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A NEW ITERATION PROCESS FOR APPROXIMATION OF FIXED POINTS FOR SUZUKI'S GENERALIZED NON-EXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract: This paper is proposed to introduce a new three step iteration process, called AR-iteration process to approximate fixed points of Suzuki's generalized non-expansive mappings. Some weak and strong convergence results are proved for such mappings in uniformly convex Banach spaces. Moreover, we prove analytically and with numerical example that our iteration process converges faster than some other known iteration processes. To support our claim, we use MATLAB program to approximate fixed points for Suzuki's generalized non-expansive mappings using our new iteration process and Picard, Thakur, M and Sahu iteration processes. We extend, improve and generalize several known results in the literature of iteration processes of fixed point theory.

Keywords: Suzuki's generalized non-expansive mappings; fixed points; iteration process; uniformly convex Banach space; weak and strong convergence results.

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1. INTRODUCTION

Many problems that usually occur in different fields like engineering, economics, chemistry and game theory etc. can be solved by using fixed point theory since it provides useful tools to solve equations. In this direction, Banach contraction theorem [3] is the first and important tool in the hands of authors which guarantee the existence and uniqueness of fixed points. However, once the existence of a fixed point of some mapping is established then the problem is to find the value of that fixed point and iteration processes have great achievement in this regard. The Picard iteration process for approximation of fixed points is used in well-known Banach contraction theorem. After that many iteration processes came in existence to approximate the fixed points of nonlinear mappings satisfying different contractive conditions like: Mann[9], Ishikawa[7], Agarwal[2], Noor[10], Abbas[1], SP[13], Picard Mann[8], Picard-S[6], Thakur New[20], Sahu[16], M[22], J[4].

The convergence speed is an important factor for considering an iteration process over another iteration process. Rhoades [15] made a remark that: Mann iteration process for decreasing functions converges faster than Ishikawa iteration process and in case of increasing functions, Ishikawa iteration process is better than the Mann iteration process. Also, Mann iteration process appears to be independent of the initial guess (see also [14]). Rhoades [15] also made a remark that: "It is doubtful if any global statement can be made, since there is nothing about these iteration procedures to cause their analysis to be different from that of the other approximation method."

Interest in the uniformly convex Banach spaces lies in the fact that the metric properties of the problem at hand can be analyzed more accurately and in a comparatively straightforward and easy manner. Suzuki [19] introduced the concept of generalized non-expansive mappings in 2008. These are also known as condition (C) and defined as:

Let B be a nonempty subset of a Banach space X. A mapping $T: B \to B$ is said to satisfy condition (C), if for all x, y $\in B$, we have

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\| .$$

Suzuki obtained existence of fixed points and convergence results for such mappings. Suzuki's generalized non-expansive mappings are more general than non-expansive mappings as well as contraction mappings. Suzuki also showed that mappings satisfying condition (C) are weaker than non-expansiveness and stronger than quasi non-expansiveness.

Recently, MATLAB program has become a tool to calculate and approximate the fixed points more accurately with comparatively less errors. Using fixed point algorithms, we can find the approximate values for different iterative schemes in less time and can compare the convergence speed of these iterations which can be shown graphically.

Motivated by above, in this paper, we introduce a new three step iteration process namely AR - iteration process. We prove some weak and strong convergence theorems for Suzuki's generalized non-expansive mappings in the setting of uniformly convex Banach spaces. We compare analytically and with numerical example the speed of convergence of this new iteration process that is AR - iteration process with the Picard, Thakur, M and Sahu iteration processes using MATLAB program.

2. PRELIMINARIES

A point which remains invariant under a given map is known as a fixed point of the map i.e., for a mapping T: X $\to$ X, a fixed point is a point x $\in X$ for which T (x) = x. Throughout the paper, the set $\mathbb{N}$ stands for all positive integers. Let B be a nonempty closed and convex subset of a real Banach space X and T: $B \to B$ be a nonlinear mapping. Let, $F_T = \{x \in B : Tx = x\}$, be the set of all fixed points of the mapping T on B.

The mapping T: $B \to B$ satisfies a Lipschitz condition with constant $\alpha \geq 0$ such that

$$\mathrm{d}(Tx, Ty) \leq \alpha\, d(x, y), \qquad \text{for all } x, y \in B.$$

On the basis of Lipschitz constant we have the following cases:

(i) If $\alpha < 1$, T is called a contraction mapping;

(ii) If $\alpha \leq 1$, T is called a non expansive mapping;

(iii) If $\alpha = 1$, T is called a contractive.

A mapping $T\colon B \to B$ is called quasi-nonexpansive if for all $x \in B$ and $p \in F_T$, we have $\|Tx - p\| \leq \|x - p\|$.

Now we list some definitions, propositions and lemmas which will be used in the next section:

Definition 2.1. A Banach space X is called uniformly convex [5] if for each $\epsilon \in (0,2]$ there is a $\delta > 0$ such that for x, y $\in X$,

$$\left.\begin{array}{l}\|x\| \leq 1,\\ \|y\| \leq 1,\\ \|x - y\| > \epsilon\end{array}\right\} \Longrightarrow \left\|\frac{x+y}{2}\right\| \leq \delta.$$

Definition 2.2. Let B be a nonempty closed and convex subset of a uniformly convex Banach space X. A mapping $T\colon B \to X$ is called demiclosed with respect to y $\in$ X if for each sequence $\{x_m\}$ in B and each x $\in B$, $\{x_m\}$ converges weakly at x and $\{Tx_m\}$ converges strongly at y imply that $Tx = y$.

Definition 2.3. A Banach space X is said to satisfy Opial property [11] if for each sequence $\{x_m\}$ in X, converging weakly to x $\in X$, we have

$\lim_{m\to\infty} sup \|x_m - x\| < \lim_{m\to\infty} sup \|x_m - y\|$, for all y $\in X$ such that y $\neq$ x.

Definition 2.4. Let $\{x_m\}$ be a bounded sequence in a Banach space X. For x $\in B \subset X$, we set

$$r(x, \{x_m\}) = \lim_{m\to\infty} sup \|x_m - x\|.$$

The asymptotic radius of $\{x_m\}$ relative to B is defined by

$$r(B, \{x_m\}) = \inf\{ r(x, \{x_m\}) : x \in B\},$$

The asymptotic center of $\{x_m\}$ relative to B is defined by

$$A(B, \{x_m\}) = \{x \in C : r(x, \{x_m\}) = r(B, \{x_m\})\}.$$

In a uniformly convex Banach space, $A(B, \{x_m\})$ consists exactly one point.

Proposition 2.5. [19] Let us consider B be a nonempty subset of a Banach space X and $T\colon B \to B$ be a mapping. Then

(i) If T is nonexpansive then T satisfies condition (C).

(ii) Every mapping satisfying condition (C) with a fixed point is quasi non-expansive.

(iii) If T satisfies condition (C), then

$\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$, for all x, y $\in B$.

Lemma 2.6. [19] Let T be a mapping on a subset B of a Banach space X with the Opial property. Assume that T is Suzuki's generalized non-expansive mapping. If $\{x_m\}$ converges weakly to z and $\lim_{m\to\infty} \|Tx_m - x_m\| = 0$, then Tz = z.

Lemma 2.7. [19] Let B be a weakly compact convex subset of a uniformly convex Banach space X. Let T be a mapping on B. Assume that T is Suzuki's generalized non-expansive mapping. Then T has a fixed point.

Lemma 2.8. [17] Suppose that X is a uniformly convex Banach space and $\{t_m\}$ be any real sequence such that $0 < p \leq t_m \leq q < 1$ for all m ≥ 1. Let $\{x_m\}$ and $\{y_m\}$ be any two sequences of X such that

$\lim_{m\to\infty} sup \|x_m\| \leq$ r,

$\lim_{m\to\infty} sup \|y_m\| \leq$ r and

$\lim_{m\to\infty} sup \|t_m x_m + (1 - t_m) y_m\| =$ r; hold for some r ≥ 0.

Then $\lim_{m\to\infty} \|x_m - y_m\| = 0$.

Definition 2.9. [18] The notion of mappings satisfying condition (I), introduced by Senter and Dotson [17] is defined as:

A mapping $T: B \to B$ is said to satisfy condition (I), if there is a non decreasing function g: [0,∞) → [0,∞) with g(0) = 0 and g(c) > 0, for all c > 0 such that d(x, Tx) ≥ g(d(x, F_T)), for all x ∈ B, where d(x, F_T) = inf$\{d(x,p) : p \in F_T\}$.

Some Iteration processes which already exist in literature are listed here to revise this concept more accurately.

Here we will denote the iterative sequences by $\{x_m\}$ and let B be a nonempty subset of Banach space X and T be a mapping on B. Throughout this section we have m ∈ ℕ and $\{a_m\}$, $\{b_m\}$ and $\{c_m\}$ are sequences in [0,1], satisfying appropriate conditions.

In 1890, Picard [12] introduced the iteration process as:

$$\begin{cases} x_0 \in B, \\ x_{m+1} = Tx_m. \end{cases} \tag{1}$$

In 1953, Mann [9] introduced the iteration process as:

$$\begin{cases} x_0 \in B, \\ x_{m+1} = (1 - a_m)x_m + a_m Tx_m. \end{cases} \tag{2}$$

In 1974, Ishikawa [7] introduced ,the first two-step iteration process as:

$$\begin{cases} x_0 \in B, \\ y_m = (1 - a_m)x_m + m Tx_m, \\ x_{m+1} = (1 - b_m)x_m + b_m Ty_m. \end{cases} \tag{3}$$

In 2000, Noor [10] introduced the first three-step iteration process as:

$$\begin{cases} x_0 \in B, \\ z_m = (1 - a_m)x_m + a_m Tx_m, \\ y_m = (1 - b_m)x_m + b_m Tz_m, \\ x_{m+1} = (1 - c_m)x_m + c_m Ty_m. \end{cases} \tag{4}$$

In 2007, Aggarwal [2] introduced the S-iteration process as:

$$\begin{cases} x_0 \in B, \\ y_m = (1 - a_m)x_m + a_m Tx_m, \\ x_{m+1} = (1 - b_m)Tx_m + b_m Ty_m. \end{cases} \tag{5}$$

In 2014, Abbas [1] introduced the iteration process as:

$$\begin{cases} x_0 \in B, \\ z_m = (1 - a_m)\, x_m + a_m Tx_m, \\ y_m = (1 - b_m)\, Tx_m + b_m Tz_m, \\ x_{m+1} = (1 - c_m)\, Ty_m + c_m Tz_m, \end{cases} \tag{6}$$

In 2016, Thakur et. al. [20] introduced the iteration process as:

$$\begin{cases} x_0 \in B, \\ z_m = (1 - a_m)\, x_m + a_m Tx_m, \\ y_m = T((1 - b_m)\, x_m + b_m z_m), \\ x_{m+1} = Ty_m. \end{cases} \tag{7}$$

In 2016, Sahu [16] and Thakur et al. [21] introduced the same iteration process for non-expansive mappings in uniformly convex Banach space as:

$$\begin{cases} x_0 \in B, \\ z_m = (1 - a_m)\, x_m + a_m Tx_m, \\ y_m = (1 - b_m)\, z_m + b_m Tz_m, \\ x_{m+1} = (1 - c_m)Tz_m + c_m Ty_m. \end{cases} \tag{8}$$

In 2018, K. Ullah and M. Arshad [22] introduced the M iteration process as:

$$\begin{cases} x_0 \in B, \\ z_m = (1 - a_m) x_m + a_m T x_m, \\ y_m = T z_m, \\ x_{m+1} = T y_m. \end{cases} \tag{9}$$

The authors proposes M iteration for approximation of fixed points for Suzuki's generalized non-expansive mappings in the setting of uniformly convex Banach spaces. The authors numerically showed that this iteration process converges faster than Picard-S and S iteration processes.

Recently, in 2019, J.D. Bhutia And K.Tiwary [4] introduced J iteration process as:

$$\begin{cases} x_0 \in B, \\ z_m = T((1 - a_m) x_m + a_m T x_m), \\ y_m = T((1 - b_m) z_m + b_m T z_m), \\ x_{m+1} = T y_m, \end{cases} \tag{10}$$

The authors numerically showed that J iteration process converges faster than other iteration processes. The authors also discuss stability results of J iteration process and prove some results in the context of uniformly convex Banach space for Suzuki's generalized non expansive mappings.

3. MAIN RESULTS

We now introduce a new iteration process "AR - iteration process" as follows:

$$\begin{cases} x_0 \in B, \\ z_m = (1 - a_m) x_m + a_m T x_m, \\ y_m = T((1 - b_m) \mathrm{T} x_m + b_m T z_m), \\ x_{m+1} = T((1 - c_m) y_m + c_m T y_m), \end{cases} \tag{11}$$

where $\{a_m\}$, $\{b_m\}$ and $\{c_m\}$ are sequences in [0,1], satisfying appropriate conditions.

In this section, we proved that the sequence generated by AR - iteration process converges to a fixed point of the mapping T. We also proved weak and strong convergence results for Suzuki's generalized non-expansive mappings in uniformly convex Banach spaces and develop a new example to show that AR - iteration process converges faster than Picard, Thakur, M and Sahu, Thakur iteration processes. We extend and improve the known results of existing literature using our new iteration process (11).

Lemma 3.1. Let B be a nonempty closed and convex subset of a uniformly convex Banach space X, let $T\colon B \to B$ be a Suzuki's generalized non-expansive mapping with $F_T \neq \emptyset$. Let $\{x_m\}$ be a sequence generated by iteration process (11), for arbitrary $x_0 \in B$, then $\lim_{m\to\infty} \|x_m - p\|$ exists for all p $\in F_T$.

Proof. Let p $\in F_T$ and x $\in B$. Since T is Suzuki's generalized non-expansive mapping, so it satisfies condition (C), so by proposition 2.5, T is quasi non-expansive mapping i.e,

$$\|Tx - p\| \leq \|x - p\| \text{ , for all p} \in F_T \text{ and for all x} \in B. \tag{12}$$

Now using iteration process (11), we have

$$\begin{aligned} \|z_m - p\| &= \|(1 - a_m)\, x_m + a_m\, Tx_m - p\| \\ &\leq (1 - a_m)\|x_m - p\| + a_m\|Tx_m - p\| \\ &\leq (1 - a_m)\|x_m - p\| + a_m\|x_m - p\| \\ &= \|x_m - p\| \ . \end{aligned} \tag{13}$$

and,

$$\begin{aligned} \|y_m - p\| &= \left\|T\big((1 - b_m)Tx_m + b_m\, Tz_m\big) - p\right\| \\ &\leq \|(1 - b_m)\, Tx_m + b_m\, Tz_m - p\| \\ &\leq (1 - b_m)\|Tx_m - p\| + b_m\|Tz_m - p\| \\ &\leq (1 - b_m)\|x_m - p\| + b_m\|z_m - p\| \\ &\leq (1 - b_m)\|x_m - p\| + b_m\|x_m - p\| \\ &= \|x_m - p\| \ . \end{aligned} \tag{14}$$

Using (13) and (14), we get

$$\begin{aligned} \|x_{m+1} - p\| &= \left\|T\big((1 - c_m)y_m + b_m\, Ty_m\big) - p\right\| \\ &\leq \|(1 - c_m)y_m + c_m\, Ty_m - p\| \\ &\leq (1 - c_m)\|y_m - p\| + c_m\|Ty_m - p\| \\ &\leq (1 - c_m)\|y_m - p\| + c_m\|y_m - p\| \\ &\leq \|y_m - p\| \\ &\leq \|x_m - p\|. \end{aligned} \tag{15}$$

This implies that the sequence $\{\|x_m - p\|\}$ is non-increasing and bounded below for all p $\in F_T$.

Hence $\lim_{m\to\infty} \|x_m - p\|$ exists.

Lemma 3.2. Let B be a nonempty closed and convex subset of a uniformly convex Banach space X, let T: $B \to B$ be a Suzuki's generalized non-expansive mapping. Let $\{x_m\}$ be a sequence generated by iteration process (11). Then $F_T \neq \emptyset$ if and only if $\{x_m\}$ is bounded and $\lim_{m\to\infty} \|x_m - Tx_m\| = 0$.

Proof. Let $F_T \neq \emptyset$ and p $\in F_T$.

Then by Lemma 3.1, $\lim_{m\to\infty} \|x_m - p\|$ exists and $\{x_m\}$ is bounded.

Put $\lim_{m\to\infty} \|x_m - p\| = c$. (16)

From (13), (14), (16) , we have

$$\lim_{m\to\infty} sup \|z_m - p\| \le \lim_{m\to\infty} sup \|x_m - p\| \le c. \quad (17)$$

$$\lim_{m\to\infty} sup \|y_m - p\| \le \lim_{m\to\infty} sup \|x_m - p\| \le c. \quad (18)$$

Since T satisfies condition (C), we have

$\|Tx_m - p\| = \|Tx_m - Tp\| \le \|x_m - p\|$.

$$\Longrightarrow \lim_{m\to\infty} sup \|Tx_m - p\| \le \lim_{m\to\infty} sup \|x_m - p\| \le c. \quad (19)$$

Similarly,

$$\lim_{m\to\infty} sup \|Ty_m - p\| \le \lim_{m\to\infty} sup \|y_m - p\| \le c. \quad (20)$$

$$\lim_{m\to\infty} sup \|Tz_m - p\| \le \lim_{m\to\infty} sup \|z_m - p\| \le c. \quad (21)$$

On the other hand, by using (13), we have

$$\begin{aligned}
\|x_{m+1} - p\| &= \left\|T\left((1 - c_m)y_m + b_m Ty_m\right) - p\right\| \\
&\le \|(1 - c_m)y_m + c_m Ty_m - p\| \\
&\le (1 - c_m)\|y_m - p\| + c_m\|Ty_m - p\| \\
&\le (1 - c_m)\|y_m - p\| + c_m\|y_m - p\| \\
&\le \|y_m - p\|
\end{aligned}$$

$$= \left\|T\left((1-b_m)Tx_m + b_m Tz_m\right) - p\right\|$$
$$\leq \|(1-b_m)Tx_m + b_m Tz_m - p\|$$
$$\leq (1-b_m)\|Tx_m - Tp\| + b_m\|Tz_m - Tp\|$$
$$\leq (1-b_m)\|x_m - p\| + b_m\|z_m - p\|$$
$$= \|x_m - p\| - b_m\|x_m - p\| + b_m\|z_m - p\|.$$

implies that

$$\frac{\|x_{m+1} - p\| - \|x_m - p\|}{b_m} \leq \|z_m - p\| - \|x_m - p\|$$

So,

$$\|x_{m+1} - p\| - \|x_m - p\| \leq \frac{\|x_{m+1} - p\| - \|x_m - p\|}{b_m} \leq \|z_m - p\| - \|x_m - p\|$$

implies that

$$\|x_{m+1} - p\| \leq \|z_m - p\|$$

Taking lim inf on both sides, we get

$$c = \lim_{m\to\infty} inf \|x_{m+1} - p\| \leq \lim_{m\to\infty} inf \|z_m - p\|. \quad (22)$$

Using (17) and (22)

$$c = \lim_{m\to\infty}\|z_m - p\|$$
$$= \lim_{m\to\infty}\|(1-a_m)x_m + a_m Tx_m - p\|$$
$$= \lim_{m\to\infty}\|(1-a_m)(x_m - p) + a_m(Tx_m - p)\| \quad (23)$$

Therefore, by using (16), (19), (23) and Lemma 2.8, we have

$$\lim_{m\to\infty}\|x_m - Tx_m\| = 0.$$

Conversely,

Assume that $\{x_m\}$ is bounded and $\lim_{m\to\infty}\|x_m - Tx_m\| = 0$.

Let p ∈ A(B, $\{x_m\}$), then by Proposition 2.5, we have,

$$r(Tp, \{x_m\}) = \lim_{m\to\infty} sup \|x_m - Tp\|$$
$$\leq \lim_{m\to\infty} sup \left(3\|Tx_m - x_m\| + \|x_m - p\|\right)$$

$= \lim_{m\to\infty} sup\ \|x_m - p\|$

$= \ r(p, \{x_m\}) \ = \ r(B, \{x_m\})$

$\Longrightarrow Tp \in A(B, \{x_m\})$.

Since X is uniformly convex, $A(B, \{x_m\})$ is singleton, hence we have $Tp = p$.

Theorem 3.3. Let B be a nonempty closed and convex subset of a uniformly convex Banach space X, let T: $B \to B$ be a Suzuki's generalized non-expansive mapping. Let $\{x_m\}$ be a sequence generated by iteration process (11). Assume that X satisfies Opial's condition, then $\{x_m\}$ converges weakly to a point of F_T.

Proof. Let p $\in F_T$, then by Lemma 3.1 $\lim_{m\to\infty}\|x_m - p\|$ exists. We now prove that $\{x_m\}$ has unique weak sub-sequential limit in F_T. Let p_1 and p_2 be weak limits of the subsequences $\{x_{m_r}\}$ and $\{x_{m_s}\}$ of $\{x_m\}$ respectively. From Lemma 3.2, $\lim_{m\to\infty}\|x_m - Tx_m\| = 0$ and I – T is demiclosed at zero by Lemma 2.6. This implies that (I – T) $p_1 = 0 \Longrightarrow p_1 = Tp_1$, similarly $Tp_2 = p_2$.

Now we show uniqueness. If $p_1 \neq p_2$, then by using Opial's condition,

$$\lim_{m\to\infty}\|x_m - p_1\| = \lim_{m_r\to\infty}\|x_r - p_1\|$$

$$< \lim_{m_r\to\infty}\|x_{m_r} - p_2\|$$

$$= \lim_{m\to\infty}\|x_m - p_2\|$$

$$= \lim_{m_s\to\infty}\|x_{m_s} - p_2\|$$

$$< \lim_{m_s\to\infty}\|x_{m_s} - p_1\|$$

$$= \lim_{m\to\infty}\|x_m - p_1\|.$$

which is a contradiction, so $p_1 = p_2$. Hence, $\{x_m\}$ converges weakly to a point of F_T.

Theorem 3.4. Let B be a nonempty compact and convex subset of a uniformly convex Banach space X, let T: $B \to B$ be a Suzuki's generalized non-expansive mapping. Let $\{x_m\}$ be a sequence generated by iteration process (11). Then $\{x_m\}$ converges strongly to a fixed point of

T.

Proof. Since by Lemma 2.7, $F_T \neq \emptyset$, so by Lemma 3.2, we have $\lim_{m\to\infty} \|x_m - Tx_m\| = 0$. Since B is compact, so there exists a subsequence $\{x_{m_r}\}$ of $\{x_m\}$ such that $\{x_{m_r}\}$ converges strongly to p for some p $\in B$. By Proposition 2.5 ,we have

$\|x_{m_r} - Tp\| \leq 3\, \|Tx_{m_r} - x_{m_r}\| + \|x_{m_r} - p\|$, for all $r \geq 1$.

Letting r $\to \infty$, we get Tp = p i.e., p $\in F_T$. Also, $\lim_{m\to\infty} \|x_m - p\|$ exists by Lemma 3.1. Thus p is the strong limit of the sequence $\{x_m\}$ itself.

Now using the condition (I), we prove the strong convergence result.

Theorem 3.5. Let B be a nonempty closed and convex subset of a uniformly convex Banach space X, let $T\colon B \to B$ be a Suzuki's generalized non-expansive mapping satisfying condition (I). Let $\{x_m\}$ be a sequence generated by iteration process (11). Then $\{x_m\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.2,

$$\lim_{m\to\infty} \|x_m - Tx_m\| = 0. \tag{24}$$

From condition (I) and (24), we have

$0 \leq \lim_{m\to\infty} \mathrm{g}\big(\mathrm{d}(x_m, F_T)\big) \leq \lim_{m\to\infty} \|x_m - Tx_m\| = 0$

$$\Longrightarrow \lim_{m\to\infty} \mathrm{g}\big(\mathrm{d}(x_m, F_T)\big) = 0. \tag{25}$$

Since g: $[0,\infty) \to [0,\infty)$ is a non decreasing function with g(0) = 0 and g(c) > 0, for all c > 0 , so from (25) we have

$\lim_{m\to\infty} \mathrm{d}(x_m, F_T) = 0.$

Thus, we have a subsequence $\{x_{m_r}\}$ of $\{x_n\}$ and a sequence $\{y_r\} \subset F_T$ such that

$\|x_{m_r} - y_r\| < \frac{1}{2^r}$ for all r $\in \mathbb{N}$.

So using (11), we get

$\|x_{m_{r+1}} - y_r\| \leq \|x_{m_r} - y_r\| < \frac{1}{2^r}$.

Hence

$$\|y_{r+1} - y_r\| \leq \|y_{r+1} - x_{r+1}\| + \|x_{r+1} - y_r\|$$

$$\leq \frac{1}{2^{r+1}} + \frac{1}{2^r}$$

$$< \frac{1}{2^{r-1}} \to 0, \text{ as } r \to \infty.$$

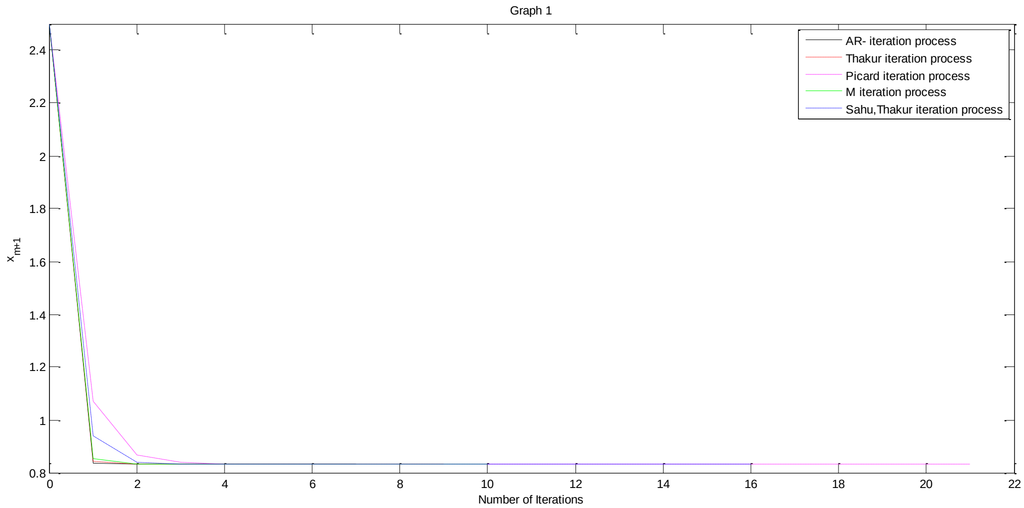
This shows that $\{y_r\}$ is a Cauchy sequence in F_T and so it converges to a point p. Since F_T is closed, therefore p ∈ F_T and then $\{x_{m_r}\}$ converges strongly to p. Since $\lim_{m\to\infty} \|x_m - p\|$ exists, we have that $x_m \to p \in F_T$. Hence the theorem is proved.

The following new numerical example is used to compare the convergence of our iteration process with other iteration processes like Picard, Thakur, M and Sahu,Thakur . We obtain the values in comparison table and graph for various iteration processes using Matlab program.

Example 3.6. Let A = [0,3] ⊆ X = $\mathbb{R}$ and T : A → A be defined by Tx = $\frac{x+5}{7}$ for all x ∈ A. Choose $a_m = b_m = c_m = 0.5$ for each m ∈ $\mathbb{N}$ with the initial value $x_1 = 2.5$. Table 1 and graph 1 given below show that our AR iteration process (11) converges faster than all of Picard, Thakur, M and Sahu, Thakur Iteration Processes:

Table 1. A comparison table of AR, Picard, Thakur, M and Sahu,Thakur iteration processes:

M	AR	Picard	Thakur	M	Sahu,Thakur
1	2.50000000000000000	2.50000000000000000	2.50000000000000000	2.50000000000000000	2.50000000000000000
2	0.83551496400309400	1.07142857142857140	0.84443981674302371	0.85276967930029157	0.94023323615160348
3	0.83333618904076090	0.86734693877551017	0.83340734571757114	0.83355999626006161	0.84018988686686669
4	0.83333333707139268	0.83819241982507287	0.83333382654372390	0.83333597663277037	0.83377311227717510
5	0.83333333333822635	0.83402748854643893	0.83333333662003306	0.83333336415898274	0.83336154072914825
6	0.83333333333333981	0.83343249836377697	0.83333333335523552	0.83333333369281615	0.83333514255405616
7	**0.83333333333333326**	0.83334749976625389	0.83333333333347936	0.83333333333752546	0.83333344937664500
8	0.83333333333333326	0.83333535710946482	0.83333333333333426	0.83333333333338222	0.83333334077634458
9	0.83333333333333326	0.83333362244420928	**0.83333333333333326**	0.83333333333333393	0.83333333381072761
10	0.83333333333333326	0.83333337463488710	0.83333333333333326	**0.83333333333333326**	0.83333333336395343
11	0.83333333333333326	0.83333333923355535	0.83333333333333326	0.83333333333333326	0.83333333333529724
12	0.83333333333333326	0.83333333417622224	0.83333333333333326	0.83333333333333326	0.83333333333345927
13	0.83333333333333326	0.83333333345374605	0.83333333333333326	0.83333333333333326	0.83333333333334147
14	0.83333333333333326	0.83333333335053517	0.83333333333333326	0.83333333333333326	0.83333333333333381
15	0.83333333333333326	0.83333333333579074	0.83333333333333326	0.83333333333333326	0.83333333333333337
16	0.83333333333333326	0.83333333333368442	0.83333333333333326	0.83333333333333326	**0.83333333333333326**
17	0.83333333333333326	0.83333333333338355	0.83333333333333326	0.83333333333333326	0.83333333333333326
18	0.83333333333333326	0.83333333333334048	0.83333333333333326	0.83333333333333326	0.83333333333333326
19	0.83333333333333326	0.83333333333333426	0.83333333333333326	0.83333333333333326	0.83333333333333326
20	0.83333333333333326	0.83333333333333337	0.83333333333333326	0.83333333333333326	0.83333333333333326
21	0.83333333333333326	**0.83333333333333326**	0.83333333333333326	0.83333333333333326	0.83333333333333326
22	0.83333333333333326	0.83333333333333326	0.83333333333333326	0.83333333333333326	0.83333333333333326



Graph 1. Convergence of AR iteration process with other iteration processes.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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