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# A NOVEL FEEDBACK CONTROL SYSTEM TO STUDY THE STABILITY IN STATIONARY STATES

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**Abstract.** In the last few decades, the stabilization in stationary states has emerged as a new auspicious campaigner in chaos theory and found a celebrated place through various control techniques such as predictive control, delayed feedback control, constant proportional feedback control and oscillating feedback control system. Generally, it is accepted that the superiority of control systems is not only to quash the irregular distribution of stationary states, but also to illustrate its basin of attractions as large as possible depending on the numerical as well as analytical observance. In this article, the universal stabilization in unstable stationary states is studied through superior fixed point feedback control system for a family of one-dimensional maps. Further, it is interesting to know that the novel system provides freedom in the control parameter $\gamma$ due to which the stabilization increases more rapidly for the lager range of parameter $\gamma$ in $[0,1]$. The analytical as well as numerical simulations are demonstrated to examine the behavior of parameter $\gamma$ for which the unstable stationary state admits universal stability.

**Keywords:** stationary states; fixed point feedback system; chaos; stability.

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## 1. Introduction

The term stabilization plays a central role in the dynamics of nonlinear dynamical systems and especially in unstable systems and automation. It is believed that this concept was emerged

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from the study of the equilibrium state of mechanical systems in 1644, when E. Torricelli an Italian physicist and mathematician examined the stationary point of a rigid body under the gravitational forces. In 1892, it was Lyapunov [14] who gave the fundamental ideas and techniques leading to basic research and applications of stability of nonlinear systems. But in the nineteenth century, the concept of stabilization in nonlinear dynamical systems is adopted due to Ott et al. [17] and Pyragas [21], that is poured by predictive feedback control and delayed feedback control systems, respectively. The modern stability theory is completely influenced by the tremendous work of Ushio and Yamando [29], which deals with the predictive based control in discrete chaotic system. Afterward, various control techniques were introduced to stabilize the chaos such as delayed feedback control technique to stabilize unstable periodic orbits [6, 16, 23], oscillating feedback process to stabilize the chaos [20], stabilization using predictive control techniques [19], etc.

From last two decades, the theory of control and stability is used in wide range applications of science and engineering such as ecology, biology, cryptography and traffic control models. Generally, three types of stabilizations are mentioned in nonlinear dynamical systems: (a) the stability in a system with respect to its equilibria, (b) the stability of its orbit of a system output trajectory, and (c) the structural stability of a system itself. In 1991, Ditto et al. [10] studied the stability in chaos by controlling unstable periodic points of order one and two using first kind stability. Also, Azevedo et al. [4] examined that the chaos control plays an important role in microwave-pumped spin-wave-instability experiments and studied that the irregular behavior reduces into periodicity for some cautiously selected amplitude and modulation frequency. In the next year, a control system was used to stabilize cardiac arrhythmias in rabbit ventrile by Garfinkel et al. [11]. Further, in 1997, Sinha [27] examined the stabilization in unstable behavior of biological reactions using various control schemes (see also [18, 24]). Also, it is interesting to know that the irregular distribution in traffic flow on the road is considered as one of the most powerful examples of nonlinear dynamical system. It was first observed by Jarett and Zhang [30] and then in 1997 using carfollow model the stabilization was carried out in traffic system on road. In 2012, using queue model an efficient traffic flow model was established by Grether et al. [12] depending on two parameters: traffic signal and travelers. Further, for the detailed applications on the stability of irregular systems one may refer to Boccaletti et al. [5], Devaney [9], Shang et al. [25], Chugh et al. [7], Sharkovsky et al. [26] and Holmgren [13].

Further, in 2011, the fuzzy algorithm to control chaotic behavior in nonlinear dynamical systems using a minimum entropy approach was studied by Sadeghian et al. [22]. Recently, in 2019 Ashish et al. [3] established the stability in standard logistic map through superior fixed point feedback iterative technique showing application in the traffic flow model (see also [1, 2]).

The article has been divided into five major sections. Section 1 contains the literature review on the stability of stationary states with applications in science and engineering. While Section 2 has the basic entities of chaos theory which are used in further sections. In Sections 3 and 4 the main results are studied followed by theorems, corollaries, remarks and examples. The universal stability of unstable fixed states is illustrated in Section 3 and the stability of unstable periodic states is established in Section 4. Finally the article is concluded in Section 5.

## 2. PRELIMINARIES

In this section, we recall some essential entities on stabilization and control in chaos theory which are used in further sections to establish the universal stabilization of stationary states in one-dimensional discrete maps. First, we present the basic results on Schwarzian derivative given by Singer [28] and then provide the essential entities on stabilization and control. Throughout this article $Sh(x)$ denotes the Schwarzian derivative for an one-dimensional map $h(x)$ and $SM(\gamma,x)$ shows the Schwarzian derivative in superior fixed point feedback control system $M(\gamma,x)$, where $\gamma \in (0,1)$.

**Definition 2.1.** Let $h(x)$ be an one-dimensional real valued map. Then, the Schwarzian derivative for $h(x)$ at a point $x$ is defined by:

$$Sh(x) = \frac{h'''(x)}{h'(x)} - \frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2,$$

where $h'(x)$, $h''(x)$, and $h'''(x)$ are assumed as the first, second and third continuous derivatives for the map $h(x)$ [28].

**Definition 2.2.** Let $h(x)$ be an one-dimensional map defined on $[0,r]$, where $r$ is a positive real number. Then, the map $h(x)$ is said to be $C^3$ one- dimensional map if it satisfy the following axioms:

(1) $h(x)$ has a unique critical point $c$ less then the stationary state $P$ such that $h'(x) > 0$, for all $x \in (0,c)$ and $h'(x) < 0$, for all $x \in (c,r)$.
(2) $h(x)$ has two stationary states 0 and $P$ such that $h'(0) > 1$ and $P > 0$.
(3) $h(x)$ has a negative Schwarzian derivative for each $x \in [0,r] - c$, that is, $Sh(x) < 0$, for $x \neq c$.

**Definition 2.3.** Let $h(x)$ be an one-dimensional map, then a point $x$ is said to be periodic stationary state of period $p$ if it satisfies $h^p(x) = x$, where $p$ is a positive integer which stands for the $p^{th}$ iterate for the map $h(x)$ [8].

**Definition 2.4.** Let $\{x_0, x_1, \ldots, x_{p-1}\}$ be an iterated sequence of period $p$ for the map $h(x)$, then the first order derivative for the $p^{th}$ iterate of the map $h(x)$ is given by:

$$(h^p)'(x_0) = h'(x_{p-1}) \ldots h'(x_1).h'(x_0)$$

and is known as the chain rule along a cycle for periodic stationary states [9].

**Definition 2.5.** Let $x \in X$ be a fixed point for an one-dimensional differentiable system $h(x)$, then the point $x$ is said to be attracting when $|h'(x)| < 1$, repelling when $|h'(x)| > 1$ and is said to be inactive when $|h'(x)| = 1$ [9].

**Definition 2.6.** Let $h(x)$ be an one-dimensional map defined on a non-empty set $X$ , then the sequence $\{x_n\}$ for an initiator $x_0 \in X$ defined by:

$$x_n = x_{n-1} + \gamma_{n-1}(h(x_{n-1}) - x_{n-1}),$$

where $\gamma_{n-1} \in [0,1]$ and $n \in N$, is said to be superior feedback control system or Mann iterative procedure for the map $h(x)$ [15].

**Corollary 2.7.** *Let $h(x)$ be an one-dimensional map defined on a closed interval $[0,1]$ then, $Sh(x) < 0$ for all $x$ implies $Sh^p(x) < 0$, for each positive integer $p$* [28].

## 3. Stability in Stationary States using Superior Feedback Control

In this section, we deal with the stability of unstable stationary states for a family of $C^3$-one dimensional maps. Therefore, let $h(x)$ be an original $C^3$-one dimensional map, then using the Definition 2.6, the relation

$$x+\gamma(h(x)-x)=M(\gamma,x) \quad \text{(say)} \tag{1}$$

where $\gamma \in (0,1)$ is a control parameter and $x \in [0,r]$, where $r$ is a positive real number, is known as the superior feedback control system since it takes predictive iterates as input on discrete time intervals. To examine the main result on stabilization of stationary states of the system $M(\gamma,x)$ we start the section with the following preliminary results:

**Theorem 3.1.** *Let us assume $h'(x)<0$ and $M'(\gamma,x)<0$ for all $x \in I \subset [0,r]$, where $h(x)$ is an one-dimensional map and $M(\gamma,x)$ is the superior feedback control system defined on $I$. If $h(x)$ has a negative Schwarzian derivative, that is, $Sh(x)<0$ for each $x \in I$ then, show that $M(\gamma,x)$ also has a negative Schwarzian derivative, that is, $SM(\gamma,x)<0$ for each $x \in I$ and $\gamma \in (0,1)$.*

*Proof.* Let us consider $Sh(x)<0$, then from Definition 2.1, for Schwarzian derivative, we can say

$$\frac{h'''(x)}{h'(x)}-\frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2<0,$$

$$\text{that is,} \qquad 2h'''(x)<3\frac{(h''(x))^2}{h'(x)}. \tag{2}$$

Then, using superior feedback control system $M(\gamma,x)=x+\gamma(h(x)-x)$, we obtain

$$M'(\gamma,x)=1+\gamma(h'(x)-1), \tag{3}$$

$$M''(\gamma,x)=\gamma h''(x), \tag{4}$$

$$\text{and} \qquad M'''(\gamma,x)=\gamma h'''(x). \tag{5}$$

Multiplying Equation (5) by 2 and then using inequality (2), we get

$$2M'''(\gamma,x)<3\gamma\frac{(h''(x))^2}{h'(x)}. \tag{6}$$

Now, using the Equations (3) and (4) in the inequality (6), we obtain

$$2M'''(\gamma,x) < 3\gamma \frac{\left(\frac{(M''(\gamma,x)}{\gamma}\right)^2}{\frac{M'(\gamma,x)-(1-\gamma)}{\gamma}}. \tag{7}$$

Then, solving inequality (7), we have

$$\frac{M'''(\gamma,x)}{M'(\gamma,x)} - \frac{3}{2}\left(\frac{M''(\gamma,x)}{M'(\gamma,x)}\right)^2 < 0,$$

$$\text{that is,} \qquad SM(\gamma,x) < 0, \qquad\qquad \text{(by Definition 2.1)}$$

for all $x \in I$ and $\gamma \in (0,1)$. Hence, the Schwarzian derivative for the superior feedback control system $M(\gamma,x)$ is also negative. This completes the proof. □

**Remark 3.2.** *Since the Schwarzian derivative for the superior feedback control system $M(\gamma,x)$ is also negative on $I$, then from the Definition 2.2 there exists a point $c \in [0,r]$ such that $M'(\gamma,c) = 0$, that means, the superior feedback control system has a unique critical point.*

Now, we formulate the following preliminary result for the existence of period-doubling bifurcation for a particular value of the parameter $\gamma$ in the superior system $M(\gamma,x)$:

**Theorem 3.3.** *Let $h(x)$ be a $C^3$-one dimensional map and $h'(P) < -1$, where $P$ is a stationary state. Then, the superior feedback control system $M(\gamma,x)$, where $\gamma \in (0,1)$ undergoes a period-doubling bifurcation at $\gamma_0 = \frac{2}{1-h'(P)}$, where $M'(\gamma_0,P) = -1$.*

*Proof.* Since $M(\gamma_0,x) = x + \gamma_0(h(x)-x)$, then for the stationary state $P$, we obtain

$$M'(\gamma_0,P) = 1 + \gamma_0(h'(P)-1).$$

Now, taking $M'(\gamma_0,P) = -1$, we get

$$1 + \gamma_0(h'(P)-1) = -1,$$

$$\text{that is,} \qquad \gamma_0 = \frac{2}{1-h'(P)}.$$

Thus, the superior feedback control system admits a period-doubling bifurcation at $\gamma_0 = \frac{2}{1-h'(P)}$. This completes the proof. □

**Remark 3.4.** *From Theorem 3.3, it is analyzed that the superior feedback control system $M(\gamma,x)$ admits universal stabilization for the unstable stationary state $P$ of $C^3$-one dimensional maps for $\gamma \in (0, \frac{2}{1-h'(P)})$.*

Next, we examine the main result of this section to examine the stabilization of stationary state $P$ in superior feedback control system $M(\gamma,x)$:

**Theorem 3.5.** *Let $h(x)$ be a $C^3$-one dimensional map and $M(\gamma,x)$ be the superior feedback control. If $M'(\gamma,P) \geq -1$ then the stationary point $P$ is universal stable in $M(\gamma,x)$ for each $x \in (0,r)$.*

*Proof.* Let us consider $M(\gamma,x) = x + \gamma(h(x) - x)$ be the superior feedback control system, such that

$$M'(\gamma,x) = \gamma h'(x) + (1-\gamma). \tag{8}$$

Then, from Equation (8) it is clear that $h'(0) > 1$ implies $M'(\gamma,0) > 1$, that means, from Definition 2.5 the stationary state 0 is an unstable stationary state in superior feedback control $M(\gamma,x)$. Also, it is observed that $M'(\gamma,x) = 0$ if and only if $h^{'}(x) = \frac{(\gamma-1)}{\gamma}$, for each $\gamma \in (0,1)$. Further, to examine the stability in stationary state $P$ the following cases are studied:

*Case-1*: When $M(\gamma,x)$ has no critical point. Then, obviously the function will be strictly increasing and therefore the stationary state $P$ is always globally attracting since $M(\gamma,x) > x$ for each $x$ belongs $(0,P)$ and $M(\gamma,x) < x$ for each $x$ belongs to $(P,r)$. The conditions holds for each $h^{'}(x) > \frac{(\gamma-1)}{\gamma}$ and $x \in (0,r)$.

*Case-2*: When $M(\gamma,x)$ has a critical point. Then, the relation $SM(\gamma,x) < 0$ for all $x \in [0,r] - c$ implies $h(x)$ may have at most one inflexion point in $(c,r)$. Then, the following subcases arise:

When $h(x)$ has no inflexion point in $(c,r)$. Then, the map $h'(x)$ strictly decreases in the interval $(c,r)$ and there exists at most one point $e > c$ such that $M'(\gamma,e) = 0$. Therefore, the point $e$ is assumed as a local maxima for $M(\gamma,x)$ because $M'(\gamma,x) > 0$ in $(0,e)$ and $M'(\gamma,x) < 0$ in $(e,r)$. So, if the point $e > P$ then $M'(\gamma,P) > 0$ shows stationary state $P$ is universal stable. Now, let us consider $c < e < P$. But we know $M(\gamma,x) \leq M(\gamma,e)$ for each $x \in (0,e]$ and also from Theorem 3.1, $SM(\gamma,x) < 0$ on $(e,r)$. Then, from the Corollary, "Let $h(x)$ be the continuous map defined on $(0,r)$ into $[0,r]$. Let $p,q,u,v$ and $w$ be the points satisfying $p \leq u < w < v \leq q$ such that the map $h(x)$ in $(u,v)$ has at most one turning point and also $h(x) \leq h(u)$ for $x \leq u$ and $h(x) \geq h(v)$ for $x \geq v$. If the map $h(x)$ decreases at point $w$ with $Sh(x) < 0$ for $x \in (u,v)$ except at most one critical point and also $h'(w) \geq -1$. Then, the point $w$ is universal stable for the map $h(x)$," taking $u = e$, $v = r$ and $w = P$, the stationary state $P$ is universal stable for the superior control system $M(\gamma,x)$.

When $h(x)$ has an inflexion point, say, $y$ in $(c,r)$ then, the map $h^{'}(x)$ attains a global minima at $h^{'}(y)$. Therefore, if $h'(y) \geq \frac{(\gamma-1)}{\gamma}$, then the superior system $M(\gamma,x)$ increases strictly and trivially $P$ is universal stable. Now, let us consider $h'(y) < \frac{(\gamma-1)}{\gamma}$, then there may exists at least one, or at most two critical points $c_1$ and $c_2$ such that $c_1 < y < c_2$ for the superior system $M(\gamma,x)$. For one critical point the result already have been studied. Also, for two critical points the system $M(\gamma,x)$ increases in the interval $(0,c_1) \cup (c_2,r)$ and decreasing in the interval $(c_1,c_2)$. Then, again from Theorem 3.1, taking $u = c_1$, $v = c_2$ and $w = P$, the stationary state $P$ is universal stable. This completes the proof. □

**Example 3.6.** *Let $h(x) = \lambda x(1-x)$ be an original $C^3$-one dimensional map, where $\lambda \in [0,4]$ and $x \in [0,1]$. Then, determine the stability for the trivial stationary states of $h(x)$ using superior feedback control system $M(\gamma,x)$ for some prescribed range of control parameter $\gamma$.*

*Solution.* The map $h(x) = \lambda x(1-x)$ is a well-defined model of population growth with trivial stationary states 0 and $1-\frac{1}{\lambda}$, where $\lambda \in (0,4]$. Also, the stationary state $1-\frac{1}{\lambda}$ admits local stability for $1 < \lambda \leq 3$ and unstability for $\lambda > 3$, that is, when $\lambda > 3$ the system first undergoes a period-doubling bifurcation and then for $\lambda > 3.56$ approaches to an irregular distribution as shown in Figure 3. Therefore, the universal stabilization in stationary state $1-\frac{1}{\lambda}$ makes sense when $\lambda > 3$. From Theorem 3.3 and Remark 3.4, it is illustrated that the stationary state $P$ for the map $h(x)$ in superior system undergoes an unstable behavior at $\gamma = \frac{2}{\lambda-1}$, that is, the stationary state undergos a universal stabilization for all $\gamma \in (0, \frac{2}{\lambda-1})$.

In particular, when $\lambda = 4$ the stationary state $P = 0.75$ for the system $h(x)$ undergoes a complete universal stabilization in $\gamma \in (0,0.67)$. Figure 1 shows the unstable distribution of an original system for $\lambda > 3$. While Figure 2 gives the plotting for stable orbit of stationary state controlled by superior feedback control $M(\gamma,x)$ for $\lambda = 4, 3.5$ and 3 corresponding to the control parameter $\gamma = 0.4, 0.5$ and $0.6$ respectively. Therefore, from both the Figures it is analyzed that the stationary state $1-\frac{1}{\lambda}$ is universal stabilized for each $\lambda > 3$ in a prescribed range of parameter $\gamma$. Further, Figures 3 and 4 shows the period-doubling plotting for the universal stability of stationary state. In Figure 3 we see that for $\lambda > 3$ the stationary state admits an unstability diverting into periodic as well as irregular distribution. While Figure 4 shows that the unstabilized distribution is stabilized in the stationary state $1-\frac{1}{\lambda}$ using superior feedback control when $\lambda = 4$ and $\gamma = 0.6$.

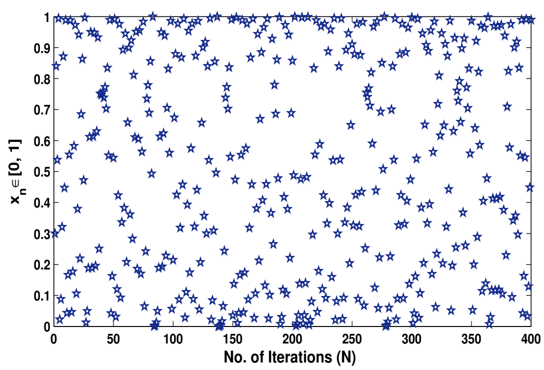


FIGURE 1. Unstable distribution for stationary state $1-\frac{1}{\lambda}$ of $\lambda x(1-x)$

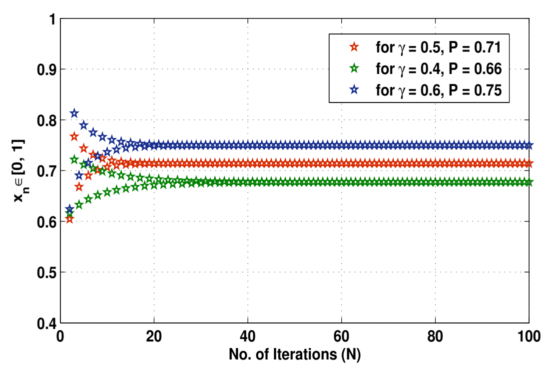


FIGURE 2. Stability for stationary state $1-\frac{1}{\lambda}$ at $\lambda = 4, 3.5$ and 3

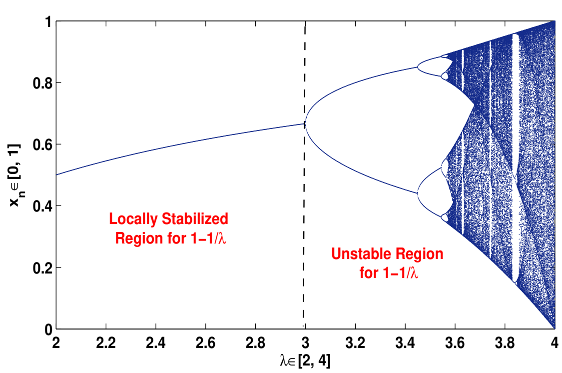


FIGURE 3. Complete bifurcation plot of $\lambda x(1-x)$ for $\lambda \in [0,4]$

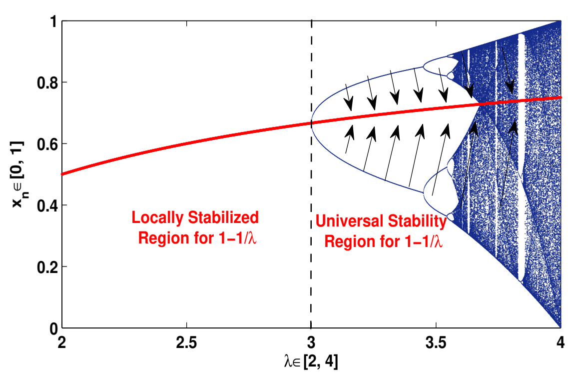


FIGURE 4. Bifurcation plot with stability in stationary state $1-\frac{1}{\lambda}$ at $\lambda = 4$

**Example 3.7.** *Let $xe^{\lambda(1-x)}$ be the ricker map defined on $[0,1]$. Then, examine the universal stability for trivial stationary state $P$ using superior system $M(\gamma,x)$ for some $\gamma \in (0,1)$.*

*Solution.* The relation $h(x) = xe^{\lambda(1-x)}$ is a popular one-dimensional map with $P = 1$ as its trivial stationary state which is locally stable when $\lambda \leq 2$, unstable when $\lambda > 2$ and chaotic when $\lambda > 2.69$.

Therefore, the universal stabilization for an unstable stationary state 1 is illustrated for $\lambda > 2$. Thus, from Theorem 3.3 and Remark 3.4, it is clear that the stationary state 1 for the ricker map $xe^{\lambda(1-x)}$ in superior feedback control undergoes an unstable behavior at $\gamma = \frac{2}{\lambda}$, that is, the stationary state undergos a universal stabilization when $\gamma \in (0, \frac{2}{\lambda})$. In particular, at $\lambda = 3$ the stationary state 1 undergoes a universal stabilization when $\gamma \in (0, 0.6)$.

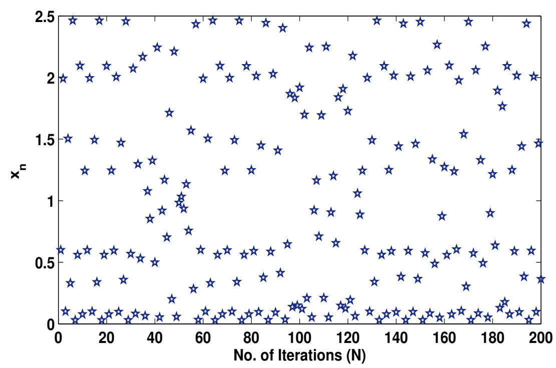


FIGURE 5. Unstable distribution for periodic stationary state 1 of $xe^{\lambda(1-x)}$

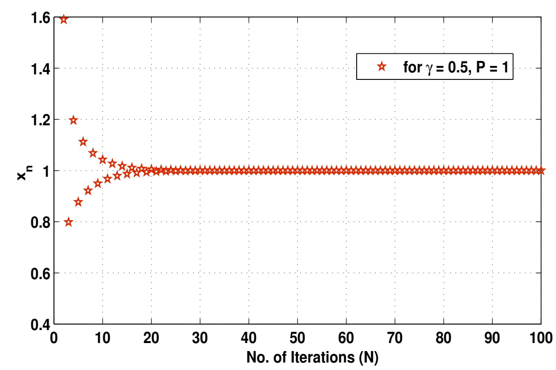


FIGURE 6. Universal stability of periodic stationary state 1 at $\lambda = 3$

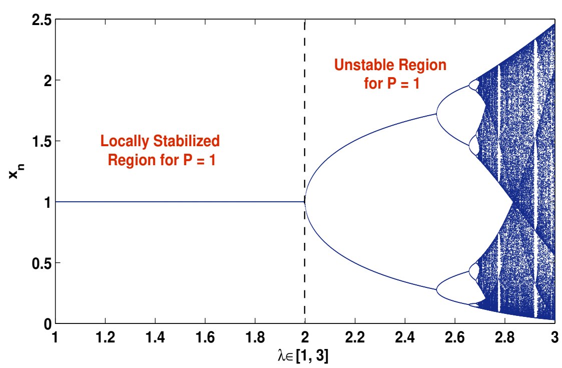


FIGURE 7. Complete bifurcation plot of $xe^{\lambda(1-x)}$ $x \in [0, 1]$ and $\lambda \in [0, 3]$

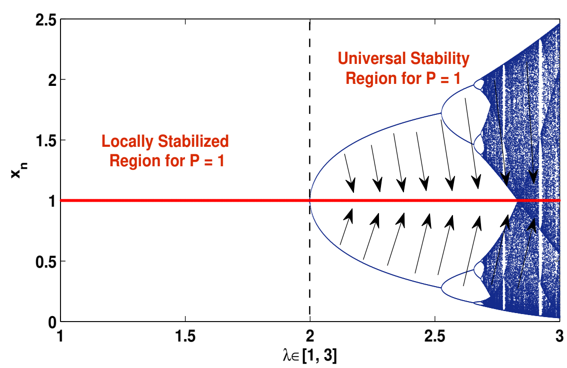


FIGURE 8. Bifurcation plot versus stability in periodic state 1 at $\lambda = 3$

Figure 5 gives an unstable distribution of stationary state 1 of an original map $xe^{\lambda(1-x)}$ for $\lambda > 2$ and this unstable distribution is stabilized using superior feedback control $M(\gamma, x)$ for $\gamma \in (0, 0.6)$ as shown in Figure 6. Further, the Figures 7 and 8 show the stabilization using period-doubling bifurcation plot. Figure 7 gives an original distribution of iterative orbits of the system showing local stability for $\lambda \in [1,2]$ and unstability for $\lambda > 2$. While Figure 8 represents the local as well as universal stabilization of stationary states when $\lambda \in [2,3]$.

## 4. STABILITY IN PERIODIC STATES USING SUPERIOR FEEDBACK CONTROL

In this section, we illustrate the stabilization of unstable periodic stationary states for a family of $C^3$-one dimensional maps using superior feedback control (9). Sometimes the stability in periodic stationary states plays a crucial role in various applications of science and engineering such as traffic flow models, ecology and cryptography. Therefore, using the following feedback control system the periodicity is stabilized for some particular range of the control parameter $\gamma \in (0,1)$. Thus, the relation

$$x + \gamma(h^p(x) - x) = M^p(\gamma, x), \qquad \text{(say)} \tag{9}$$

where $h^p(x)$ denotes the $p^{th}$ iterate of $h(x)$, $x \in [0,r]$ and $\gamma \in (0,1)$ is known as superior feedback control system for periodic stationary states. Therefore, to examine the universal stabilization for periodic stationary states we present the following without proof results because all the results may be derived directly from previous results for stationary states.

**Theorem 4.1.** *Let us consider $(h^p)'(x) < 0$ and $(M^p)'(\gamma,x) < 0$ for all $x \in I \subset [0,r]$, where $h(x)$ is a $C^3$-one dimensional map defined on $I$ and $M^p(\gamma,x)$ is the superior feedback control. If the Schwarzian derivative for $h^p(x)$ is negative then the Schwarzian derivative for $M^p(\gamma,x)$ is also negative for all $x \in I$ and $\gamma \in (0,1)$.*

*Proof.* Similar to the proof of Theorem 3.1 in Section 3. □

**Corollary 4.2.** *If $h(x)$ is an original $C^3$-one dimensional map and $(h^p)'(Q) < -1$, where $Q$ is a periodic stationary state of period-$p$. Then, the superior system $M^p(\gamma,x)$ undergoes a period-doubling bifurcation at $\gamma = \frac{2}{1-(h^p)'(Q)}$, for $(M^p)'(\gamma,Q) = -1$.*

*Proof.* Let $M^p(\gamma,x) = x + \gamma(h^p(x) - x)$ be a superior feedback control, then we determine

$$(M^p)'(\gamma,Q) = \gamma(h^p)'(Q) + 1 - \gamma. \tag{10}$$

Now, taking $(M^p)'(\gamma, Q) = -1$ in Equation (10), we get

$$\gamma(h^p)'(Q) + 1 - \gamma = -1,$$

$$\text{and,} \qquad \gamma((h^p)'(Q) - 1) = -2,$$

$$\text{that is,} \qquad \gamma = \frac{2}{1-(h^p)'(Q)}.$$

This complete the proof. □

**Remark 4.3.** *It is examined from Corollary 4.2, that the superior feedback control $M^p(\gamma, x)$ admits the universal stabilization for unstable periodic stationary states of $C^3$-one dimensional map $h(x)$ for all $\gamma \in (0, \frac{2}{1-(h^p)'(Q)})$.*

Next, we formulate the following main theorem of this section to establish the universal stabilization of periodic stationary states:

**Theorem 4.4.** *Let $h(x)$ be an original $C^3$-one dimensional map and $M^p(\gamma, x)$ be the superior feedback control for periodic stationary state $Q$. If $(M^p)'(\gamma, x) \geq -1$, then, the periodic state $Q$ of order $p$ is universal stable for each $x \in (0, r)$.*

*Proof.* Proof of this theorem is quite similar to the proof of Theorem 3.5. □

**Example 4.5.** *Let $\lambda x(1-x)$ be the $C^3$-one dimensional map, where $\lambda \in [0,4]$ and $x \in [0,1]$. Then, determine the universal stability of periodic stationary states of period 2 using the superior system $M^p(\gamma, x)$ in some prescribed range of parameter $\gamma$.*

*Solution.* It is well-known in the dynamics of the one-dimensional map $\lambda x(1-x)$ that it has the following two periodic stationary states of period-2:

$$Q_1 = \frac{(\lambda+1) + \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda} \qquad \text{and} \qquad Q_2 = \frac{(\lambda+1) - \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda}.$$

Further, it is assumed that $Q_1$ and $Q_2$ admits local stability for $3 \leq \lambda < 3.45$ and for greater then 3.45 it settle down into periodic points of higher order and finely approaches to chaos. That means, when $\lambda$ approaches through 3.45 the periodic states $Q_1$ and $Q_2$ approaches to unstable distribution as shown in Figure 9. Therefore, using Remarks 4.3 at $\lambda = 4$ we get the periodic stationary state $Q_1 = \frac{5+\sqrt{5}}{8}$ and $Q_2 = \frac{5-\sqrt{5}}{8}$ which are universal stabilized using superior feedback control (9) for $\gamma \in (0, 0.4)$ as shown in Figure 10.

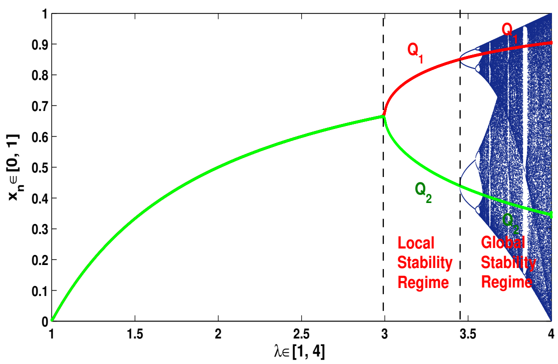


FIGURE 9. Bifurcation plot with stability in $Q_1$ and $Q_2$ states at $\lambda = 4$

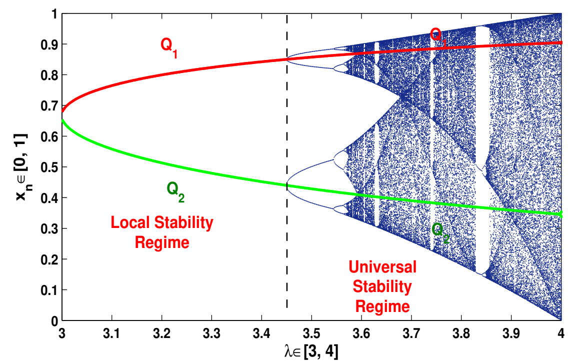


FIGURE 10. Zoomed bifurcation plot with stability in $Q_1$ and $Q_2$ states

## 5. CONCLUSION

In this article, using superior feedback control system to $C^3$-one dimensional maps the universal stability of unstable stationary states is studied. The whole article focus on the relation $x + \gamma(h(x) - x)$ given by Mann [15] in 1953 and stated as superior feedback control. As compared to other predictive procedures the novel control system determines the universal stability for a family of $C^3$-maps depending on an extra credential of control parameters $\gamma$. Therefore, the following results are illustrated:

(1) The Schwarzian derivative: a road map to examine the dynamical properties of one dimensional maps is used to study the universal stability of unstable stationary states. In Section 3, Theorem 3.1 illustrates that the Schwarzian derivative is also negative in superior feedback control as compared to an original $C^3$-one dimensional map.

(2) Theorem 3.3, determines the necessary condition to stabilize an unstable stationary state using superior feedback control, that is, $|M'(\gamma, P)| < 1$ for some $\gamma \in (0, 1)$. In sequel, from Theorem 3.3 and Remark 3.4, it is analyzed that the superior feedback control (1) admits a universal stability for an unstable stationary state $P$ of $C^3$-maps for $\gamma \in (0, \frac{2}{1-h'(P)})$.

(3) For simplicity the results are followed by the Examples 3.6 and 3.7 for logistic and Ricker maps and the stability is achieved in the prescribed range of parameter $\gamma$ such as $(0, \frac{2}{\lambda-1})$ and $(0, \frac{2}{\lambda})$, respectively. But it is observed that as the value of parameter $\lambda$ decreases the length of the interval of control parameter increases rapidly as shown in Example 3.6 for logistic map.

(4) Further, in Section 4 the universal stability for unstable periodic stationary states using superior control $x+\gamma(h^p(x)-x)$ is determined followed by Theorems 4.4. Further, the Corollary 4.2, determines the value of control parameter $\gamma$ for which the unstable periodic state of period-$p$ is stabilized. Thus, from Remark 4.3 the chaotic system is universal stabilized when $\gamma \in (0, \frac{2}{1-(h^p)'(Q)})$ followed by Example 4.5.

Further, it is strongly recommended that the superior feedback control system $x+\gamma(h(x)-x)$ may be taken over in various applications of nonlinear dynamical systems to improve the quality of a system.

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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