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THE RESTRICTED DETOUR POLYNOMIALS OF A HEXAGONAL CHAIN AND A LADDER GRAPH

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Abstract. The restricted detour distance $D^{*}(u,v)$ between two vertices u and v of a connected graph G is the length of a longest $u-v$ path P in G such that $<V(P)>=P$. The restricted detour polynomial of G, is a graph distance polynomial defined on restricted detour distance. The restricted detour polynomials and restricted detour indices of hexagonal graphs and ladder graphs are obtained in this paper.

Keywords: Restricted detour distance, restricted detour index, restricted detour polynomial, hexagonal graph, ladder graph.

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1. Introduction

Let G be a connected graph, and let u and v be any two vertices of G.The (standard) distance $d(u,v)$ between u and v in G is the length of a **shortest** $u-v$ path P in G [8]. It is clear that the induced subgraph $<V(P)>$ is P itself. Based on this observation, Chartrand, et al [4],in 1993 defined the detour distance $d^{*}(u,v)$ between vertices u and v as the length of a **longest** $u-v$ path P for which

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$<V(P)>=P$. Later on, Chartrand, et al and any other authors (see [5] and [6]). Defined the concepts of detour distance $D(u,v)$ between vertices u and v in G, as the length of a longest $u-v$ path P, without assuming the induced condition $<V(P)>=P$. Therefore, in order to differentiate between the two concepts, we shall call the detour distance with the induced condition, the **restricted detour distance** between u and v, and denote it by $D_G^*(u,v)$ or simply $D^*(u,v)$. From this definition of the concept D^* on the vertex set $V(G)$, we notice that $D^*(u,v)=0$ if and only if $u=v$, and $D^*(u,v)=1$ if and only if uv is an edge of G. However, the triangle inequality does not hold in general [4], therefore the restricted detour distance is not metric on $V(G)$.

An induced $u-v$ path of length $D^*(u,v)$ will be called a **restricted** (or **an induced**) **detour path**. Moreover, a connected graph G is called a **restricted detour graph** if $D^*(u,v)=d(u,v)$ for every pair u,v of vertices in G. It is clear that all trees, complete graphs, and complete bipartite graphs are restricted detour graphs. However, every cycle of order $p\geq 5$ is not restricted detour.

For more properties and results on restricted detour distances, one may see [4].

2. Restricted Detour Polynomials

Let G be a (p,q) connected graph. The concept of Hosoya polynomial $H(G;x)$ is based on standard distance, (See [7], [9], and [10]), and the concept of detour polynomials $D(G;x)$ of G, (See [2] and [3]) is based on detour distance. On the same line, the concept of **restricted detour polynomial**, denoted by $D^*(G;x)$ or $H^*(G;x)$, see [1], is defined as follows:

(2.1) $$D^*(G;x)=\sum_{u,v} x^{D_G^*(u,v)},$$

where the summation is taken over all unordered pairs u,v of vertices of G. The **index** of G with respect to restricted detour distance is denoted by $dd^*(G)$ and defined by

(2.2) $$dd^*(G)=\sum_{u,v} D_G^*(u,v),$$

and will be called **restricted detour index** of G.

It is clear that

(2.3) $dd^*(G) = \frac{d}{dx} D^*(G;x)\Big|_{x=1}$.

One can easily notice that

(2.4) $D^*(G;x) = \sum_{k \geq 0} C^*(G,k)x^k$,

in which $C^*(G,k)$ is the number of unordered pairs of vertices u, v of G such that $D_G^*(u,v) = k$.

Let u be any vertex of G, and let $C^*(u,G;k)$ be the number of vertices v of G such that $D^*(u,v) = k$. Then, the polynomial is defined by

(2.5) $D^*(u,G;x) = \sum_{k \geq 0} C^*(u,G;k)x^k$,

is called the **restricted detour polynomial of vertex** u.

It is clear that

(2.6) $D^*(G;x) = \frac{1}{2}\left(\sum_{u \in V(G)} D^*(u,G;x) + p \right)$.

We illustrate these concepts in the next example.

Example: Let Q_3 be the 3-cube graph, and let u be any vertex in Q_3 as shown in Fig.2.1. From the symmetry of Q_3,we have

$$D^*(Q_3;x) = 4[1 + D^*(u,Q_3;x)].$$

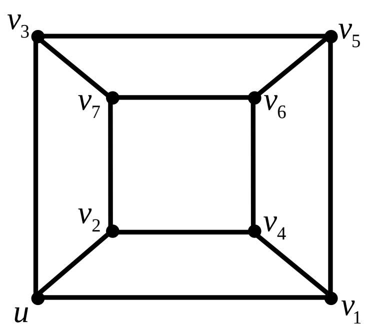

Fig.2.1. The 3-cube Q_3.

By direct calculation using Fig.2.1,we obtain the restricted detour distances from vertex u to the other vertices $v_1, v_2, ..., v_7$ which, respectively, are 1, 1, 1, 4, 4, 3, 4. Thus

$$D^*(u,Q_3;x) = 1 + 3x + x^3 + 3x^4,$$

and so

$$D^*(Q_3;x) = 8 + 12x + 4x^3 + 12x^4,$$

and

$dd^*(Q_3) = 72$.

In 2010, Abdullah and Muhammed-Saleh [1] obtained the restricted detour polynomials and restricted detour indices of some special graphs.

In this paper, we obtain the restricted detour polynomial and index of a hexagonal graph consisting of one row of m hexagons.

3. Restricted Detour Polynomials of Hexagonal Graphs

Let J_m, $m \geq 1$, be a hexagonal chain consisting of one row of m hexagons h_1, h_2, …, h_m as depicted in Fig.3.1. Then, $p(J_m) = 4m+2$, $q(J_m) = 5m+1$.

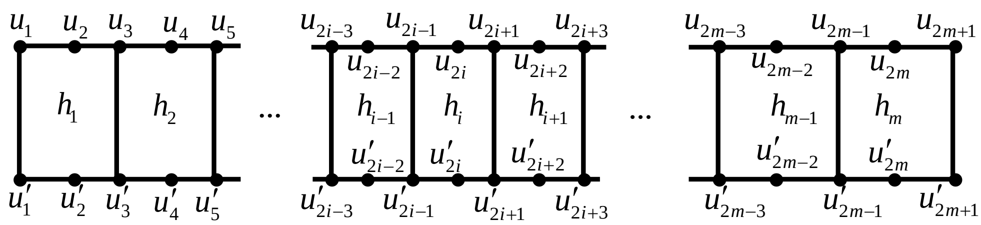

Fig.3.1. A hexagonal graph J_m.

From Fig.3.1 and taking care of the symmetry of J_m, we have the following reduction formula:

(3.1) $D^*(J_m; x) = D^*(J_{m-1}; x) + F_m(x), \quad m \geq 2$,

in which

(3.2) $F_m(x) = 2D^*(u_1, J_m; x) + 2D^*(u_2, J_m; x) - (3x + x^3 + 2x^4)$.

We shall find $D^*(u_i, J_m; x)$, $i = 1, 2$.

Remark. All restricted detour distance $D^*(\ ,\ )$ in this section are calculated in the graph J_m.

Proposition 3.1. *For* $m \geq 2$ *and* $i = 2, 3, \ldots, m$

(1) $D^*(u_1, u_{2i+1}) = 2i + 2\left\lceil \frac{i}{2} \right\rceil$,

(2) $D^*(u_1, u_{2i}) = 2i + 1 + 2\left\lfloor \frac{i}{2} \right\rfloor$,

(3) $D^*(u_1, u'_{2i+1}) = 2i + 1 + 2\left\lfloor \frac{i}{2} \right\rfloor$,

(4) $D^*(u_1, u'_{2i}) = 2i + 2\left\lceil \frac{i}{2} \right\rceil$.

Proof.

(1) From Fig.3.1, one may easily see that $D^*(u_1,u_5)=6$, $D^*(u_1,u_7)=10$, $D^*(u_1,u_9)=12$, and for each $2\le i\le m$, a (u_1,u_{2i+1}) restricted detour is

u_1, u_1', u_2', u_3', u_3, u_4, u_5, u_5', …, $(u_{2i-1}, u_{2i}, u_{2i+1})$ (or … u_{2i-1}', u_{2i}', u_{2i+1}', u_{2i+1}),

which is of length $2i+2\left\lceil \frac{i}{2}\right\rceil$.

(2) We notice that $D^*(u_1,u_4)=7$, $D^*(u_1,u_6)=9$, $D^*(u_1,u_8)=13$, …; and for $4\le i\le m$ a (u_1,u_{2i}) restricted detour is

u_1, u_1', u_2', u_3', u_3, u_4, u_5, u_5', …, $(u_{2i-3}, u_{2i-3}', u_{2i-2}', u_{2i-1}', u_{2i-1}, u_{2i})$ (or … u_{2i-2}', u_{2i-1}', u_{2i}', u_{2i+1}', u_{2i+1}, u_{2i}),

which is of length $2i+1+2\left\lfloor \frac{i}{2}\right\rfloor$.

Parts (3) and (4) are proved using similar ways. ■

Proposition 3.2. *For* $m\ge 2$,

(3.3) $D^*(u_1,J_m;x)=1+2x+x^3+2x^4+2(x+1)\sum_{i=2}^{m}x^{3i}$.

Proof.

From Fig.3.1, we get

$$D^*(u_1,J_m;x)=D^*(u_1,J_1;x)+\sum_{i=2}^{m}[x^{D^*(u_1,u_{2i+1})}+x^{D^*(u_1,u_{2i})}+x^{D^*(u_1,u_{2i+1}')}+x^{D^*(u_1,u_{2i}')}].$$

Since

(3.4) $D^*(u_1,J_1;x)=1+2x+x^3+2x^4$,

then, from Preposition 3.1, we obtain

$$D^*(u_1,J_m;x)=1+2x+x^3+2x^4+2\sum_{i=2}^{m}x^{2i}[x^{1+2\left\lfloor \frac{i}{2}\right\rfloor}+x^{2\left\lceil \frac{i}{2}\right\rceil}]$$

$$=1+2x+x^3+2x^4+2\sum_{i=2}^{m}x^{2i}[x^{i}+x^{i+1}]=1+2x+x^3+2x^4+2\sum_{i=2}^{m}(1+x)x^{3i}. \ ■$$

Proposition 3.3 *For* $m\ge 4$, *we have*

(1) $D^*(u_2,u_{2i+1})=2i+1+\left\lfloor \frac{i}{2}\right\rfloor$, for $i=2,3,...,m$.

(2) $D^*(u_2, u_{2i}) = 2i + 2\left\lceil \frac{i}{2} \right\rceil$, for $i = 3, 4, \ldots, m$.

(3) $D^*(u_2, u'_{2i+1}) = 2i + 2\left\lceil \frac{i}{2} \right\rceil$, for $i = 2, 3, \ldots, m$.

(4) $D^*(u_2, u'_{2i}) = 2i + 1 + \left\lfloor \frac{i}{2} \right\rfloor$, for $i = 4, 5, \ldots, m$.

Proof.

It is similar to that of proof Proposition 3.1. ■

Proposition 3.4 *For* $m \geq 4$,

(3.5) $D^*(u_2, J_m; x) = 1 + 2x + x^3 + 2x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + 2(x+1)\sum_{i=4}^{m} x^{3i}$.

Proof.

From Fig.3.1, (3.4), and Proposition 3.3, we get

$$D^*(u_2, J_m; x) = D^*(u_2, J_1; x) + \sum_{i=2}^{m} [x^{D^*(u_2, u_{2i+1})} + x^{D^*(u_2+u_{2i})} + x^{D^*(u_2+u'_{2i+1})} + x^{D^*(u_2+u'_{2i})}]$$

$$= D^*(u_1, J_1; x) + (x^{D^*(u_2,u_5)} + x^{D^*(u_2,u_7)}) + (x^{D^*(u_2,u_4)} + x^{D^*(u_2,u_6)}) + (x^{D^*(u_2,u'_5)} +$$

$$x^{D^*(u_2,u'_7)}) + (x^{D^*(u_2,u'_4)} + x^{D^*(u_2,u'_6)}) + 2\sum_{i=4}^{m} [x^{2i+1+\left\lfloor \frac{i}{2} \right\rfloor} + x^{2i+2\left\lceil \frac{i}{2} \right\rceil}]$$

$$= (1 + 2x + x^3 + 2x^4) + (x^7 + x^9) + (x^8 + x^{10}) + (x^6 + x^{10}) + (x^5 + x^7) +$$

$$2\sum_{i=4}^{m} x^{2i}(x^i + x^{i+1}).$$

Simplifying the expression, we get (3.5). ■

Proposition 3.5. *For* $m \geq 4$, *we have a reduction formula*

(3.6) $D^*(J_m; x) = D^*(J_{m-1}; x) + F_m(x)$,

where

$$F_m(x) = R(x) + 8(x+1)\sum_{i=4}^{m} x^{3i},$$

$$R(x) = 4 + 5x + 3x^3 + 6x^4 + 2x^5 + 6x^6 + 8x^7 + 2x^8 + 6x^9 + 8x^{10}.$$

Proof.

From (3.2), we have, for $m \geq 2$,

$$F_m(x) = 2D^*(u_1, J_m; x) + 2D^*(u_2, J_m; x) - (3x + x^3 + 2x^4).$$

From Proposition 3.2 and 3.4, we obtain, for $m \geq 4$:

$$F_m(x) = 2\{1+2x+x^3+2x^4+2(x+1)(x^6+x^9) \ +2(x+1)\sum_{i=4}^{m} x^{3i}$$

$$+1+2x+x^3+2x^4 \ x^5+x^6+2x^7+x^8+x^9+2x^{10} \ +2(x+1)\sum_{i=4}^{m} x^{3i}\} \ -(3x+x^3+2x^4)\,.$$

Simplifying the algebraic expression, we get $F_m(x)$ as given in (3.6)

Hence, the proof is completed. ■

Now, we state our main result.

Theorem 3.6 . *For* $m \geq 4$,

(3.7) $D^*(J_m;x) = 4m+2+(5m-1)\ x+3mx^3+6mx^4+(2m-2)x^5+(6m-6)x^6+$
$(8m-10)x^7+(2m-2)x^8+(6m-12)x^9+(8m-16)x^{10}+$
$8(x+1)\sum_{k=4}^{m}(m+1-k)x^{3k}$.

Proof.

From Proposition 3.5, we have

$$D^*(J_m;x) = D^*(J_{m-1};x) + R(x) + 8(x+1)\sum_{i=4}^{m} x^{3i}$$

$$= D^*(J_{m-2};x) + 2R(x) + 8(x+1)\left[\sum_{i=4}^{m-1} x^{3i} + \sum_{i=4}^{m} x^{3i}\right].$$

Thus solving our reduction formula, we obtain

(3.8) $D^*(J_m;x) = D^*(J_3;x) + (m-3)R(x) + 8(x+1)\sum_{k=4}^{m}\ \sum_{i=4}^{k} x^{3i}$

$$= D^*(J_3;x) + (m-3)R(x) + 8(x+1)\sum_{k=4}^{m}(m+1-k)x^{3k}\ .$$

By direct calculation, we get

$$D^*(J_3;x) = 14+16x+9x^3+18x^4+4x^5+12x^6+14x^7+4x^8+6x^9+8x^{10}.$$

Therefore, substituting $R(x)$, from (3.6), and $D^*(J_3;x)$ in (3,8) and simplifying, we get the required result (3.7). ■

Theorem 3.7. *For* $m \geq 4$ *, the restricted detour index of* J_m *is given by*

$$dd^*(J_m) = 8m^3 + 28m^2 - 2m + 9$$

Proof.

Taking the derivative of $D^*(J_m;x)$ with respect to x, we get

$$D^{*\prime}(J_m;x) = (5m+1) + 9mx^2 + 24mx^3 + (10m-10)x^4 + (36m-36)x^5 +$$

$$(56m-70)x^6 + (16m-16)x^7 + (54m-108)x^8 + (80m-160)x^9 +$$

$$8\sum_{k=4}^{m}(m+1-k)x^{3k} + 8(x+1)\sum_{k=4}^{m}3k(m+1-k)x^{3k-1} .$$

Putting $x=1$, we get

$$D^{*\prime}(J_m;1) = 290m-399+ 8\sum_{k=4}^{m}(m+1+6mk+5k-6k^2)$$

$$= 290m-399+8\left[(m+1)(m-3)+(6m+5)\sum_{k=4}^{m}k-6\sum_{k=4}^{m}k^2\right]$$

$$= 290m-399+8(m^2-2m-3)+ 8(6m+5)(\frac{m+4}{2})(m-3)-$$

$$48\left[\frac{1}{6}m(m+1)(2m+1)-14\right] = 8m^3+28m^2-2m+9 .$$

Corollary 3.8 . *For* $m\geq 3$ *, the restricted detour diameter of* J_m *is* $3m+1$.

Proof.

It is clear that the highest power of x in $D^*(J_m;x)$ is $3m+1$.■

Moreover, one may notice that $D^*(J_m;x)$ does not contain the terms x^2 and x^{3k-1} for $4\leq k\leq m$.

4. The Restricted Detour Polynomial of the Ladder L_n

Let P_n be a path of order $n, n\geq 2$. The ladder graph L_n is $K_2\times P_n$. It is clear that $p(L_n)=2n$, $q(L_n)=3n-2$, and diam $L_n=n$. It is known [7] that the Hosoya polynomial of P is given by

(4.1) $H(P_n;x)=\sum_{k=0}^{n-1}(n-k)x^k$.

Let the vertices of P_n be $u_1, u_2, ..., u_n$, and let the vertices of L_n be labeled as shown in Fig. 4.1.

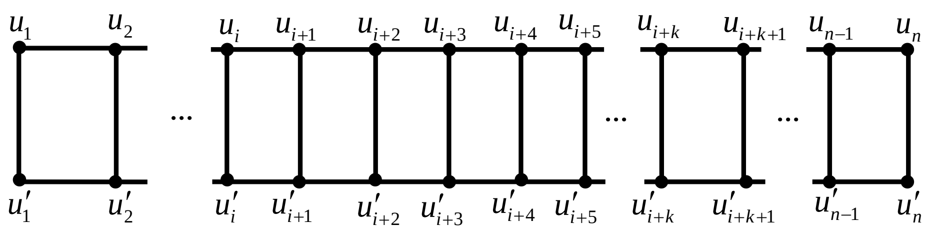

Fig.4.1. The ladder L_n, $n\geq 2$.

Proposition 4.1. *For* $k\geq 0$,

(4.2) $D^*_{L_n}(u_i,u_{i+k}) = k+2\left\lceil\frac{k-1}{4}\right\rceil$, (4.3) $D^*_{L_n}(u_i, u'_{i+k}) = k+1+2\left\lfloor \frac{k}{4} \right\rfloor$.

Proof.

It is clear that $d_{P_n}(u_i, u_{i+k}) = k$. From Fig.4.1, we notice that there is a restricted detour between vertices u_i and u_{i+k}, for $k \geq 2$, in L_n, namely

$$u_i, u'_i, u'_{i+1}, u'_{i+2}, u_{i+2}, u_{i+3}, u_{i+4}, u'_{i+4}, u'_{i+5}, ..., u_{i+k-1}, u_{i+k} \text{ (or } ..., u'_{i+k}, u_{i+k})$$

which is of length $k+2\left\lceil \frac{k-1}{4} \right\rceil$. Hence (4.2) holds.

(b) If $k=0$, then $D^*_{L_n}(u_i, u'_i) = 1$, and if $k=1$, then $D^*_{L_n}(u_i, u'_{i+1}) = 2$. Also $D^*_{L_n}(u_i, u'_{i+2}) = 3$, $D^*_{L_n}(u_i, u'_{i+3}) = 4$, $D^*_{L_n}(u_i, u'_{i+4}) = 7$.

Thus, (4.3) holds for $k = 0,1,2,3,4$. In general, we have a restricted detour between u_i and u'_{i+k} in L_n of length $k+1+2\left\lfloor \frac{k}{4} \right\rfloor$, namely, for $k \geq 4$,

$$u_i, u'_i, u'_{i+1}, u'_{i+2}, u_{i+2}, u_{i+3}, u_{i+4}, u'_{i+4}, ..., u_{i+k}, u'_{i+k} \text{ (or} ..., u'_{i+k-1}, u'_{i+k}),$$

which is of length $k+1+2\left\lfloor \frac{k}{4} \right\rfloor$.■

Let $S = \{u_1, u_2, ..., u_n\}$, and $S' = \{u'_1, u'_2, ..., u'_n\}$. From (4.1), we notice that the number of unordered pairs of vertices which are of distance k apart in P_n is $(n-k)$.Therefore, by Proposition 4.1, the number of unordered pairs of vertices of S (or of S') which are of restricted detour distance $k+2\left\lceil \frac{k-1}{4} \right\rceil$, for $k \geq 2$, in L_n, is $(n-k)$. Also, the number of unordered pairs u, u' with $u \in S$ and $u' \in S'$, which are of restricted detour distance $1+k+2\left\lfloor \frac{k}{4} \right\rfloor$, for $k \geq 0$, in L_n, is $(n-k)$. Using this fact, we shall prove the following theorem.

Theorem 4.2. *For* $n \geq 3$,

(4.4) $D^*(L_n; x) = 2n + (3n-2)x + 2(n-1)x^2 + 2\sum_{k=2}^{n-1}(n-k)x^k \left(x^{2\left\lceil \frac{k-1}{4} \right\rceil} + x^{1+2\left\lfloor \frac{k}{4} \right\rfloor}\right)$.

Proof.

From the symmetry of L_n, we have for all i, $j \in \{1, 2, ..., n\}$,

$$D^*_{L_n}(u_i, u_j) = D^*_{L_n}(u'_i, u'_j)$$

and

$D^*_{L_n}(u_i,u'_j)=D^*_{L_n}(u'_i,u_j)$.

Since the order of L_n is $2n$ and its size is $(3n-2)$, then by Proposition 4.1, we get

$$D^*(L_n;x)=2n+(3n-2)x+2\sum_{k=2}^{n-1}(n-k)x^{k+2\left\lceil\frac{k-1}{4}\right\rceil}+2\sum_{k=1}^{n-1}(n-k)x^{k+1+2\left\lfloor\frac{k}{4}\right\rfloor}$$
$$=2n+(3n-2)x+2(n-1)x^2+2\sum_{k=2}^{n-1}(n-k)x^{k+2\left\lceil\frac{k-1}{4}\right\rceil}+2\sum_{k=2}^{n-1}(n-k)x^{k+1+2\left\lfloor\frac{k}{4}\right\rfloor}.$$

Hence, the proof is completed. ■

The next corollary determines the restricted detour diameter of L_n.

Corollary 4.3. *For* $n\geq 1$, *let* $m=\left\lfloor\frac{n}{4}\right\rfloor$, *then*

$$Diam^*(L_n)=\begin{cases}n+2m-1, & \text{if } n\equiv 0\,(\text{mod}\,4)\\ n+2m, & \text{if } n\equiv 1\,\text{or}\,2\,(\text{mod}\,4)\\ n+2m+1, & \text{if } n\equiv 3\,(\text{mod}\,4).\end{cases}$$

Proof.

Since $diam\,P_n=n-1$, then

$$Diam^*(L_n)=\max\left\{(n-1)+2\left\lceil\frac{n-2}{4}\right\rceil, 1+(n-1)+2\left\lfloor\frac{n-1}{4}\right\rfloor\right\}.$$

Let $n\equiv r\ (\text{mod}\,4)$, then $n=4m+r$, where $r=0, 1, 2\,\text{or}\,3$.

If $r=0$, then

$$Diam^*(L_n)=\max\left\{n-1+2\left\lceil\frac{4m-2}{4}\right\rceil,\ n+2\left\lfloor\frac{4m-1}{4}\right\rfloor\right\}$$
$$=\max\{n-1+2m,\ n+2(m-1)\}=n+2m-1.$$

If $r=1$, then

$$Diam^*(L_n)=\max\left\{n-1+2\left\lceil\frac{4m-1}{4}\right\rceil,\ n+2\left\lfloor\frac{4m}{4}\right\rfloor\right\}$$
$$=\max\{n-1+2m,\ n+2m\}=n+2m.$$

If $r=2$, then

$$Diam^*(L_n)=\max\left\{n-1+2\left\lceil\frac{4m}{4}\right\rceil,\ n+2\left\lfloor\frac{4m+1}{4}\right\rfloor\right\}$$
$$=\max\{n-1+2m,\ n+2m\}=n+2m.$$

If $r=3$, then

$$Diam^*(L_n)=\max\left\{n-1+2\left\lceil\frac{4m+1}{4}\right\rceil,\ n+2\left\lfloor\frac{4m+2}{4}\right\rfloor\right\}$$
$$=\max\{n-1+2(m+1),\ n+2m\}=n+2m+1. ■$$

We shall obtain the restricted detour index of L_n

Theorem 4.4. *For* $n \geq 2$, *we have*

$$dd^*(L_n) = \begin{cases} \frac{1}{2} n(2n^2 + n - 2), & \text{for even } n \\ \frac{1}{2}(2n^3 + n^2 - 6n + 5) + 4\left(\left\lceil \frac{n-2}{4} \right\rceil + \left\lfloor \frac{n-1}{4} \right\rfloor\right), & \text{for odd } n. \end{cases}$$

Proof.

Assuming $n \geq 3$ and taking the derivative of $D^*(L_n; x)$ with respect to x, and then putting $x = 1$, we get from Theorem 4.2:

$$dd^*(L_n) = 3n - 2 + 4(n-1) + 2\sum_{k=2}^{n-1}(n-k)\ (2k + 1 + 2\left\lceil \frac{k-1}{4} \right\rceil + 2\left\lfloor \frac{k}{4} \right\rfloor)$$

$$= 7n - 6 + 2\sum_{k=2}^{n-1}\left\{n + (2n-1)k - 2k^2\right\} + 4\sum_{k=2}^{n-1}(n-k)(\left\lceil \frac{k-1}{4} \right\rceil + \left\lfloor \frac{k}{4} \right\rfloor)$$

$$= 7n - 6 + 2\left\{\ n(n-2) + (2n-1)\ \frac{n+1}{2}(n-2) - 2\,[\frac{1}{6}(n-1)n(2n-1) - 1]\ \right\} + 4A,$$

where

(4.5) $$A = \sum_{k=2}^{n-1}(n-k)(\left\lceil \frac{k-1}{4} \right\rceil + \left\lfloor \frac{k}{4} \right\rfloor).$$

Therefore,

(4.6) $$dd^*(L_n) = \frac{2}{3}n^3 + n^2 - \frac{2}{3}n + 4A.$$

We shall find the value of A. Expanding the summation in (4.5), we get

$$A = \left[(n-2)(1+0) + (n-3)(1+0)\right] + \left[(n-4)(1+1) + (n-5)(1+1)\right] + \left[(n-6)(2+1) + (n-7)(2+1)\right] + \ldots$$

$$= (2n-5)(1) + (2n-9)(2) + (2n-13)(3) + \ldots = 2n(1+2+3\ldots) - (5+18+39+\ldots).$$

If $4 \leq n$ is even, then

(4.7) $$A = 2n\sum_{i=1}^{\frac{n-2}{2}} i - \sum_{i=1}^{\frac{n-2}{2}} i(4i+1) = (2n-1)\frac{1}{2}(\frac{n-2}{2})(\frac{n}{2}) - 4(\frac{1}{6})(\frac{n-2}{2})(\frac{n}{2})(n-1)$$

$$= \frac{1}{4}(\frac{1}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{3}n).$$

Thus, from (4.6) and (4.7), we get the formula for $dd^*(L_n)$ for even $n \geq 4$.

If n is odd, $n \geq 5$, then

(4.8) $$A = 2n\sum_{i=1}^{\frac{n-3}{2}} i - \sum_{i=1}^{\frac{n-3}{2}} i(4i+1) + (\left\lceil \frac{n-2}{4} \right\rceil + \left\lfloor \frac{n-1}{4} \right\rfloor)$$

$$= (2n-1)\sum_{i=1}^{\frac{n-2}{2}} i - 4\sum_{i=1}^{\frac{n-2}{2}} i^2 + (\left\lceil \frac{n-2}{4} \right\rceil + \left\lfloor \frac{n-1}{4} \right\rfloor)$$

$$= (2n-1)\frac{1}{2}(\frac{n-3}{2})(\frac{n-1}{2}) - 4(\frac{1}{6})(\frac{n-3}{2})(\frac{n-1}{2})(n-2) + (\left\lceil \frac{n-2}{4} \right\rceil + \left\lfloor \frac{n-1}{4} \right\rfloor)$$

$$= \frac{1}{4}(\frac{n^3}{3} - \frac{n^2}{2} - \frac{7n}{3} + \frac{5}{2}) + (\left\lceil \frac{n-2}{4} \right\rceil + \left\lfloor \frac{n-1}{4} \right\rfloor).$$

Thus, from (4.6) and (4.8), we obtain the required formula for $n \geq 5$.

Moreover, one may easily see that the formula for $dd^*(L_n)$ given in the theorem holds also for $n = 2$ and 3. Hence the proof is completed. ■

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