^4 + 4A_1 C_1}{k^2}, a_1 = \frac{1}{4}\frac{4A_1 C_1 + k^4}{k^2}, b_0 = 2,$$

$$a_2 = \frac{1}{4}\frac{4A_1 C_1 + k^4}{k^2}, b_2 = 1, c_0 = 0, c_1 = c_1, c_2 = -c_1, k = k, y_0 = 2, y_2 = 1, \omega = \frac{1}{4}\frac{-k^4 + 12A_1 c_1}{k} \tag{40}$$

From eq. (36) and (33), we get the exact solution of u (x, t) and v (x, t) as follows:

$$v_4 = \frac{A_1 e^{kx+\left(\frac{-k^4 + 12A_1 c_1}{4k}\right)t} - A_1 e^{-kx-\left(\frac{-k^4 + 12A_1 c_1}{4k}\right)t}}{e^{kx+\left(\frac{-k^4 + 12A_1 c_1}{4k}\right)t} + 2 + e^{-kx-\left(\frac{-k^4 + 12A_1 c_1}{4k}\right)t}},$$

$$u_4 = \frac{\frac{1}{4}\left(\frac{4A_1c_1 + k^4}{k^2}\right)e^{kx+\left(\frac{-k^4+12A_1c_1}{4k}\right)t} + \frac{1}{2}\left(\frac{-3k^4 + 4A_1c_1}{k^2}\right) + \frac{1}{4}\left(\frac{4A_1c_1 + k^4}{k^2}\right)e^{-kx-\left(\frac{-k^4+12A_1c_1}{4k}\right)t}}{e^{kx+\left(\frac{-k^4+12A_1c_1}{4k}\right)t} + 2 + e^{-kx-\left(\frac{-k^4+12A_1c_1}{4k}\right)t}},$$

$$w_4 = \frac{c_1 e^{kx+(\frac{-k^4+12A_1c_1}{4k})t} - c_1 e^{-kx-(\frac{-k^4+12A_1c_1}{4k})t}}{e^{kx+(\frac{-k^4+12A_1c_1}{4k})t} + 2 + e^{-kx-(\frac{-k^4+12A_1c_1}{4k})t}} \tag{41}$$

Case 5.

$$A_0 = A_0, A_1 = \frac{A_0(-c_1 + c_0)}{c_0}, A_2 = \frac{A_0 c_1}{c_0}, B_0 = 2, B_2 = 1, a_0 = -\frac{3k^4 c_0 + A_0(c_0 - 2c_1)^2}{2k^2 c_0},$$

$$a_1 = -\frac{-k^4 c_0 + A_0(c_0 - 2c_1)^2}{4k^2 c_0}, a_2 = -\frac{-k^4 c_0 + A_0(c_0 - 2c_1)^2}{4k^2 c_0}, b_0 = 2, b_2 = 1, c_0 = c_0,$$

$$c_1 = c_1, c_2 = -c_1 + c_0, k = k, y_0 = 2, y_2 = 1, \omega = -\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}, \tag{42}$$

From eq. (36) and (33), we get the exact solution of u (x, t) and v (x, t) as follows:

$$v_5 = \frac{\frac{A_0(-c_1 + c_0)}{c_0}e^{kx+\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t} + A_0 + \frac{A_0 c_1}{c_0}e^{-kx-\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t}}{e^{kx+\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t} + 2 + e^{-kx-\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t}},$$

$$u_5$$

$$= \frac{-\frac{-k^4 c_0 + A_0(c_0 - 2c_1)^2}{4k^2 c_0}e^{kx+\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t} - \frac{3k^4 c_0 + A_0(c_0 - 2c_1)^2}{2k^2 c_0} - \frac{k^4 c_0 + A_0(c_0 - 2c_1)^2}{4k^2 c_0}e^{-kx-\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t}}{e^{kx+\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t} + 2 + e^{-kx-\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t}},$$

$$w_5 = \frac{c_1 e^{kx+\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t} + c_0 + (-c_1 + c_0)e^{-kx-\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t}}{e^{kx+\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t} + 2 + e^{-kx-\left(-\frac{k^4 c_0 + 3A_0(c_0 - 2c_1)^2}{4kc_0}\right)t}} \tag{43}$$

Case 6.

$$A_0 = \frac{4k^4 + 2A_1(c_0 - 2c_1)}{c_0 - 2c_1}, A_1 = A_1, A_2 = A_1, B_0 = 2, B_2 = 1, b_0 = 2, b_2 = 1, c_0 = c_0, c_1 = c_1,$$

$$y_0 = 2, y_2 = 1, c_2 = c_1,$$

$$a_0 = -\frac{2k^4(7c_0 - 12c_1) + A_1(c_0 - 2c_1)^2}{4k^2(c_0 - 2c_1)}, a_1 = -\frac{-2k^4(c_0 - 4c_1) + A_1(c_0 - 2c_1)^2}{8k^2(c_0 - 2c_1)},$$

$$a_2 = -\frac{-2k^4(c_0 - 4c_1) + A_1(c_0 - 2c_1)^2}{8k^2(c_0 - 2c_1)}, \omega = -\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}, \tag{44}$$

From eq. (36) and (33), we get the exact solution of u (x, t) and v (x, t) as follows:

$$v_6 = \frac{A_1 e^{kx + \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t} + \frac{2(-2A_1 c_1 + 2k^4 + c_0 A_1)}{c_0 - 2c_1} + A_1 e^{-kx - \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t}}{e^{kx + \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t} + 2 + e^{-kx - \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t}}$$

u_6

$$= \frac{\frac{2k^4(c_0 - 4c_1) - A_1(c_0 - 2c_1)^2}{8k^2(c_0 - 2c_1)} e^{kx + \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t} - \frac{2k^4(7c_0 - 12c_1) + A_1(c_0 - 2c_1)^2}{4k^2(c_0 - 2c_1)} + \frac{2k^4(c_0 - 4c_1) - A_1(c_0 - 2c_1)^2}{8k^2(c_0 - 2c_1)} e^{-kx - \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t}}{e^{kx + \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t} + 2 + e^{-kx - \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t}}$$

$$w_6 = \frac{c_1 e^{kx + \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t} + c_0 + c_1 e^{-kx - \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t}}{e^{kx + \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t} + 2 + e^{-kx - \left(-\frac{3A_1(c_0 - 2c_1)^2 + 2k^4(c_0 + 4c_1)}{8k(c_0 - 2c_1)}\right)t}}, \qquad (45)$$
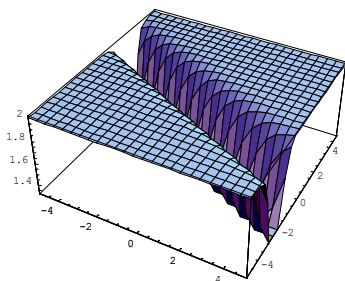
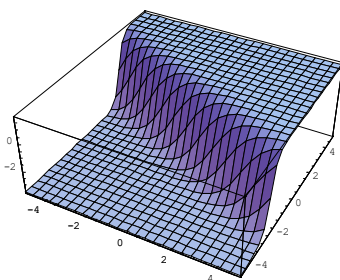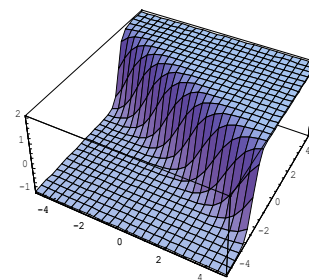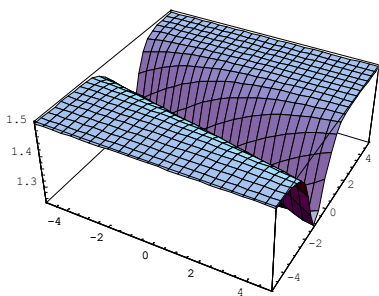Case1 fig.6a exact solution of $u_1(x, t)$ Fig.6 b exact solution of $v_1(x ,t)$ Fig.6c exact solution of $w_1(x ,t)$

$$a_2 = 2; c_0 = 1; c_2 = -1; k = 2$$

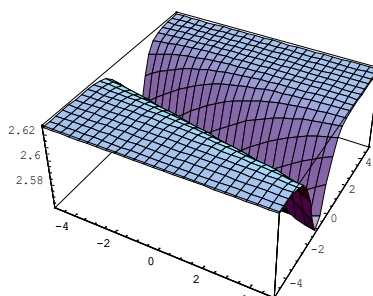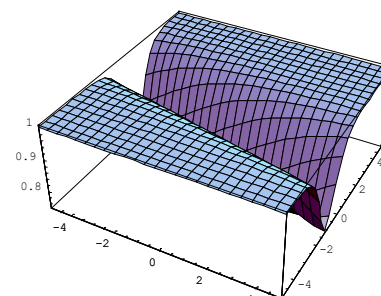Case2 fig.7a exact solution of $u_2(x ,t)$ Fig.7 b exact solution of $v_2(x ,t)$ Fig.7c exact solution of $w_2(x ,t)$

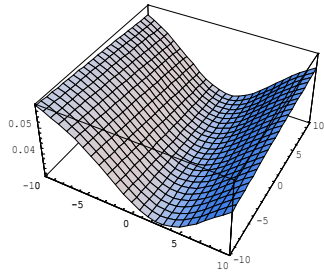$$a_0 = 2; c_2 = 1; k = -\frac{1}{2}; c_0 = 1$$

fig.8a exact solution of $u_3(x, t)$

Fig.8b exact solution of $v_3(x, t)$

Fig.8c exact solution of $w_3(x, t)$

$$y_2 = -2; b_0 = 3; c_2 = 1; k = \frac{1}{4}$$

fig.9a exact solution of $u_4(x, t)$
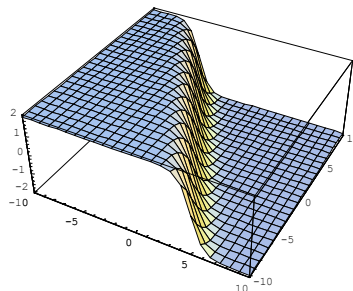
Fig.9b exact solution of $v_4(x, t)$

Fig.9c exact solution of $w_4(x, t)$

$$A_1 = 2; c_1 = 1; k = -2$$

Case5 fig.10a exact solution of $u_5(x, t)$

Fig.10b exact solution of $v_5(x, t)$

Fig.10c exact solution of $w_5(x, t)$

$$A_0 = 1; c_0 = -1; c_1 = -2; k = 2$$
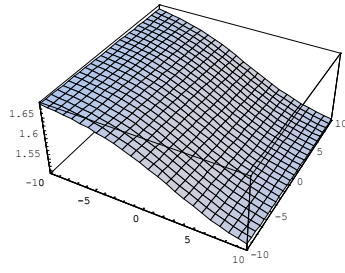

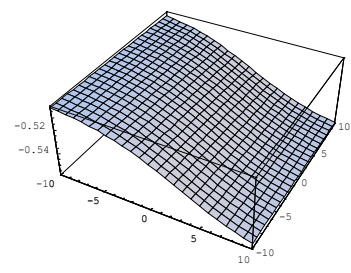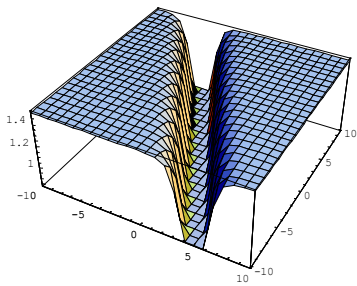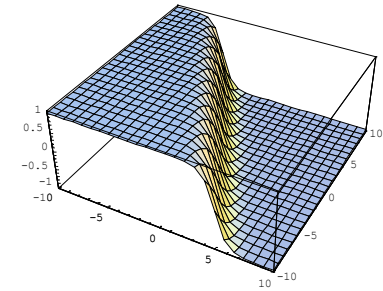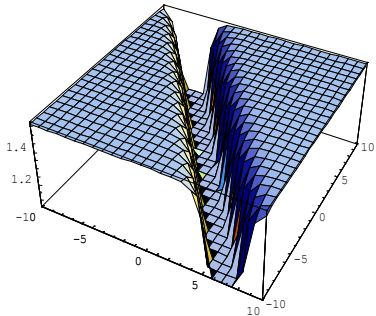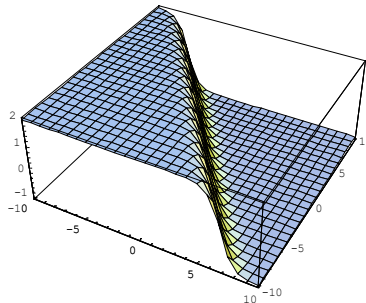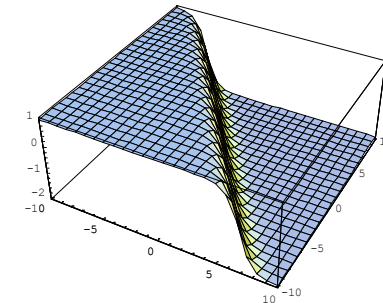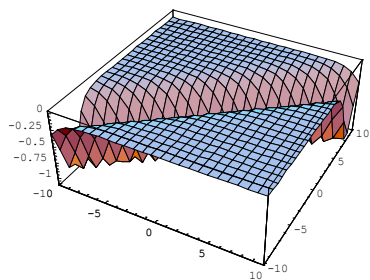Case6 fig.11a exact solution of $u_6(x, t)$

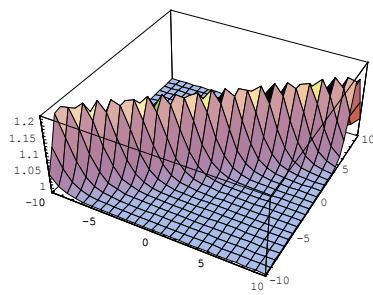Fig.11b exact solution of $v_6(x, t)$
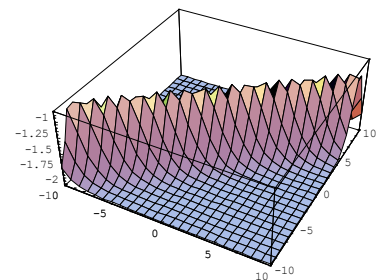
Fig.11c exact solution of $w_6(x, t)$

$$A_1 = 1; c_1 = -2; c_0 = \frac{1}{2}; k = 1$$

4. DISCUSSION

In pervious section we obtain the exact solutions for example 1 and 2 by exp-function method, and figs. (1)- (5) shows five cases of the exact solution of example 1 and figs. (6)- (11) shows six cases of the exact solutions of example 2. In this section, we obtain the corresponding numerical solutions of example 1 and 2 by the homotopy analysis method[20], [21]. In Table 1 and Table 2, display the absolute errors between the 15-term HAM solutions and the exact solutions for the coupled Hirota-Satsuma-KdV equation, and table (3-5) , display the absolute errors between the 10-term HAM solutions and the exact solutions for the Hirota–Satsuma coupled KdV system.

Table 1. Absolute error of HAM for example 1 of u (x, t)

t	$|u_{ex} - u_1 HAM|$	$|u_{ex} - u_2 HAM|$	$|u_{ex} - u_3 HAM|$	$|u_{ex} - u_4 HAM|$	$|u_{ex} - u_5 HAM|$
0.1	3.408×10^{-32}	1.502×10^{-35}	1.690×10^{-36}	9.590×10^{-27}	1.051×10^{-41}
0.2	2.252×10^{-27}	9.853×10^{-31}	1.037×10^{-31}	6.081×10^{-22}	6.816×10^{-37}
0.3	1.501×10^{-24}	6.417×10^{-28}	6.757×10^{-29}	3.868×10^{-19}	4.4260×10^{-34}
0.4	1.519×10^{-22}	6.349×10^{-26}	6.689×10^{-27}	3.741×10^{-17}	4.366×10^{-32}
0.5	5.478×10^{-21}	2.2363×10^{-24}	2.357×10^{-25}	1.289×10^{-15}	1.533×10^{-30}
0.6	1.027×10^{-19}	4.101×10^{-23}	4.325×10^{-24}	2.3146×10^{-14}	2.804×10^{-29}
0.7	1.229×10^{-18}	4.791×10^{-22}	5.056×10^{-23}	2.649×10^{-13}	3.267×10^{-28}
0.8	1.057×10^{-17}	4.025×10^{-21}	4.250×10^{-22}	2.182×10^{-12}	2.737×10^{-27}
0.9	7.0667×10^{-17}	2.628×10^{-20}	2.776×10^{-21}	1.397×10^{-11}	1.782×10^{-26}
1.0	3.873×10^{-16}	1.407×10^{-19}	1.487×10^{-20}	7.346×10^{-11}	9.519×10^{-26}

Table 2. Absolute error of HAM for example 1 of v (x, t)

t	$|v_{ex} - v_1 HAM|$	$|v_{ex} - v_2 HAM|$	$|v_{ex} - v_2 HAM|$	$|v_{ex} - v_4 HAM|$	$|v_{ex} - v_5 HAM|$
0.1	1.730×10^{-32}	7.280×10^{-36}	1.010×10^{-36}	1.918×10^{-26}	5.240×10^{-42}
0.2	1.126×10^{-27}	4.926×10^{-31}	5.185×10^{-32}	1.216×10^{-21}	3.408×10^{-37}
0.3	7.505×10^{-25}	3.2086×10^{-28}	3.378×10^{-29}	7.736×10^{-19}	2.213×10^{-34}
0.4	7.597×10^{-23}	3.1745×10^{-26}	3.344×10^{-27}	7.481×10^{-17}	2.183×10^{-32}
0.5	2.739×10^{-21}	1.118×10^{-24}	1.178×10^{-25}	2.578×10^{-15}	7.669×10^{-31}
0.6	5.139×10^{-20}	2.050×10^{-23}	2.162×10^{-24}	4.629×10^{-14}	1.402×10^{-29}
0.7	6.146×10^{-19}	2.395×10^{-22}	2.528×10^{-23}	5.298×10^{-13}	1.633×10^{-28}
0.8	5.285×10^{-18}	2.012×10^{-21}	2.125×10^{-22}	4.36×10^{-12}	1.368×10^{-27}
0.9	3.533×10^{-17}	1.314×10^{-20}	1.388×10^{-21}	2.795×10^{-11}	8.914×10^{-27}
1.0	1.936×10^{-16}	7.035×10^{-20}	7.435×10^{-21}	1.469×10^{-10}	4.759×10^{-26}

Table 3. Absolute error of HAM for example 2 of u (x, t)

t	$|u_{ex} - u_1 HAM|$	$|u_{ex} - u_2 HAM|$	$|u_{ex} - u_3 HAM|$	$|u_{ex} - u_4 HAM|$	$|u_{ex} - u_5 HAM|$	$|u_{ex} - u_6 HAM|$
0.1	4.892×10^{-10}	6.026×10^{-14}	1.550×10^{-11}	1.845×10^{-10}	4.892×10^{-10}	1.309×10^{-10}
0.2	4.892×10^{-10}	7.760×10^{-14}	1.560×10^{-11}	1.845×10^{-10}	4.892×10^{-10}	1.309×10^{-10}
0.3	4.892×10^{-10}	9.595×10^{-14}	1.540×10^{-11}	1.845×10^{-10}	4.892×10^{-10}	1.309×10^{-10}
0.4	4.892×10^{-10}	1.155×10^{-13}	1.520×10^{-11}	1.845×10^{-10}	4.892×10^{-10}	1.309×10^{-10}
0.5	4.892×10^{-10}	1.379×10^{-13}	1.370×10^{-11}	1.845×10^{-10}	4.892×10^{-10}	1.306×10^{-10}
0.6	4.892×10^{-10}	1.719×10^{-13}	2.700×10^{-12}	1.845×10^{-10}	4.892×10^{-10}	1.283×10^{-10}
0.7	4.892×10^{-10}	2.581×10^{-13}	5.310×10^{-11}	1.844×10^{-10}	4.892×10^{-10}	1.167×10^{-10}
0.8	4.893×10^{-10}	5.419×10^{-13}	2.791×10^{-10}	1.844×10^{-10}	4.892×10^{-10}	6.987×10^{-11}
0.9	4.8971×0^{-10}	1.466×10^{-12}	1.046×10^{-9}	1.844×10^{-10}	4.892×10^{-10}	9.004×10^{-11}
1.0	4.919×10^{-10}	4.216×10^{-12}	3.328×10^{-9}	1.844×10^{-10}	4.892×10^{-10}	5.663×10^{-10}

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Table 4. Absolute error of HAM for example 2 of v (x, t)

t	$|v_{ex} - v_1 HAM|$	$|v_{ex} - v_2 HAM|$	$|v_{ex} - v_2 HAM|$	$|v_{ex} - v_4 HAM|$	$|v_{ex} - v_5 HAM|$	$|v_{ex} - v_6 HAM|$
0.1	1.784×10^{-9}	7.007×10^{-10}	3.018×10^{-10}	4.625×10^{-10}	1.834×10^{-10}	7.69×10^{-10}
0.2	1.784×10^{-9}	7.008×10^{-10}	3.012×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	7.69×10^{-10}
0.3	1.784×10^{-9}	7.008×10^{-10}	3.015×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	7.69×10^{-10}
0.4	1.784×10^{-9}	7.008×10^{-10}	3.015×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	7.69×10^{-10}
0.5	1.784×10^{-9}	7.007×10^{-10}	3.008×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	7.693×10^{-10}
0.6	1.784×10^{-9}	7.007×10^{-10}	2.964×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	7.715×10^{-10}
0.7	1.784×10^{-9}	7.006×10^{-10}	2.732×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	7.831×10^{-10}
0.8	1.784×10^{-9}	7.006×10^{-10}	1.804×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	8.299×10^{-10}
0.9	1.784×10^{-9}	7.003×10^{-10}	1.357×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	9.898×10^{-10}
1.0	1.783×10^{-9}	6.993×10^{-10}	1.073×10^{-10}	4.626×10^{-10}	1.834×10^{-10}	1.466×10^{-9}

Table 5. Absolute error of HAM for example 2 of w (x, t)

t	$|w_{ex} - w_1 HAM|$	$|w_{ex} - w_2 HAM|$	$|w_{ex} - w_2 HAM|$	$|w_{ex} - w_4 HAM|$	$|w_{ex} - w_5 HAM|$	$|w_{ex} - w_6 HAM|$
0.1	1.816×10^{-9}	4.232×10^{-10}	4.74×10^{-10}	6.564×10^{-11}	2.165×10^{-10}	8.922×10^{-10}
0.2	1.816×10^{-9}	4.234×10^{-10}	4.73×10^{-10}	6.565×10^{-11}	2.165×10^{-10}	8.922×10^{-10}
0.3	1.816×10^{-9}	4.233×10^{-10}	4.77×10^{-10}	6.565×10^{-11}	2.165×10^{-10}	8.922×10^{-10}
0.4	1.816×10^{-9}	4.232×10^{-10}	4.76×10^{-10}	6.565×10^{-11}	2.165×10^{-10}	8.922×10^{-10}
0.5	1.816×10^{-9}	4.233×10^{-10}	4.70×10^{-10}	6.5651×10^{-11}	2.165×10^{-10}	8.923×10^{-10}
0.6	1.816×10^{-9}	4.232×10^{-10}	4.46×10^{-10}	6.5652×10^{-11}	2.165×10^{-10}	8.929×10^{-10}
0.7	1.816×10^{-9}	4.234×10^{-10}	3.25×10^{-10}	6.5652×10^{-11}	2.165×10^{-10}	8.958×10^{-10}
0.8	1.816×10^{-9}	4.235×10^{-10}	1.72×10^{-10}	6.5653×10^{-11}	2.165×10^{-10}	9.075×10^{-10}
0.9	1.816×10^{-9}	4.242×10^{-10}	1.857×10^{-9}	6.5653×10^{-11}	2.165×10^{-10}	9.475×10^{-10}
1.0	1.815×10^{-9}	4.263×10^{-10}	6.854×10^{-9}	6.5654×10^{-11}	2.165×10^{-10}	1.066×10^{-9}

And figures (12)-(16) shows the comparison of the exact solution by exp-function method with analytic approximation given by HAM for example 1 from case 1 to case 5, respectively and figures (17)-(22) shows the comparison of the exact solution by exp-function method with analytic approximation given by HAM for example 2 from case 1 to case 6 ,respectively .
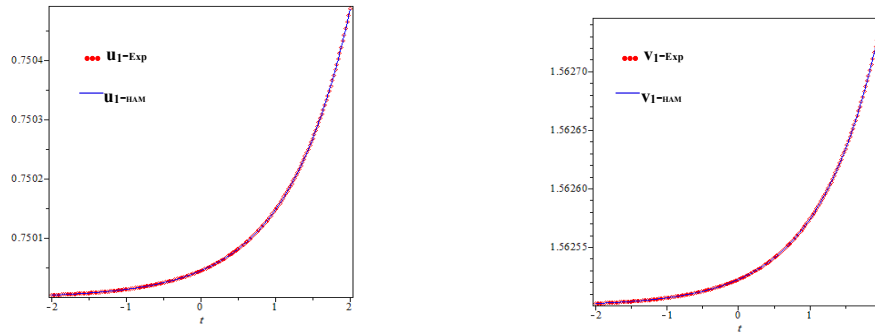

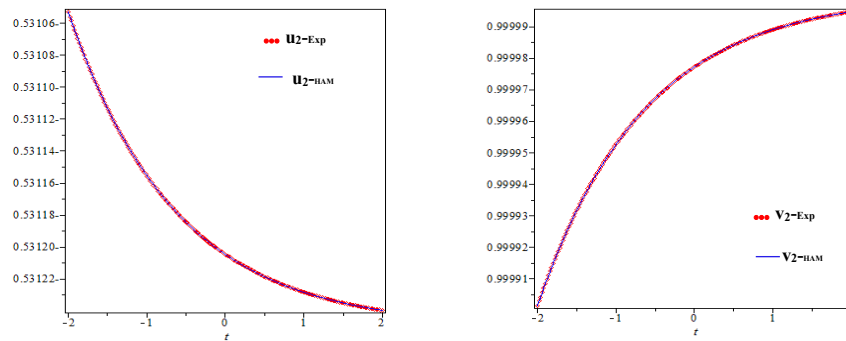Fig.12 Exact and approximation solution of u (x, t) and v (x, t) at $\hbar$=-1, (x=20, $k = -\frac{1}{2}$)

Fig.13 Exact and approximation solution of u (x, t) and v (x, t) at $\hbar$=-1, (x=20, $A_1 = 1, k = -\frac{1}{2}$)
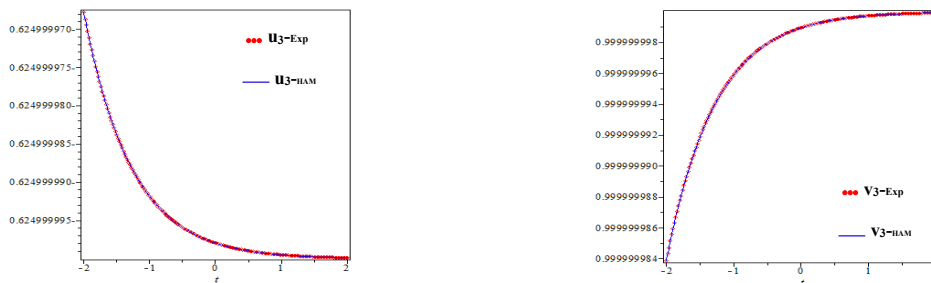
Fig.14 Exact and approximation solution of u (x, t) and v (x, t) at $\hbar$=-1, (x=20, $B_0 = 1; A_1 = 1; k = 1$)
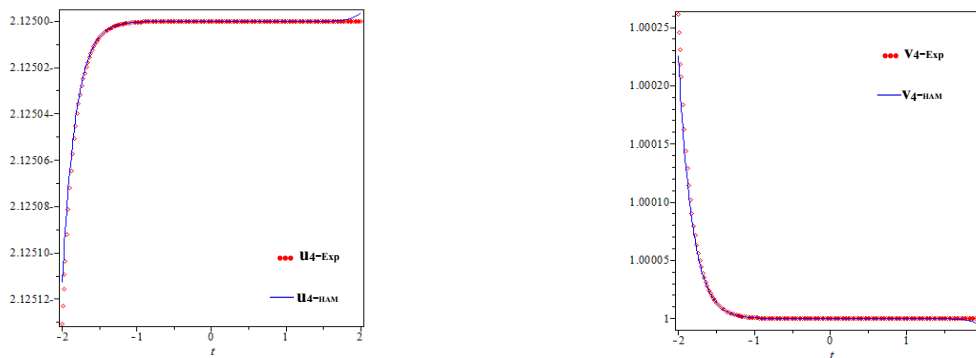
Fig.15 Exact and approximation solution of u (x, t) and v (x, t) at $\hbar$=-1, (x=20, $B_0 = -4; A_1 = -1; k = -1$)
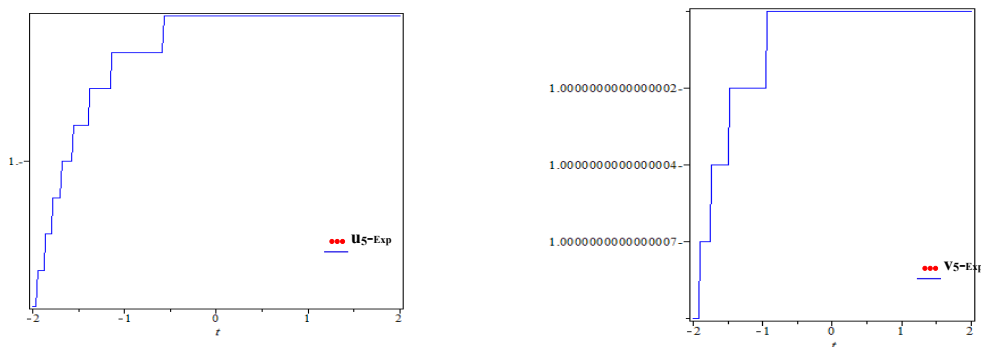
Fig.16 Exact and approximation solution of u (x, t) and v (x, t) at $\hbar$=-1, (x=20, $B_0 = 2, b_0 = -1, A_1 = 1, k = -2$)
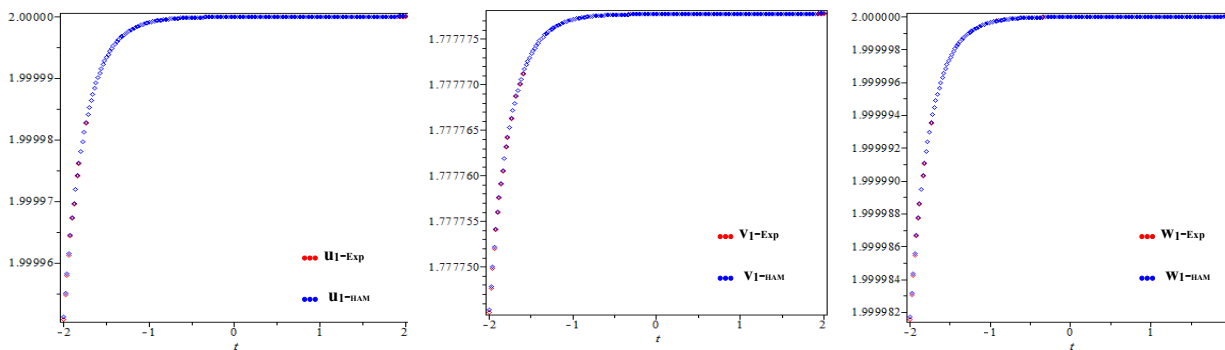

Fig.17 Exact and approximation solution of u(x, t), v(x, t) and w(x, t) at $\hbar$=-1, (x=10, $a_2 = 2, c_0 = 1, c_2 = -1, k = 2$)
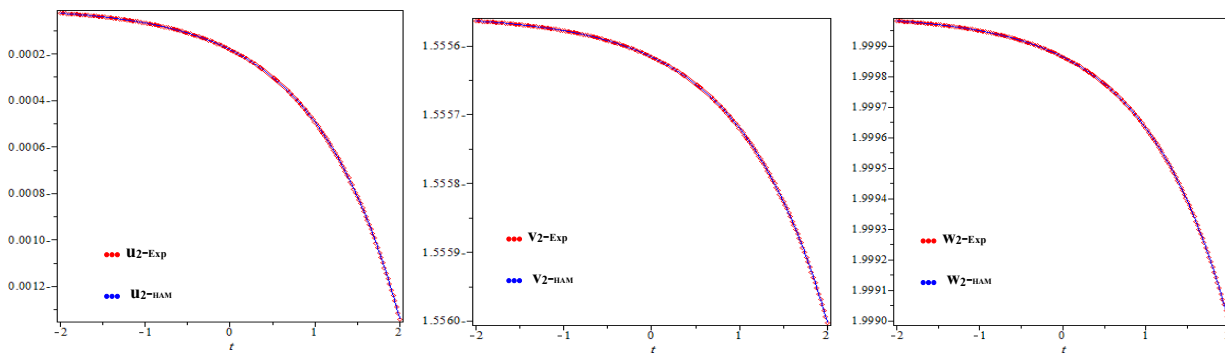

Fig.18 Exact and approximation solution of u(x, t), v(x, t) and w(x, t) at at $\hbar$=-1, (x=10 $c_2 = 2, c_0 = 1, k = 1, a_0 = -4$)
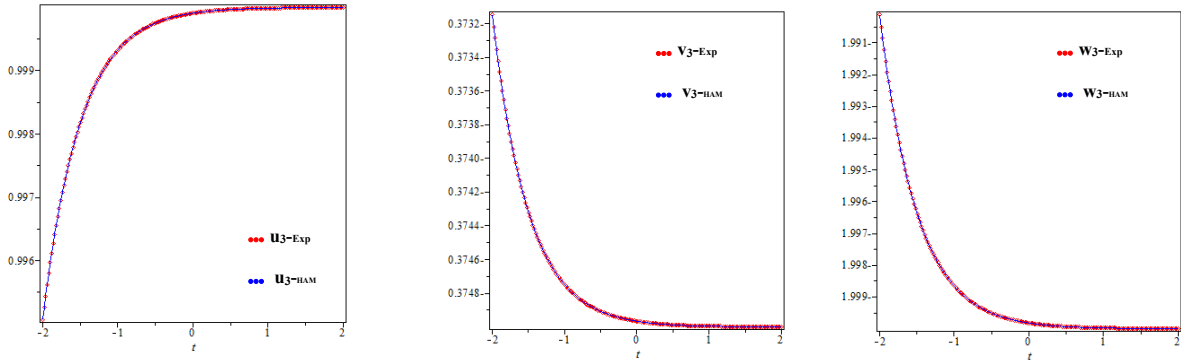
Fig.19 Exact and approximation solution of u(x, t), v(x, t) and w(x, t) at $\hbar$=-1, (x=10 $k = 1, c_2 = 2, b_0 = 2, y_2 = 1$)

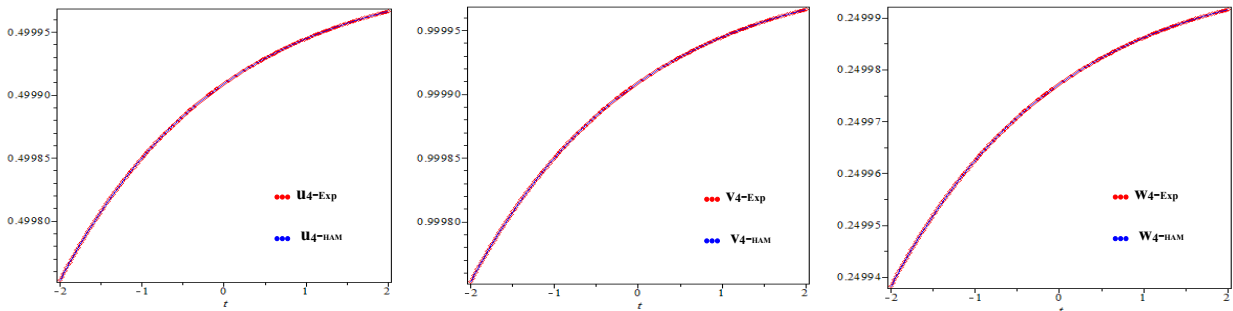Fig.20 Exact and approximation solution of u(x, t), v(x, t) and w(x, t) at at $\hbar$=-1, (x=10 $k = 1, A_1 = 1, c_1 = \frac{1}{4}$)
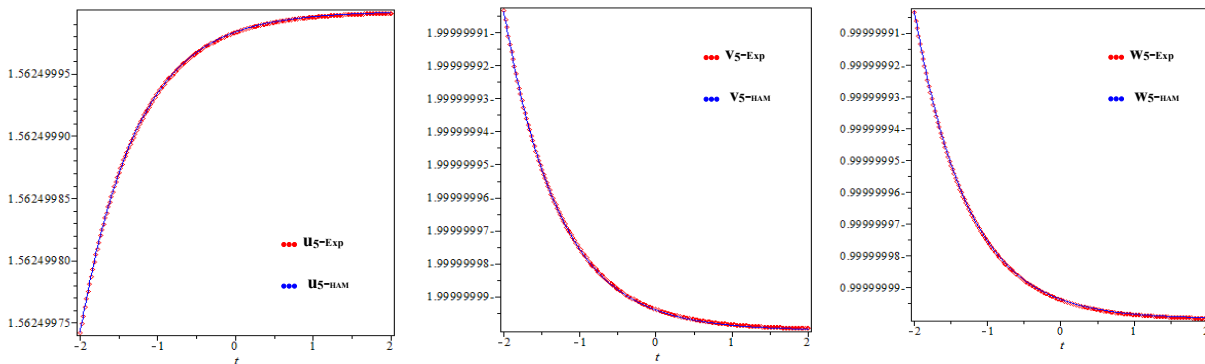

Fig.21 Exact and approximation solution of u(x, t), v(x, t) and w(x, t) at at $\hbar$=-1, (x=10, $k = 2, A_0 = -1, c_0 =, c_1 = -1$)
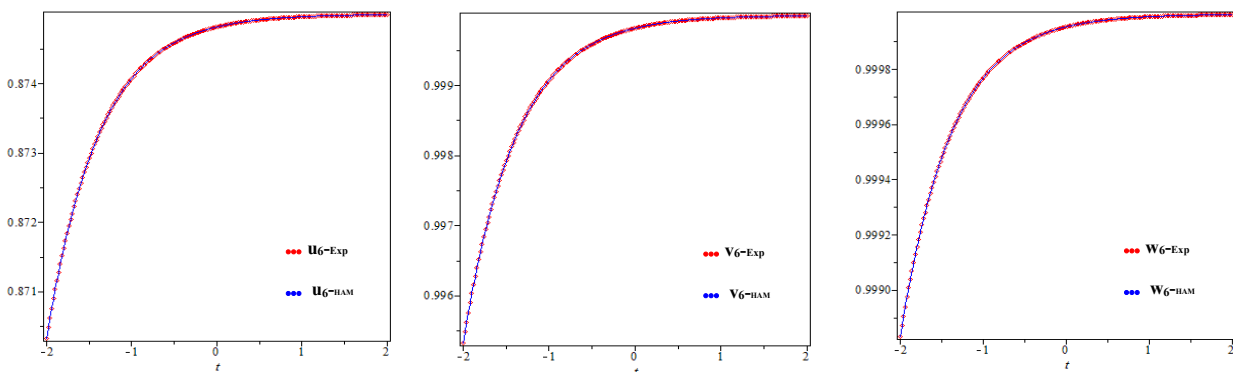

Fig.22 Exact and approximation solution of u(x, t), v(x, t) and w(x, t) at $\hbar$=-1, ($x = 10, k =, A_1 = c_0 = c_1 = 1$)

4. CONCLUSIONS

In the presented work, the Exp-function method has been applied to the coupled Hirota-Satsuma-KdV equation and the Hirota–Satsuma coupled KdV system in order to find some new types of exact solutions. This procedure is very unpretentious, and our results obtained by the Exp-function method compared with those obtained by the homotopy analysis method.

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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