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THE GENERAL SOLUTION FOR SINGULAR EQUATIONS OF $(n+1)$ ORDER USING ADOMIAN DECOMPOSITION METHOD

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Abstract. This article proposes a general and efficient modification of Adomain decomposition method (ADM) to obtain the general solutions for (n+1)- order nonlinear singular differential equations with different boundary conditions. This technique is proposed to overcome the singular behaviour of this type of problems. We study several nonlinear problems which will illustrate the efficiency of using that developed technique of the given method to clearly confirm the effectiveness and accuracy of ADM. In addition, we compare the numerical results with the exact solution to explain the rapid convergence of the approximation series as the solution.

Keywords: Adomian method; boundary conditions; (n+1) order non-linear ordinary differential equation.

2010 AMS Subject Classification: 34B15.

1. Introduction

This study aims at developing a new technique of ADM for the class of singular boundary value problems of the form;

$$y^{(n+1)}+\frac{m}{x}y^{(n)}+Ny=f(x), \tag{1}$$

under the following boundary conditions

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$$y(0) = a_0 \ ,y'(0) = a_1 \ ,...,y^{(n-1)}(0) = a_{n-1} \ ,y^{(r)}(b) = a_n.$$

Where N is a nonlinear differential operator of order smaller than n, $f(x)$ is known function and $a_0, a_1, ..., a_{n-1}, a_n, b$ are constants,
where $0 \leq r \leq n$, $n \geq 1$. Many studies have recently made about singular boundary value problems because singular boundary value problems occurs repeatedly in different fields of engineering and scientific applications. Moreover, the results of those studies have attracted the attention of many concerned researchers. The solutions of singular boundary value problems have been made through the application of various numerical methods by many researchers. These methods have been studied to address the fundamental difficulty in solving the singular value problems which is the existence of the singularity at x=0. For instance, Kanth and Aruna applied a differential transform method [9] and variational iteration method [10] to solve singular boundary value problems. chang solved types of nonlinear equations with boundary conditions by applying taylor series [3]. The integral method [4] has been applied to solve singular two point boundary value problems by El-sayed. In [11] authors used Haar wavelet method for the Lane– Emden equations. Some authors applied cubic B-spline [8] for solving non-linear singular boundary value problems. The ADM [1,2] is an analytic approximation method. in 1980s, George Adomian proposed this method. This technique is powerful and effective for solving linear and nonlinear equations of different types(ordinary differential equations, partial differential equation, algebraic equation and integral differential equations...) in the field of mathematics, physics, biology, chemistry, ect. This method depends on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series and each term of this series is a polynomial called Adomian's polynomials. Generating of Adomain polynomials for nonlinear equations relies on the formula which have been proposed by George Adomain [2]. The ADM was applied to solve singular boundary value problems such as [13,14]. Hasan Y.Q and Ming L.Z applied a modified of Adomian decomposition method to solve singular boundary value problems of higher-order, see [6,7]. other successful applications of this method were given in [5,12]. This article is an attempt to handle singular boundary value

problems of (n+1)-order by using ADM. We therefore introduced a new differential operator to solve this type of equations.

2. New Technique of (ADM)

Problem (1) in the operater form

$$Ly = f(x) - Ny, \tag{2}$$

where the differential operator L is defined by

$$L(.) = x^{-1}\frac{d^{n-r}}{dx^{n-r}}x^{1+n-m-r}\frac{d}{dx}x^{m-n+r}\frac{d^r}{dx^r}(.), \tag{3}$$

where $m \leq (n-r),\ n \geq 1$. The inverse for operator L is L^{-1} studied in a $(n+1)$ integrals defined as

$$L^{-1}(.) = \underbrace{\int_0^x \cdots \int_0^x}_{r} x^{n-m-r}\int_b^x x^{m-n-1+r}\underbrace{\int_0^x \cdots \int_0^x}_{n-r} x(.)dx\ldots dx. \tag{4}$$

By using $L^{-1}$to both sides of (2) to obtain

$$y = \gamma(x) + L^{-1}f(x) - L^{-1}(Ny), \tag{5}$$

such that

$$L\gamma(x) = 0.$$

The Adomian decomposition method assumes that solution y(x) by an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{6}$$

and the nonlinear termNy by an infinite series of polynomials

$$Ny = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently by algorithm [15,16], and the A_n are the Adomian polynomials,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[Z(\sum_{i=0}^{n}\lambda^i y_i)]_{\lambda=0},\ n = 0,1,2,3,.... \tag{8}$$

Which gives

$$A_0 = Z(y_0),$$

$$A_1 = Z'(y_0)y_1,$$

$$A_2 = Z'(y_0)y_{(2)} + \frac{1}{2}Z''(y_0)y_1^2,$$

$$A_3 = Z'(y_o)y_{(3)} + Z''(y_0)y_1y_2 + \frac{1}{3!}Z'''(y_0)y_1^3, \tag{9}$$

$$\cdots$$

from (6) , (7) and (5) we get

$$\sum_{n=0}^{\infty} y_{(n)} = \gamma(x) + L^{-1}f(x) - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{10}$$

To determine the components $y_n(x)$,we use Adomian decomposition method by using the relation

$$y_0 = \gamma(x) + L^{-1}f(x),$$

$$y_{n+1} = -L^{-1}A_n,\ n \geq 0, \tag{11}$$

therefore

$$y_0 = \gamma(x) + L^{-1}f(x),$$

$$y_1 = -L^{-1}A_0,$$

$$y_2 = -L^{-1}A_1,$$

$$y_3 = -L^{-1}A_3, \tag{12}$$

$$....$$

Using the equation (9) and (12) we can determine the components $y_n(x)$, and therefore, we can directly obtain series solution of $y(x)$ in (8).In addition, and for numerical reasons, we can be the n-term approximate

$$\Psi_n = \sum_{n=0}^{n-1} y_n(x),$$

in order to approximate the exact solution.

3. DISCUSSION OF ADM WITH NUMERICAL EXAMPLES

In this part we will discuss four types of equations and give every type a new differential operator. Then we will make examples to everyone which contain equations of the third, fourth and fifth order.

The first type put $n=1,\ r=0$, in Eq.(1) and Eq.(3) we get the same method in [6].
And when $n=1,\ r=1$, in (1),(3), we obtain the same method which has been studied in [7].
The second type when $n=2,\ r=0,1,2$, from Eq.(1) , Eq.(3) . We get

$$y^{(3)}+\frac{m}{x}y^{(2)}+Ny=f(x), \tag{13}$$

with one of the following conditions

i)$y(0)=a_0$,$y'(0)=a_1$,$y(b)=a_3$. When n=2, r=0,

ii)$y(0)=a_0$,$y'(0)=a_1$,$y'(b)=a_3$. When n=2, r=1,

iii)$y(0)=a_0$,$y'(0)=a_1$,$y''(b)=a_3$. When n=2, r=2.

Where operators L are gotten as

$$L(.)=x^{-1}\frac{d^2}{dx^2}x^{3-m}\frac{d}{dx}x^{m-2}(.), \tag{14}$$

$$L(.)=x^{-1}\frac{d}{dx}x^{2-m}\frac{d}{dx}x^{m-1}\frac{d}{dx}(.), \tag{15}$$

$$L(.)=x^{(-m)}\frac{d}{dx}x^{(m)}\frac{d^2}{dx^2}(.). \tag{16}$$

The inverse operators L^{-1} are respectively

(17) $$L^{-1}(.) = x^{2-m}\int_b^x x^{m-3}\int_0^x\int_0^x x(.)dxdxdx,$$

(18) $$L^{-1}(.) = \int_0^x x^{1-m}\int_b^x x^{m-2}\int_0^x x(.)dxdxdx,$$

(19) $$L^{-1}(.) = \int_0^x\int_0^x x^{-m}\int_b^x x^{m}(.)dxdxdx.$$

Eq.(13) with conditions (i,ii) have been studied in [6,7] so, we are going to study the equation with condition (iii) and using a new differential operator(16) and the inverse operator (19).

Example 1. First, we consider the third order boundary value problem:

(20) $$y''' - \frac{2}{x}y'' - y - y^2 = f(x),$$

$$y(0) = 0, y'(0) = 0, y''(1) = 35.3377,$$

where

$$f(x) = e^x\left(-6+6x+7x^2-e^x x^6\right).$$

We use Taylor series of f(x) with order 10

$$f(x) = -6+10x^2+9x^3+\frac{17x^4}{4}+\frac{41x^5}{30}-\frac{2x^6}{3}-\frac{325x^7}{168}-\frac{5729x^8}{2880}-\frac{20137x^9}{15120}-\frac{67181x^{10}}{100800},$$

for m=2, in the new differential operator (16) and its inverse (19) gives

$$L(.) = x^2\frac{d}{dx}x^{-2}\frac{d^2}{dx^2}(.).$$

So

$$L^{-1}(.) = \int_0^x\int_0^x x^2\int_1^x x^{-2}(.)dxdxdx.$$

In an operator form, Eq.(13) becomes

(21) $$Ly = g(x) + y + y^2.$$

By using L^{-1} on both sides of (20) we get

$$y = 35.3377x^4 + L^{-1}g(x) + L^{-1}y + L^{-1}y^2.$$

To find the solution, we use the iterative formula

$$y_0 = 35.3377x^4 + L^{-1}g(x),$$

$$y_{n+1} = L^{-1}y_n + L^{-1}A_n,\ n \geq 0, \tag{22}$$

where the nonlinear term y^2 has Adomian polynomials A_n as the following

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1, \tag{23}$$

$$A_2 = 2y_2y_0 + y_1^2,$$

so, from (22) and (23) we get

$$y_0 = x^3 + 1.17165x^4 + 0.5x^5 + 0.15x^6 + 0.0337302x^7 + 0.00610119x^8 - 0.00185185x^9$$
$$-0.00358245x^{10} - 0.00258342x^{11} - 0.00126119x^{12} - 0.0004747x^{13} + \ldots,$$
$$y_1 = -0.188897x^4 + 0.x^5 + 0.0166667x^6 + 0.0092988x^7 + 0.00223214x^8 + 0.00319444x^9$$
$$+0.0044019x^{10} + 0.00308943x^{11} + 0.00139185x^{12} + 0.000473912x^{13} + \ldots,$$
$$y_2 = 0.0159126x^4 - 0.00149918x^7 + 0.x^8 + 0.0000462963x^9 - 0.000682397x^{10}$$
$$-0.000571959x^{11} - 0.000144289x^{12} + 3.83584\,10^{-6}x^{13} + \ldots,$$

the solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 = x^3 + 0.998665x^4 + 0.5x^5 + 0.166667x^6 + 0.0415298x^7 + 0.00833333x^8$$
$$+0.00138889x^9 + 0.000137057x^{10} + \ldots.$$

observe, we can write the series of true solution $y(x) = x^3e^x$ is

$$y(x) = x^3 + x^4 + 0.5x^5 + 0.166667x^6 + 0.0416667x^7 + 0.00833333x^8$$

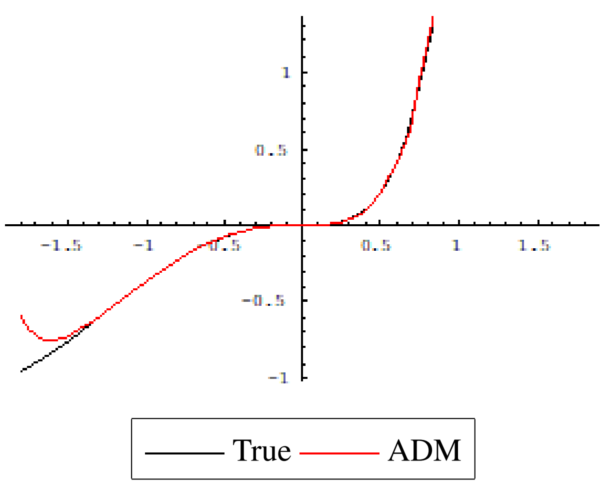
$$+0.00138889x^9+0.000198413x^{10}+\dots$$

In table 1, we give the true solutions and the ADM solution in[0,1]

Table 1.Numerical results for Example 1

x	True solution	ADM solution	Absolute Error
0.0	0.000000000	0.00000000	000000000
0.1	0.00110517	0.00110504	$1.33483\times10^{-}7$
0.2	0.00977122	0.00976908	$2.13727\times10^{-}6$
0.3	0.03644618	0.03643534	0.0000108415
0.4	0.09547678	0.09544237	0.0000344025
0.5	0.20609015	0.20600557	0.00008458
0.6	0.39357766	0.39340025	0.000177402
0.7	0.69071717	0.69038295	0.000334221
0.8	1.13947695	1.13889451	0.00058244
0.9	1.79305066	1.79210043	0.00095022
1	2.71828182	2.71684714	0.00143468

In Fig 1, we have plotted $\sum_0^2 y_i(x)$, which is similar to the true solution $y(x)=x^3e^x$.



The third type, when $n=3$, then r take one of the values $0,1,2,3$ yields

$$y^{(4)}+\frac{m}{x}y^{(3)}+Ny=f(x),$$

under one of the following boundary conditions

i)$y(0)=a_0, y'(0)=a_1, y''(0)=a_2, y(b)=a_4$, when n=3, r=0,

ii)$y(0)=a_0$,$y'(0)=a_1$,$y''(0)=a_2$,$y'(b)=a_4$, when n=3, r=1,

iii)$y(0)=a_0$,$y'(0)=a_1$,$y''(0)=a_2$,$y''(b)=a_4$, when n=3, r=2,

iv)$y(0)=a_0$,$y'(0)=a_1$,$y''(0)=a_2$,$y'''(b)=a_4$.when n=3, r=3.

The differential operators L are given, respectively as

(24) $$L(.)=x^{-1}\frac{d^3}{dx^3}x^{4-m}\frac{d}{dx}x^{m-3}(.),$$

(25) $$L(.)=x^{-1}\frac{d^2}{dx^2}x^{3-m}\frac{d}{dx}x^{m-2}\frac{d}{dx}(.),$$

(26) $$L(.)=x^{(-1)}\frac{d}{dx}x^{2-m}\frac{d}{dx}x^{m-1}\frac{d^2}{dx^2}(.).$$

(27) $$L(.)=x^{-m}\frac{d}{dx}x^{m}\frac{d^3}{dx^3}(.).$$

The inverse operators L^{-1} are respectively

(28) $$L^{-1}(.)=x^{3-m}\int_b^x x^{m-4}\int_0^x\int_0^x\int_0^x x(.)dxdxdxdx,$$

(29) $$L^{-1}(.)=\int_0^x x^{2-m}\int_b^x x^{m-3}\int_0^x\int_0^x x(.)dxdxdxdx,$$

(30) $$L^{-1}(.)=\int_0^x\int_0^x x^{1-m}\int_b^x x^{m-2}\int_0^x x(.)dxdxdxdx.$$

(31) $$L^{-1}(.)=\int_0^x\int_0^x\int_0^x x^{-m}\int_b^x x^{m}(.)dxdxdxdx.$$

Note that, the operators (24), (25) under the conditions (i),(ii) have been studied in [6,7]. So we will apply the method under the conditions (iii), (iv). as the following example:

Example 2. Next, we examine the boundary value problem of the kind

$$y^{(4)} - \frac{4}{x} y^{(3)} = -8(9+57x^4-137x^8+7x^{12})e^{-4y}, \tag{32}$$

with one of the following boundary conditions

$$y(0)=0, y'(0)=0, y''(0)=0, y''(1)=2,$$

or

$$y(0)=0, y'(0)=0, y''(0)=0, y'''(1)=-8.$$

To study Eq. (32) with first condition, we substitute m=-4 in (26), (30) to obtain

$$L(.) = x^{-1}\frac{d}{dx}x^{6}\frac{d}{dx}x^{-5}\frac{d^2}{dx^2}(.),$$

and the inverse operator is

$$L^{-1}(.) = \int_0^x \int_0^x x^5 \int_1^x x^{-6} \int_0^x x(.)dxdxdxdx.$$

In an operator form, Eq. (32) becomes

$$Ly = -8(9+57x^4-137x^8+7x^12)e^{-4y}. \tag{33}$$

By using L^{-1} on both sides of (33) we get

$$y = \frac{2}{42}x^7 + L^{-1}(-8(9+57x^4-137x^8+7x^12)e^{-4y}).$$

To find the solution, we use the iterative formula

$$y_0 = 0.047619x^7,$$

$$y_{n+1} = L^{-1}(-8(9+57x^4-137x^8+7x^12))A_n,\ n \geq 0, \tag{34}$$

where the nonlinear term e^{-4y} has A_n

$$A_0 = e^{-4y_0},$$

(35) $$A_1 = -4y_1 e^{-4y_0},$$

$$A_2 = 4(-y_2 + 2y_1^2)e^{-4y_0},$$

so, from (34) and (35) we have

$$y_1 = x^4 + 1.00697x^7 - 1.35714x^8 + 0.0034632x^{11} + 0.166061x^{12} + 0.00397698x^{15}$$

$$-0.00185185x^{16} - 0.0000242521x^{18} - 0.00299223x^{19} - 0.0000596833x^{22} + \dots,$$

$$y_2 = -1.24791x^7 + 0.857143x^8 + 0.0732345x^{11} + 0.217143x^{12} + 0.0816328x^{15}$$

$$-0.225251x^{16} - 0.00102569x^{18} - 0.0670808x^{19} + 0.0728304x^{20} - 0.00248791x^{22} + \dots,$$

thus

$$y(x) = x^4 - 0.193314x^7 - 0.5x^8 + 0.0766977x^{11} + 0.383203x^{12} + 0.0856098x^{15}$$

$$-0.227103x^{16} - 0.00104994x^{18} - 0.070073x^{19} + 0.0728304x^{20} - 0.00254759x^{22} + \dots.$$

for consider Eq. (32) under first condition, we put m=-4 in the new differential operator (27), and in the inverse operator (31) we get

$$L(.) = x^4 \frac{d}{dx} x^{-4} \frac{d^3}{dx^3}(.),$$

and

$$L^{-1} = \int_0^x \int_0^x \int_0^x x^4 \int_{\frac{1}{6}}^x x^{-4}(.)dxdxdxdx.$$

In an operator form, Eq.(32) becomes

(36) $$Ly = -8(9 + 57x^4 - 137x^8 + 7x^12)e^{-4y}.$$

By using L^{-1} on both sides of (36)

$$y = -\frac{4}{105}x^7 + L^{-1}(-8(9 + 57x^4 - 137x^8 + 7x^12)e^{-4y}).$$

To get the solution we use the iterative formula

$$y_0 = -0.0380952x^7,$$

(37) $$y_{n+1} = L^{-1}(-8(9 + 57x^4 - 137x^8 + 7x^12))A_n, n \geq 0,$$

where the nonlinear term e^{-4y} has are Adomian polynomials A_n as the following

$$A_0 = e^{-4y_0},$$

$$A_1 = -4y_1 e^{-4y_0}, \tag{38}$$

$$A_2 = 4(-y_2 + 2y_1^2)e^{-4y_0},$$

so, from (37) and (38) we get

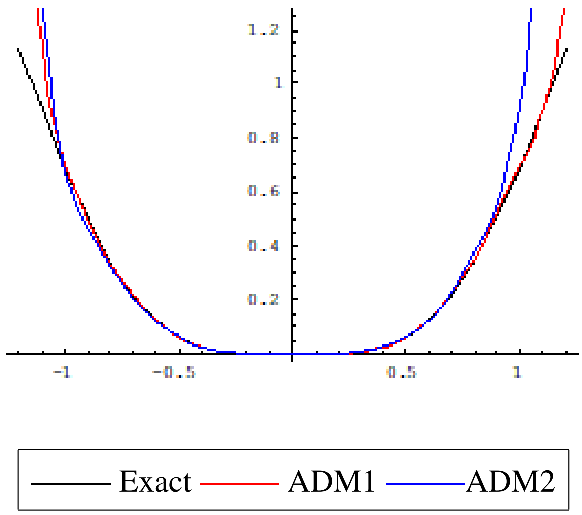
$$y_1 = x^4 + 1.03365x^7 - 1.35714x^8 - 0.00277056x^{11} + 0.166061x^{12} - 0.00318158x^{15}$$
$$-0.00185185x^{16} + 0.00239378x^{19} - 0.0000501914x^{23}.$$
$$y_2 = -0.927291x^7 + 0.857143x^8 + 0.0751746x^{11} + 0.217143x^{12} + 0.0882998x^{15}$$
$$-0.225251x^{16} + 0.000842292x^{18} - 0.0619068x^{19} + 0.0728304x^{20} + 0.00207196x^{22} + \dots,$$
$$\dots$$

Table 2 shows the Exact solution, the ADM solution and Absolute Error in example (2),

Table 2. Numerical results for Example 2

x	EXACT solution	ADM1 solution with the first condition	Absolute Error	ADM2 solution with the second condition	Absolute Error
0.0	00000000	00000000	00000000	00000000	0000000
0.1	0.00009999	0.0001	6.19879×10^{-9}	0.00009997	3.4×10^{-8}
0.2	0.00159872	0.00159951	7.93172×10^{-7}	0.00159559	3.1×10^{-6}
0.3	0.00806737	0.00808090	0.00001353	.00801392	0.00005344
0.4	0.0252778	0.0253787	0.000100899	0.0.02487751	0.00040029
0.5	0.06062462	0.0610999	0.00047525	0.05871607	0.00190855
0.6	0.12186358	0.12351336	0.00164977	0.115016	0.0068494
0.7	0.21519202	0.21966408	0.00447206	0.19490591	0.0202861
0.8	0.34330597	0.35254001	0.00923404	0.0002335	0.0527535
0.9	0.50446544	0.51653381	0.01206837	0.0378866	0.125599
1	0.69314718	0.69461426	0.00146709	0.41716996	0.275977

In Fig 2, we have plotted $\sum_{i=0}^{3} y_i(x)$, which is almost equal to the exact solution $y(x) = \ln(1+x^4)$.



The fourth type, when $n = 4,\ r \in \{0,1,2,3,4\}$ we get

$$y^{(5)} + \frac{m}{x} y^{(4)} + Ny = f(x),$$

by using one of the following boundary value problems

i) when $n = 4,\ r = 0,$

$y(0) = a_0, y'(0) = a_1, y''(0) = a_2, y'''(0) = a_3, y(b) = a_4,$

ii) when $n = 4,\ r = 1,$

$y(0) = a_0, y'(0) = a_1, y''(0) = a_2, y'''(0) = a_3, y'(b) = a_4,$

iii)when $n = 4,\ r = 2,$

$y(0) = a_0, y'(0) = a_1, y''(0) = a_2, y'''(0) = a_3, y''(b) = a_4,$

iv)when $n = 4,\ r = 3,$

$y(0) = a_0, y'(0) = a_1, y''(0) = a_2, y'''(0) = a_3, y'''(b) = a_4,$

v) when $n=4,\ r=4,$

$y(0)=a_0, y'(0)=a_1, y''(0)=a_2, y'''(0)=a_3, y^{(4)}(b)=a_4.$

The differential operators respectively are

$$L(.)=x^{-1}\frac{d^4}{dx^4}x^{5-m}\frac{d}{dx}x^{m-4}(.), \tag{39}$$

$$L(.)=x^{-1}\frac{d^3}{dx^3}x^{4-m}\frac{d}{dx}x^{m-3}\frac{d}{dx}(.), \tag{40}$$

$$L(.)=x^{-1}\frac{d^2}{dx^2}x^{3-m}\frac{d}{dx}x^{m-2}\frac{d^2}{dx^2}(.), \tag{41}$$

$$L(.)=x^{-1}\frac{d}{dx}x^{2-m}\frac{d}{dx}x^{m-1}\frac{d^3}{dx^3}(.), \tag{42}$$

$$L(.)=x^{-m}\frac{d}{dx}x^{m}\frac{d^4}{dx^4}(.)., \tag{43}$$

And the inverse operators L^{-1} respectively as below

$$L^{-1}(.)=x^{4-m}\int_b^x x^{m-5}\int_0^x\int_0^x\int_0^x\int_0^x x(.)dxdxdxdxdx, \tag{44}$$

$$L^{-1}(.)=\int_0^x x^{3-m}\int_b^x x^{m-4}\int_0^x\int_0^x\int_0^x x(.)dxdxdxdxdx, \tag{45}$$

$$L^{-1}(.)=\int_0^x\int_0^x x^{2-m}\int_b^x x^{m-3}\int_0^x\int_0^x x(.)dxdxdxdxdx. \tag{46}$$

$$L^{-1}(.)=\int_0^x\int_0^x x\int_0^x x^{1-m}\int_b^x x^{m-2}\int_0^x x(.)dxdxdxdxdx. \tag{47}$$

$$L^{-1}(.)=\int_0^x\int_0^x\int_0^x\int_0^x x^{-m}\int_b^x x^{m}(.)dxdxdxdxdx. \tag{48}$$

In the following example, the fifth order boundary value problem study under term (iii), (iv), (v) only. The another conditions have been studied in [6,7] .

Example 3. Let us consider the fifth order boundary value problem

$$y^{(5)}-\frac{3}{x}y^{(4)}=-\frac{12(192+80x^2-220x^4+7x^6)e^{4y}}{x(4+x^2)}, \tag{49}$$

under the boundary conditions

$y(0)=\log(\frac{1}{4}), y'(0)=0, y''(0)=-\frac{1}{2}, y'''(0)=0, y''(2)=0,$

or

$y(0)=\log(\frac{1}{4}), y'(0)=0, y''(0)=\frac{-1}{2}, y'''(0)=0, y'''(2)=\frac{1}{8},$

or

$y(0)=\log(\frac{1}{4})y'(0)=0, y''(0)=-\frac{1}{2}, y'''(0)=0, y''''(2)=-\frac{3}{16},$

with exact solution $y=\log(\frac{1}{4+x^2})$.

To study Eq.(49) with first condition, we put m=-3 in the new differential operator (41), and in the inverse operator (46) we get

$$L(.)=x^{-1}\frac{d^2}{dx^2}x^6\frac{d}{dx}x^{-5}\frac{d^2}{dx^2}(.),$$

$$L^{-1}(.)=\int_0^x\int_0^x x^5\int_2^x x^{-6}\int_0^x\int_0^x x(.)dxdxdxdxdx.$$

Eq.(49) can be written as operator form

$$Ly=-\frac{12(192+80x^2-220x^4+7x^6)}{x(4+x^2)}e^{4y}. \tag{50}$$

Taking L^{-1} on both sides of (50) and using the boundary condition gives

$$y(x)=-1.38629-0.25x^2+0.000372024x^7-12L^{-1}(\frac{192+80x^2-220x^4+7x^6}{x(4+x^2)}e^{4y}).$$

To get tho solution we use the iterative formula

$$y_0=-1.38629-0.25x^2+0.000372024x^7.$$

$$y_{n+1}=-12L^{-1}\frac{(192+80x^2-220x^4+7x^6)}{x(4+x^2)}A_n,\ n\geq 0, \tag{51}$$

where the nonlinear term e^{4y} has Adomian polynomials A_n as the following

$$A_0=e^{4y_0},$$

(52) $$A_1 = 4y_1e^{4y_0},$$

$$A_2 = 4y_2e^{4y_0} + 8e^{4y_0}y_1^2,$$

...

So, by substituting (52) into (51)and it must observe that, to compute y_1, we use the Taylor series for $(\frac{192+80x^2-220x^4+7x^6}{x(4+x^2)})$ with order 10.

In this case we obtain

$$y_0 = -1.38629 - 0.25x^2 + 0.000372024x^7,$$

$$y_1 = 0.03125x^4 - 0.00520833x^6 - 0.00053857x^7 + 0.00114397x^8 - 0.000213914x^{10},$$

$$y_2 = 0.00022131x^7 - 0.000167411x^8 + 0.0000186012x^{10} + \dots$$

$$y_3 = 4.61597 \times 10^{-6}x^7 - 1.15597 \times 10^{-7}x^{11} - 2.70563 \times 10^{-7}x^{12} + \dots,$$

this means that the solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 = -1.38629 - 0.25x^2 + 0.03125x^4$$

$$-0.00520833x^6 + 0.0000593791x^7 + 0.000976563x^8$$

$$-0.000195313x^{10} - \dots.$$

The series of exact solution $y(x) = \log(\frac{1}{4+x^2})$ is as the following

$$y(x) = -1.38629 - 0.25x^2 + 0.03125x^4 - 0.00520833x^6 + 0.000976563x^8$$

$$-0.000195313x^{10} + \dots$$

To study Eq(49) with second condition, we substitute m=-4 in (42), (47) to get the new differential operator

$$L(.) = x^{-1}\frac{d}{dx}x^5\frac{d}{dx}x^{-4}\frac{d^3}{dx^3}(.),$$

and the inverse operator

$$L^{-1} = \int_0^x \int_0^x x \int_0^x x^4 \int_2^x x^{-5} \int_0^x x(.)dxdxdxdxdx.$$

By using operator form, Eq.(49) is

(53) $$Ly = -\frac{12(192+80x^2-220x^4+7x^6)}{x(4+x^2)}e^{4y}.$$

Taking L^{-1} on (53) and using the condition gives

$$y(x) = -1.1.38629 - 0.25x^2 + 0.0000372024x^7 - 12L^{-1}(\frac{192+80x^2-220x^4+7x^6}{x(4+x^2)}e^{4y}).$$

To determine the components $y_n(x)$ we use the iterative formula

$$y_0 = -1.1.38629 - 0.25x^2 + 0.0000372024x^7.$$

$$y_{n+1} = -12L^{-1}\frac{(192+80x^2-220x^4+7x^6)}{x(4+x^2)}A_n,\ n \geq 0, \tag{54}$$

where the nonlinear term e^{4y} has A_n as the following

$$A_0 = e^{4y_0},$$

$$A_1 = 4y_1e^{4y_0}, \tag{55}$$

$$A_2 = 4y_2e^{4y_0} + \frac{16}{2}e^{4y_0}y_1^2,$$

...

So, from (54) and (55) we get

$$y_0 = -1.38629 - 0.25x^2 - 0.0000279018x^7$$

$$y_1 = 0.03125x^4 - 0.00520833x^6 - 0.000163265x^7 + 0.00114397x^8$$

$$-0.000213914x^{10} - 1.05689\,10^{-8}x^{11} + 0.0000377473x^{12} + \dots,$$

$$y_2 = 0.000233239x^7 - 0.000167411x^8 + 0.0000186012x^{10} + 4.63822 \times 10^{-8}x^{11}$$

$$+3.2134 \times 10^{-6}x^{12} + \dots,$$

the solution in a series form is given by

$$y(x) = -1.38629 - 0.25x^2 + 0.03125x^4 - 0.00520833x^6 - 0.00110687x^7$$

$$+0.000976563x^8 - 0.000195313x^{10} - 0.000016927x^{11} + 0.000040986x^{12} + \dots$$

To study Eq. (49) with the third condition, we put m=-3 in the new differential operator (43), and in the inverse operator (48) to get

$$L(.) = x^3\frac{d}{dx}x^{-3}\frac{d^4}{dx^4}(.),$$

and

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x x^3 \int_2^x x^{-3} x(.) dxdxdxdxdx.$$

In an operator form, Eq.(47) becomes

$$Ly = -\frac{12(192+80x^2-220x^4+7x^6)}{x(4+x^2)} e^{4y}. \tag{56}$$

Taking L^{-1} on (56) and using the boundary condition gives

$$y = -1.38629 - 0.25x^2 - 0.0000279018x^7 - 12L^{-1}\frac{192+80x^2-220x^4+7x^6}{x(4+x^2)}e^{4y}.$$

To determine the components $y_n(x)$, we use the iterative formula

$$y_0 = -1.38629 - 0.25x^2 - 0.0000279018x^7.$$

$$y_{n+1} = -12L^{-1}\frac{(192+80x^2-220x^4+7x^6)}{x(4+x^2)}A_n, n \geq 0, \tag{57}$$

where A_n are Adomian polynomials of nonlinear term e^{4y},as below,

$$A_0 = e^{4y_0},$$

$$A_1 = 4y_1 e^{4y_0}, \tag{58}$$

$$A_2 = 4y_2 e^{4y_0} + 8e^{4y_0} y_1^2,$$

...

So, from (57) and (58) we have

$$y_0 = -1.38629 - 0.25x^2 - 0.0000279018x^7,$$

$$y_1 = 0.03125x^4 - 0.00520833x^6 - 0.000107195x^7 + 0.00114397x^8 - 0.000213914x^{10}$$

$$+7.92664 \times 10^{-9}x^{11} + \dots,$$

$$y_2 = 0.000162925x^7 - 0.000167411x^8 + 0.0000186012x^{10} + 3.0453 \times 10^{-8}x^{11}$$

$$+3.2134 \times 10^{-6}x^{12} - 7.80847\, 10^{-9}x^{13} - 2.57975 \times 10^{-6}x^{14} - 3.0248 \times 10^{-9}x^{15} + \dots,$$

$$y_3 = -0.0000442476x^7 - 4.62856\times 10^{-8}x^{11} - 2.70563\times 10^{-7}x^{12} + 1.18681\times 10^{-8}x^{13}$$
$$+1.13487\times 10^{-7}x^{14} + 5.23813\times 10^{-9}x^{15} + \ldots,$$

this means that the solution in a series form is given by

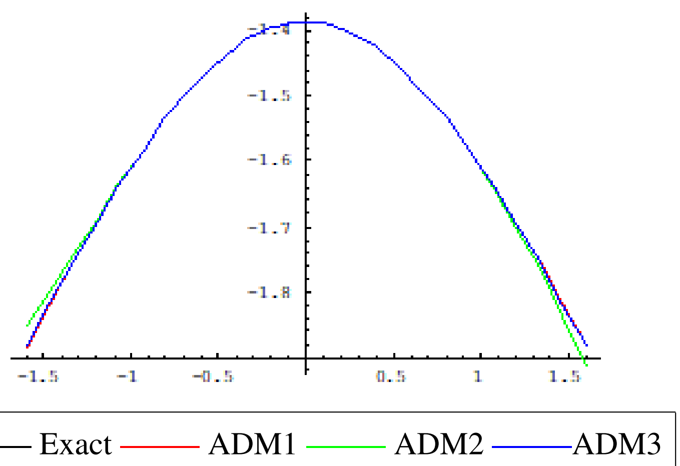
$$y(x) = y_0 + y_1 + y_2 + y_3 = -1.38629 - 0.25x^2 + 0.03125x^4 - 0.00520833x^6$$
$$-0.0000164186x^7 + 0.000976563x^8 - 0.000195313x^{10} - \ldots,$$

the series of exact solution $y(x) = \log(\frac{1}{4+x^2})$ is as follows

$$y(x) = -1.38629 - 0.25x^2 + 0.03125x^4 - 0.00520833x^6 + 0.000976563x^8$$
$$-0.000195313x^{10} + \ldots.$$

Noted that, when we continue finding the approximate solution for the above problem by using the ADM method, we easily get the exact solution.

In Fig 3, we have plotted $\sum_{i=0}^{3} y_i(x)$, which is almost equal to the exact solution $y = \log(\frac{1}{4+x^2})$.



4. CONCLUSION

In this paper, a reliable modification of ADM has been proposed for solving (n+1)-order of non-linear singular boundary value problems. Throughout all the illustrative examples given in this paper, it can be concluded that the proposed modification of (ADM) for solving singular boundary value problems of (n+1) order gives more reliable, and exact solutions. We have presented a generalization by using such new technique of ADM because it becomes so clear that the solutions approach to the exact solution as in example (1,2) and sometimes the solutions equal the exact solution as example (3). This study has generally arrived at a proposed generalization which shows the suggested method is reliable and efficient. The graphics as well as the results obtained in this study validate performance, consistency, and rapid convergence of the technique and we see that the ADM is a powerful tool for linear and nonlinear ordinary differential equations.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] R. Rach, A. Baghdasarian, G. Adomian, Differential equations with singular coefficients, Appl. Math. Comput. 47 (1992), 179–184.

[2] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1988), 501–544.

[3] S.-H. Chang, Taylor series method for solving a class of nonlinear singular boundary value problems arising in applied science, Appl. Math. Comput. 235 (2014), 110–117.

[4] S.M. El-Sayed, Integral methods for computing solutions of a class of singular two-point boundary value problems, Appl. Math. Comput. 130 (2002), 235–241.

[5] O. González-Gaxiola, J.A. Santiago, J.R. de Chávez, Solution for the Nonlinear Relativistic Harmonic Oscillator via Laplace-Adomian Decomposition Method, Int. J. Appl. Comput. Math. 3 (2017), 2627–2638.

[6] Y.Q. Hasan, L.Z. Ming, Solving singular boundary value problems of higher -order ordinary differential equation by modified Adomian decomposition method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009), 2592-2596.

[7] Y.Q. Hasan, L.Z. Ming, A note on the use of modified Adomian decomposition method for solving singular boundary value problems of higher-order ordinary differential equation, Commun. Nonlinear. Sci. Numer. Simul. 14(2009), 3261-3265.

[8] M.K. Iqbal, M. Abbas, N. Khalid, New cubic B-spline approximation for solving non-linear singular boundary value problems arising in physiology, Commun. Math. Appl. 9 (2018), 377-392.

[9] A.S. Kanth, K. Aruna, A novel approach for a class of higher order nonlinear singular boundary value problems, Int. J. Pure Appl. Math. 84 (2013), 321-329.

[10] A. Kanth , K. Aruna, He's variational iteration method for treating nonlinear singular boundary value problems, Comput. Math. Appl. 60 (2010), 821–829.

[11] R. Singh, H. Garg, V. Guleria, Haar wavelet collocation method for Lane–Emden equations with Dirichlet Neumann and Neumann-Robin boundary conditions, Comput. Appl. Math. 346 (2019), 150–161.

[12] M. Salai Mathi Selvi, G. Hariharan, K. Kannan, M.H. Heydari, Two reliable computational methods pertaining to steady state substrate concentration of an immobilized enzyme system, Alex. Eng. J. 57 (2018), 2377–2385.

[13] Umesh, M. Kumar, Numerical solution of singular boundary value problems using advanced Adomian decomposition method, Engineering with Computers. (2020). https://doi.org/10.1007/s00366-020-00972-6.

[14] A.-M. Wazwaz, R. Rach, J.-S. Duan, Solving New Fourth–Order Emden–Fowler-Type Equations by the Adomian Decomposition Method, Int. J. Comput. Meth. Eng. Sci. Mech. 16 (2015), 121–131.

[15] A.-M. Wazwaz, A first course in integral equations, World Scientific, Singapore; River Edge, NJ, 1997.

[16] A.-M. Wazwaz, A new algorithm for calculating adomian polynomials for nonlinear operators, Appl. Math. Comput. 111 (2000), 33–51.