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UNIQUENESS OF L-FUNCTION AND ITS CERTAIN DIFFERENTIAL MONOMIAL CONCERNING SMALL FUNCTIONS

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Abstract. Concerning Small functions and weighted sharing we study the uniqueness of L-function and its certain differential monomial. Our results in this paper improve and extend some earlier results.

Keywords: L-function; uniqueness; small function; weighted sharing; differential monomial.

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1. INTRODUCTION

For a long time a lot of attention have been given by many scholars on the Riemann hypothesis. The Riemann zeta function is defined by the following infinite series $\zeta(s) = \sum_{m=1}^{\infty} 1/m^s = \prod_p (1 - 1/p^s)^{-1}$ where $s = \sigma + it$, $\sigma > 1$ and p denotes prime number and the product is taken over all prime numbers. Throughout the paper an L-function L means an L-function L in the Selberg class. Such an L-function is defined by $L(s) = \sum_{m=1}^{\infty} a(m)/m^s$ satisfying the following hypothesis

(i) Ramanujan hypothesis: For every $\varepsilon > 0$, $a(m) \ll m^{\varepsilon}$.

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(ii) Analytic continuation: There exists a nonnegative integer l such that $(s-1)^l L(s)$ is an entire function of finite order.

(iii) Every L-function satisfies the functional equation

$$\lambda_L(s) = \omega\overline{\lambda_L(1-\overline{s})},$$

where

$$\lambda_L(s) = L(s)Q^s \prod_{i=1}^{k} \Gamma(\mu_i s + \nu_i)$$

with positive real numbers Q, μ_i and complex numbers ν_i, ω with $Re\nu_i \geq 0$ and $\mid \omega \mid = 1$.

(iv) Euler product: $L(s)$ satisfies $L(S) = \prod_p L_p(s)$, where $L_p(s) = exp(\sum_{m=1}^{\infty} b(p^m)/p^{ms})$ with coefficients $b(p^m)$ satisfying $b(p^m) \ll p^{m\theta}$ for some $\theta < 1/2$ and p denotes prime number.

Let F and G be two nonconstant meromorphic functions in the open complex plane $\mathbb{C}$. We denote by $S(r,F)$ any function satisfying $S(r,F) = o(T(r,F))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure. A meromorphic function ρ is said to be a small function of F if $T(r,\rho) = S(r,F)$.

If $F - z_0$ and $G - z_0$ have the same set of zeros with the same multiplicities, we say that F and G share z_0 CM (counting multiplicities) and we say that F and G share z_0 IM (ignoring multiplicities) if we do not consider the multiplicities where $z_0 \in \mathbb{C} \cup \{\infty\}$.

In this paper we prove our results using Nevanlinna's value distribution theory. Here we use the standard notations and definitions of the value distribution theory [3].

2. Preliminaries

Definition 2.1. [6] *Let ξ be a meromorphic function defined in the complex plane. Let m be a positive integer and $c \in \mathbb{C} \cup \{\infty\}$. By $N(r,c;\xi \mid\leq m)$ we denote the counting function of the c points of ξ with multiplicity $\leq k$ and by $\overline{N}(r,c;\xi \mid\leq m)$ the corresponding one for which we do not count the multiplicity. Also by $N(r,c;\xi \mid\geq m)$ we denote the counting function of the c points of ξ with multiplicity $\geq m$ and by $\overline{N}(r,c;\xi \mid\geq m)$ the corresponding one for which we do not count the multiplicity. We define*

$$N_m(r,c;\xi) = \overline{N}(r,c;\xi) + \overline{N}(r,c;\xi \mid\geq 2) + \cdots + \overline{N}(r,c;\xi \mid\geq m).$$

Definition 2.2. [6] *Let ξ be a meromorphic function defined in the complex plane and ρ be a small function of ξ. Then we denote by $N(r,\rho;\xi \mid\leq m)$, $\overline{N}(r,\rho;\xi \mid\leq m)$, $N(r,\rho;\xi \mid\geq m)$, $\overline{N}(r,\rho;\xi \mid\geq m)$, $N_m(r,\rho;\xi)$ etc. the counting functions $N(r,0;\xi-\rho \mid\leq m)$, $\overline{N}(r,0;\xi-\rho \mid\leq m)$, $N(r,0;\xi-\rho \mid\geq m)$, $\overline{N}(r,0;\xi-\rho \mid\geq m)$, $N_m(r,0;\xi-\rho)$ etc. respectively.*

In 2007 Steuding [9] proved the following uniqueness theorem.

Theorem A. [9] *Let L_1 and L_2 be two L-functions with $a(1) = 1$ and $z_0 \neq \infty$ be a complex number. If L_1 and L_2 share z_0 CM, then $L_1 \equiv L_2$.*

Remark 2.1. [4] *In 2016 Hu and Li taking $L_1 = 1+2/4^s$ and $L_2 = 1+3/9^s$ proved that Theorem A is not true for $z_0 = 1$.*

In 2010 Li [7] proved the following theorem.

Theorem B. [7] *If a meromorphic function F having finitely many poles and a nonconstant L-function L share α CM and β IM then $L \equiv F$, where α and β are two distinct finite values.*

In 2017, considering uniqueness problem of L-functions, Liu, Li and Yi [8] proved the following theorem.

Theorem C. [8] *Let $k \geq 1$ and $j \geq 1$ be integers such that $k > 3j+6$. Also let L be an L-function and F be a nonconstant meromorphic function. If $\{F^k\}^{(j)}$ and $\{L^k\}^{(j)}$ share 1 CM then $F \equiv dL$ for some constant d satisfying $d^k = 1$.*

Definition 2.3. [5] *Let ξ be a meromorphic function defined in the complex plane and m be an integer (≥ 0) or infinity . For $c \in \mathbb{C} \cup \{\infty\}$ we denote by $E_{m)}(c;\xi)$ the set of all zeros of $\xi - c$ with multiplicities not exceeding m, where a zero is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(c;\xi)$ the set of all zeros of $\xi - c$ with multiplicities not exceeding m, where a zero is counted ignoring multiplicity.*

Definition 2.4. [5] *Let ξ and χ be two meromorphic functions defined in the complex plane and m be an integer (≥ 0) or infinity . For $c \in \mathbb{C} \cup \{\infty\}$ we denote by $E_m(c;\xi)$ the set of all zeros of $f-c$ where a zero of multiplicity k is counted k times if $k \leq m$ and $m+1$ times if $k > m$. If $E_m(c;\xi) = E_m(c;\chi)$, we say that ξ, χ share the value c with weight m.*

The definition implies that if ξ, χ share a value c with weight m then z_o is a c-point of ξ with multiplicity $k(\leq m)$ if and only if it is a c-point of χ with multiplicity $k(\leq m)$ and z_o is a c-point of ξ with multiplicity $k(> m)$ if and only if it is a c-point of χ with multiplicity $n(> m)$ where k is not necessarily equal to n.

We write ξ, χ share (c,m) to mean that ξ, χ share the value c with weight m. Clearly if ξ , χ share (c,m) then ξ, χ share (c,j) for all integers $j, 0 \leq j < m$. Also we note that ξ, χ share a value c IM or CM if and only if ξ, χ share $(c,0)$ or (c,∞) respectively.

Definition 2.5. *Let ξ be a meromorphic function defined in the complex plane and ρ be a small function of ξ. Then we denote by $E_{m)}(\rho;\xi)$, $\overline{E}_{m)}(\rho;\xi)$ and $E_m(\rho;\xi)$ the sets $E_{m)}(0;\xi-\rho)$, $\overline{E}_{m)}(0;\xi-\rho)$ and $E_m(0;\xi-\rho)$ respectively.*

Using weighted sharing in 2015, Wu and Hu [10] proved the following result.

Theorem D. [10] *Let L and H be two L-functions, and let $\alpha,\beta \in \mathbb{C}$ be two distinct values. Take two positive integers m_1, m_2 with $m_1m_2 > 1$. If $E_{m_1}(\alpha,L) = E_{m_1}(\alpha,H)$, and $E_{m_2}(\alpha,L) = E_{m_2}(\alpha,H)$, then $L \equiv H$.*

Considering weighted sharing in 2018 Hao and Chen [2] proved the following theorem.

Theorem E. [2] *Let L be an L-function and F be a meromorphic function defined in the complex plane $\mathbb{C}$ with finitely many poles. Let $\alpha_1,\alpha_2 \in \mathbb{C}$ be distinct and m_1,m_2 be positive integers such that $m_1m_2 > 1$. If $E_{m_j}(\alpha_j,F) = E_{m_j}(\alpha_j,L)$, $j = 1,2$, then $L \equiv F$.*

3. Main Results

In this paper, considering small function and weighted sharing we prove the following uniqueness theorem.

Theorem 3.1. *Let L be a nonconstant L-function and ρ be a small function of L such that $\rho \not\equiv 0,\infty$. If $\overline{E}_{4)}(\rho;L) = \overline{E}_{4)}(\rho;(L^m)^{(k)})$, $E_{2)}(\rho;L) = E_{2)}(\rho;(L^m)^{(k)})$ and*

$$2N_{2+k}(r,0;L^m) \le (\sigma + o(1))T(r,L), \tag{3.1}$$

where $m \ge 1$, $k \ge 1$ are integers and $0 < \sigma < 1$, then $L \equiv (L^m)^{(k)}$.

4. Lemmas

In this section, we present some results which we employ in the proof of our main results.

Let Φ and Ψ be two nonconstant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by Ω the following function

$$\Omega = \left(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}\right) - \left(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}\right). \tag{4.1}$$

Lemma 4.1. [1] *If $\overline{E}_{4)}(1;\Phi) = \overline{E}_{4)}(1;\Psi)$, $E_{2)}(1;\Phi) = E_{2)}(1;\Psi)$ and $\Omega \not\equiv 0$, then*

$$T(r,\Phi) + T(r,\Psi) \le 2\{N_2(r,0;\Phi) + N_2(r,\infty;\Phi) + N_2(r,0;\Psi) + N_2(r,\infty;\Psi)\} + S(r,\Phi) + S(r,\Psi).$$

Lemma 4.2. {*Theorem 2.5* [3]} *Let Φ be a meromorphic function. Then*

$$T(r,\Phi) \le \overline{N}(r,\infty;\Phi) + \overline{N}(r,a;\Phi) + \overline{N}(r,b;\Phi) + S(r,\Phi),$$

where a and b are small functions of Φ.

Lemma 4.3. [12] *Let Φ be a nonconstant meromorphic function and k, p are two positive integers. Then*

$$N_p(r,0;\Phi^{(k)}) \le T(r,\Phi^{(k)}) - T(r,\Phi) + N_{p+k}(r,0;\Phi) + S(r,\Phi)$$

and

$$N_p(r,0;\Phi^{(k)}) \le N_{p+k}(r,0;\Phi) + k\overline{N}(r,\infty;\Phi) + S(r,\Phi)$$

Lemma 4.4. [11] *Let Φ be a nonconstant meromorphic function and n be a positive integer. Let $P(\Phi) = a_n\Phi^n + a_{n-1}\Phi^{n-1} + ... + a_1\Phi$ where a_i for $i = 1,2,....,n$ are meromorphic functions such that $T(r,a_i) = S(r,\Phi)$ for $i = 1,2,...,n$ and $a_n \not\equiv 0$. Then*

$$T(r,P(\Phi)) = nT(r,\Phi) + S(r,\Phi).$$

Lemma 4.5. [9] *Let L be an L-function with degree d. Then*

$$T(r,L) = \frac{d}{\pi} r \log r + O(r).$$

Lemma 4.6. *Let L be an L-function. Then $N(r,\infty;L) = S(r,L)$.*

Proof. Clearly L has at most one pole in the complex plane. Hence $N(r,\infty;L) = O(\log r)$. Hence by lemma 4.5 we have $N(r,\infty;L) = S(r,L)$. This completes the proof of the lemma. □

5. PROOF OF THE THEOREM 3.1

Proof. Let $\Phi = \frac{L}{\rho}$ and $\Psi = \frac{(L^m)^{(k)}}{\rho}$.

Clearly $\overline{E}_{4)}(1;\Phi) = \overline{E}_{4)}(1;\Psi)$, $E_{2)}(1;\Phi) = E_{2)}(1;\Psi)$ except possibly for the zeros and poles of $\rho = \rho(z)$, since $\overline{E}_{4)}(\rho;L) = \overline{E}_{4)}(\rho;(L^m)^{(k)})$, $E_{2)}(\rho;L) = E_{2)}(\rho;(L^m)^{(k)})$. Now we have to consider the following two cases.

CASE 1. Let $\Omega \equiv 0$.

Hence

$$\left(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}\right) - \left(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}\right) \equiv 0. \tag{5.1}$$

Integrating from (5.1) we get

$$\frac{1}{\Phi - 1} \equiv \frac{A}{\Psi - 1} + B, \tag{5.2}$$

where A and B are constants and $A \neq 0$.

From (5.2) it is clear that Φ and Ψ share 1 CM. We now claim that $B = 0$.

If possible let $B \neq 0$. Then from (5.2) we get

$$\frac{1}{\Phi - 1} = \frac{B(\Psi - 1 + A/B)}{\Psi - 1}. \tag{5.3}$$

Clearly from (5.3) we have

$$\overline{N}(r,0;\Psi - 1 + A/B) = \overline{N}(r,\infty;\Phi) = S(r,L) \tag{5.4}$$

If $A \neq B$, then by (5.4), lemma 4.2 and lemma 4.6 we have

$$\begin{aligned} T(r,\Psi) &\leq \overline{N}(r,\infty;\Psi)+\overline{N}(r,0;\Psi)+\overline{N}(r,0;\Psi-1+A/B)+S(r,L) \\ &\leq \overline{N}(r,0;\Psi)+S(r,L) \\ &\leq T(r,\Psi)+S(r,L). \qquad (5.5) \end{aligned}$$

Hence by lemma 4.3, lemma 4.6 and (5.6) we have

$$\begin{aligned} T(r,\Psi) &= \overline{N}(r,0;\Psi)+S(r,L) \\ &= \overline{N}(r,0;(L^m)^{(k)})+S(r,L) \\ &= N_1(r,0;(L^m)^{(k)})+S(r,L) \\ &\leq T(r,(L^m)^{(k)})-T(r,L^m)+N_{1+k}(r,0;L^m)+S(r,L). \end{aligned}$$

So $mT(r,L) \leq N_{1+k}(r,0;L^m)+S(r,L)$, which contradicts (3.1).

If $A=B$, then from (5.2) we get $-\frac{\rho^2}{L^m(BL-B\rho-\rho)} \equiv \frac{(L^m)^{(k)}}{L^m}$.
So by (5.2), lemma 4.4 and lemma 4.6 we get

$$\begin{aligned} (m+1)T(r,L) &= T(r,\frac{(L^m)^{(k)}}{L^m})+S(r,L) \\ &\leq N(r,\infty;\frac{(L^m)^{(k)}}{L^m})+S(r,L) \\ &\leq k\overline{N}(r,\infty;L)+mN(r,0;L)+S(r,L) \\ &\leq mT(r,L)+S(r,L), \end{aligned}$$

which is impossible. Hence $B=0$ and so from (5.2) we get

$$\frac{\Psi-1}{\Phi-1} \equiv A. \qquad (5.6)$$

If $A \neq 1$, then from (5.6) we get

$$\overline{N}(r,0;\Psi+A-1)=\overline{N}(r,0;\Phi) \qquad (5.7)$$

Now by lemma 4.2, lemma 4.3, lemma 4.6 and (5.7) we get

$$T(r,\Psi) \leq \overline{N}(r,0;\Psi)+\overline{N}(r,\infty;\Psi)+\overline{N}(r,0;\Psi+A-1)+S(r,\Psi)$$

and so

$$\begin{aligned} T(r,(L^m)^{(k)}) &\leq \overline{N}(r,\infty;L)+\overline{N}(r,0;(L^m)^{(k)})+\overline{N}(r,0;L)+S(r,L) \\ &\leq T(r,(L^m)^{(k)})-T(r,L^m)+N_{k+1}(r,0;L^m)+\overline{N}(r,0;L)+S(r,L) \end{aligned}$$

Hence

$$mT(r,L) \leq 2N_{k+2}(r,0;L^m)+S(r,L),$$

which contradicts (3.1). Therefore $A=1$. Hence from (5.6) we get $L \equiv (L^m)^{(k)}$.

CASE 2. Let $\Omega \not\equiv 0$.

Since $\overline{E}_{4)}(1;\Phi)=\overline{E}_{4)}(1;\Psi)$, $E_{2)}(1;\Phi)=E_{2)}(1;\Psi)$, by lemma 4.1 we get

$$T(r,\Phi)+T(r,\Psi) \leq 2\{N_2(r,0;\Phi)+N_2(r,\infty;\Phi)+N_2(r,0;\Psi)+N_2(r,\infty;\Psi)\}+S(r,\Phi)+S(r,\Psi)$$

Using Lemma 4.3 and lemma 4.6 we have

$$\begin{aligned} T(r,L)+T(r,(L^m)^{(k)}) &\leq 2\{N_2(r,0;L)+N_2(r,\infty;L)+N_2(r,0;(L^m)^{(k)}) \\ &+ N_2(r,\infty;(L^m)^{(k)})\}+S(r,L)+S(r,(L^m)^{(k)}) \\ &\leq 2N_2(r,0;L)+N_2(r,0;(L^m)^{(k)})+N_2(r,0;(L^m)^{(k)})+S(r,L) \\ &\leq 2N_2(r,0;L)+T(r,(L^m)^{(k)})-T(r,L^m)+N_{2+k}(r,0;L^m) \\ &+ N_{2+k}(r,0;L^m)+k\overline{N}(r,\infty;L^m)+S(r,L) \\ &\leq 2N_2(r,0;L)+T(r,(L^m)^{(k)})-mT(r,L) \\ &+ 2N_{2+k}(r,0;L^m)+S(r,L) \\ &\leq T(r,(L^m)^{(k)})-mT(r,L) \\ &+ 4N_{2+k}(r,0;L^m)+S(r,L) \end{aligned}$$

That is

$$(m+1)T(r,L) \leq 4N_{2+k}(r,0;L^m)+S(r,L),$$

which contradicts (3.1).

This completes the proof of the theorem. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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