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TWO DISTANCE FORCING NUMBER OF A GRAPH

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Abstract. Motivated from the graph parameters namely zero forcing number, k-forcing number and the connected k-forcing number, in this article, we introduce a new parameter known as the 2-distance forcing number. Assume that each vertex of a graph $G = (V(G), E(G))$ is colored as either white or black. Consider the set Z_{2d} of black colored vertices of the graph G. The color change rule changes the color of a white vertex v to black if the white vertex v is the only 2-distance white neighbor of a black vertex u. The set Z_{2d} is called a two distance forcing set of G if all vertices of the graph G will be turned black after limited applications of the color change rule. The 2-distance forcing number of G, denoted by $Z_{2d}(G)$, is the minimum of $\mid Z_{2d} \mid$ over all 2- distance forcing sets $Z_{2d} \subseteq V(G)$. This manuscript is intended to study the 2-distance forcing number of some graphs. We find the exact value of the 2-distance forcing number of graphs such as the pineapple graph, gear graph, jelly fish graph, helm graph, sunflower graph, comet graph and the n-pan graph.

Keywords: 2-distance forcing number; graph; diameter; k-forcing number.

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1. Introduction

Throughout this paper, we consider only connected, simple and finite graphs. Assume that we have a simple graph G in which each vertex is colored as either black or white. A subset

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$Z \subseteq V(G)$ of initially colored black vertices of a graph G is called a zero forcing set if we can change entire vertices in the graph as black by iteratively applying the color change rule. At each step, if any black vertex which has a unique white vertex neighbor, then change the color of this white neighbor vertex as black. The zero forcing number of G is the minimum cardinality of a zero forcing set in G. The zero forcing number was defined and studied in detail in [8]. A generalization of the zero forcing number called the k-forcing number of a graph was introduced in [1] and was studied in some details in [9, 5]. The connected k-forcing number was defined in [6]. In this paper, we discuss a generalization of zero forcing set based on the distance in graphs. We use the following definitions for the further development of this manuscript.

- Open neighborhood and closed neighborhood. The set of all vertices adjacent to a vertex v excluding the vertex v is called the open neighborhood of v and is denoted by $N(v)$. The set of all vertices adjacent to a vertex v including the vertex v is called the closed neighborhood of v and is denoted by $N[v]$, that is $N[v] = \{v \cup N(v)\}$.
- The length of a $u-v$ path in a graph G is the number of edges in the path $u-v$. The distance between two vertices u and v in a graph G is the length of the shortest paths between u and v.
- The 2-distance open neighborhood of a vertex u in a graph G is the set of all vertices which are at a distance at most two from u excluding the vertex u and is denoted by $N_{2d}(u)$. The 2- distance closed neighborhood of a vertex u is denoted as $N_{2d}[u]$, and is defined as $N_{2d}[u] = \{u \cup N_{2d}(u)\}$.
- Let $G = (V,E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If a vertex v of G lies at a distance at most two from the vertex u of G, then we say that v is a 2-distance neighbor of the vertex u. For example, consider the graph G depicted in Figure 1. In Figure 1, the vertices p,q,u,v are the 2-distance neighbors of the vertex r. Therefore, the 2-distance closed neighborhood of the vertex r is $N_{2d}[r] = \{r, p, q, u, v\}$. Hence $|N_{2d}[r]| = 5$.
- Color change rule: Let G be a graph in which each vertex is colored either black or white. If a black colored vertex has at most one white colored 2-distance neighbor, then change the color of that white vertex to black. When the color change rule is applied

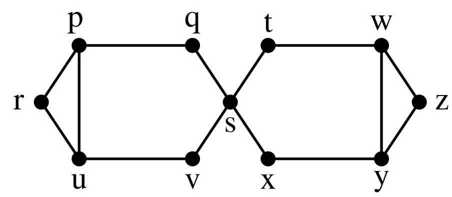

FIGURE 1. The graph G

to an arbitrary vertex v to alter the color of the vertex u to black, then we say that the vertex v forces the vertex u to black and we denote it as $v \to u$ to black.

A set containing minimum number of initially colored black vertices to change the color of the remaining white vertices of G to black by applying the above color change rule is known as the 2-distance forcing set of G, denoted by Z_{2d}, and the minimum cardinality of such a set Z_{2d} is called the 2-distance forcing number of the graph G. The 2-distance forcing number of a graph G is denoted by $Z_{2d}(G)$. In this manuscript, we initiate the study of the 2-distance forcing number of a graph. For more definitions on graphs, we refer the book [4].

2. EXACT VALUES OF $Z_{2d}(G)$

In this section, we consider some simple graphs like path, cycle, wheel graph, friendship graph, star graph, the ladder graph and the complete bipartite graph and find their 2-distance forcing number $Z_{2d}(G)$. We begin with the path graph P_n.

Proposition 1. *Let P_n be the path graph on n vertices, $n \geq 3$. Then the 2-distance forcing number of P_n is 2. That is, $Z_{2d}(P_n) = 2$.*

Proof. Represent the vertices of the path P_n, $n \geq 3$, by $u_1, u_2, \ldots, u_n$. The end vertices u_1 and u_n have the least number of 2-distance vertices. Note that the vertices u_2 and u_3 are the only 2-distance vertices of u_1. Let us start with the vertices u_1 and u_2. Assign black color to the vertices u_1 and u_2. Now the vertex $u_1 \to u_3$ to black. Consider the black vertex u_2. The 2-distance vertices of u_2 are u_1, u_3, u_4 of which u_1, u_3 are already black. So the black vertex $u_2 \to u_4$ to black and so on. Therefore, the set $Z_{2d} = \{u_1, u_2\}$ forms a 2-distance forcing set for the path P_n. The number of elements in Z_{2d} is two. Hence, $Z_{2d}(P_n) = 2$. We wish to say that with only one black vertex, obtaining a 2-distance forcing set for the path P_n ($n \geq 3$) is not possible. □

Next we consider the cycle graph C_n and find its 2-distance forcing number.

Proposition 2. *Let G be the cycle graph C_n, on $n \geq 5$ vertices. Then $Z_{2d}(G) = 4$.*

Proof. Let the vertices of the graph G be denoted by the $u_1, u_2, \ldots, u_n$. Every vertex $u_i, (i = 1, 2, \ldots, n)$ in G has four 2-distance vertices. From the graph, we can easily observe that with any three black vertices we cannot obtain a derived coloring for G. So, $Z_{2d}(G) \geq 4$.

On the other hand, we take four arbitrary adjacent vertices u_{n-1}, u_n, u_1 and u_2. Allow them to have black color. Consider the vertex u_1. The vertices u_{n-1}, u_n, u_2 and u_3 are the 2-distance vertices of the vertex u_1 and out of which the three vertices u_{n-1}, u_n, u_2 are already black. So, the black vertex $u_1 \to u_3$ to black. Again, the vertices u_n, u_1, u_3 and u_4 are at a distance at most two from u_2. Since u_1, u_n and u_3 are black, the black vertex $u_2 \to u_4$ to black. This process continues till we get the derived coloring for G. Hence the set $Z_{2d} = \{u_{n-1}, u_n, u_1, u_2\}$ generates a 2-distance forcing set for G. So, $Z_{2d}(G) \leq 4$. Therefore, it follows that $Z_{2d}(G) = 4$. □

The maximum distance between any pair of vertices in a graph G is called the diameter of G and is denoted by $diam(G)$.

Theorem 3. *Let G be a connected graph of order $n \geq 3$ with $diam(G) = 2$. Then $Z_{2d}(G) = n - 1$.*

Proof. Without loss of generality assume that G is a connected graph of order $n \geq 3$ and $diam(G) = 2$. It can be easily verified that in G if we color exactly one vertex as white and the remaining vertices as black. Then any black vertex will force the white vertex as black. Therefore, $Z_{2d}(G) \leq n - 1$.

To prove the reverse inequality, assume that there exists a two distance forcing set consisting of $n - 2$ black vertices. Let it be the vertices $v_1, v_2, \ldots, v_{n-2}$. Since the diameter of the graph G is 2 all black vertices $v_i, 1 \leq i \leq n - 2$, will have 2 white neighbors which are at a distance at most two from v_i. Therefore, $Z_{2d}(G) \geq n - 1$. □

Corollary 4. *i) If G is the Wheel graph* $W_n, (n \geq 4)$ *vertices, then* $Z_{2d}(W_n) = n-1$.

ii) If G is the Petersen graph, then $Z_{2d}(G) = n-1$.

iii) If G is the star graph $K_{1,n}(n \geq 2)$*, then* $Z_{2d}(K_{1,n}) = n$.

iv) If G is the friendship graph $F_n(n \geq 5)$*, then* $Z_{2d}(F_n) = n-1$.

v) For the complete graph K_n *of order* $n \geq 2$, $Z_{2d}(K_n) = n-1$.

One more class of graphs with $Z_{2d}(G) = n-1$ can be obtained as follows.

Definition (See[10]). The pineapple graph K_m^k, is the graph formed by coalescing any vertex of the complete graph K_m with the star graph $K_{1,k}$ ($m \geq 3, k \geq 2$). The number of vertices in K_m^k is $m+k$, the number edges in K_m^k is $\frac{m^2-m+2k}{2}$ and diameter of K_m^k is 2.

Corollary 5. *Let G be the pineapple graph* K_m^k *.Then* $Z_{2d}(G) = m+k-1$.

Proof. Since the graph G is connected with $n \geq 3$ and diam(G)=2, the result follows as an immediate consequence of Theorem 3. □

The *fan graph*, denoted by F_n, is the graph obtained by the join $K_1 + P_n$, where P_n is the path on n vertices and K_1 is the empty graph. The order of the fan graph G is $n+1$ and the diam(G)=2. Therefore we have the following

Corollary 6. *For a fan graph* F_n*, the 2-distance forcing number is n.*

Proof. Since the diam(F_n)=2 and the graph F_n is connected with $n \geq 3$, the result follows by Theorem 3. □

Theorem 7. *Let G be a disconnected graph of order* $n \geq 3$ *and let* $G_1, G_2, \ldots, G_k$ *be the connected components of G. If* $diam(G_i) = 2$ *for* $1 \leq i \leq k$*, then* $Z_{2d}(G) = n-k$.

Proof. Assume that each connected component of G is of order n_i. Since $diam(G_i) = 2$ for each connected component, therefore, we have from Theorem 3 that $Z_{2d}(G_i) = n_i - 1$, where n_i is the order of the graph G_i. Now $Z_{2d}(G) = (n_1-1)+(n_2-1)+\ldots+(n_k-1) = (n_1+n_2+\ldots+n_k)-k = n-k$. □

3. Graphs for Which $2 < diam(G) < 5$

We start this section by computing the 2-distance forcing number of graphs having diameter 3.

Theorem 8. *If G is the graph obtained by appending one pendant edge to each vertex of the complete graph K_m on $m \geq 3$ vertices, then $Z_{2d}(G) = 2(m-1)$.*

Proof. Let $u_1, u_2, \ldots, u_m$ be the vertices of the complete graph K_m and let $v_1, v_2, \ldots, v_m$ be the vertices attached to the vertices $u_1, u_2, \ldots, u_m$ respectively in G. It suffices to construct a 2-distance forcing set for G consisting of $2(m-1)$ black vertices. With out loss of generality, take the vertex v_1. The vertex v_1 has m 2-distance vertices $u_1, u_2, \ldots, u_m$. Assign black color to the vertex v_1. To start the color change rule from the vertex v_1, we have to color at least $m-1$ vertices out of $u_1, u_2, \ldots, u_m$ to black. Let $u_1, u_2, \ldots, u_{m-1}$ be the black vertices. Then $v_1 \to u_m$ to black. Now consider the black vertex u_1. The vertices $v_2, v_3, \ldots, v_m$ are the 2-distance white vertices of u_1. Let $v_2, v_3, \ldots, v_{m-1}$ be black. Then the black vertex $u_1 \to v_m$ to black. There fore, the set $Z_{2d} = \{v_1, u_1, u_2, \ldots, u_{m-1}, v_2, v_3, \ldots, v_{m-1}\}$ forms a 2-distance forcing set for G. The cardinality of the set Z_{2d} is $1 + m - 1 + m - 2 = 2(m-1)$. Moreover, we can easily observe that a set consisting less than $2(m-1)$ black vertices will never form a 2-distance forcing set for G. Hence, $Z_{2d}(G) = 2(m-1)$. □

Next we find out the 2-distance forcing number of some graphs G with $diam(G) = 4$.

Definition 9. A gear graph or the bipartite wheel graph, denoted by G_g, is the graph derived from the wheel graph W_n by attaching a vertex between every pair of adjacent vertices of the n-cycle. The gear graph G_g contains $2n+1$ vertices and $3n$ edges (See [2]).

Theorem 10. *Let G_g be the gear graph. Then $Z_{2d}(G_g) = n+2$, where n is the number of vertices in the outer cycle of the Wheel graph W_n and $g = 2n+1$.*

Proof. Let $A = \{u, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the vertex set of G_g, where u is the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the n-cycle of the Wheel graph W_n and $v_1, v_2, \ldots, v_n$ are the vertices with $u_i v_i, v_i u_{i+1} \in E(G_g)$, where i is taken modulo n. We generate a 2-distance forcing set as follows.

Consider the vertices $u_1, u_2, \ldots, u_n, u, v_1$ of G_g and color these vertices as black. Now we can easily verify that the vertex $u_2 \to v_2$ to black, $u_3 \to v_3$ to black and so on. Hence $u_n \to v_n$ to black. Therefore the set $Z_{2d} = \{u_1, u_2, \ldots, u_n, u, v_1\}$ forms a 2-distance forcing set for G_g and $| Z_{2d} | = n+2$. Hence

$$Z_{2d}(G_g) \leq n+2 \tag{1}$$

Note that each vertex $v_i, (i = 1, 2, \ldots, n)$ has five 2-distance vertices. Therefore the minimum 2-distance degree of the graph G_g is 5, that is $\delta_{2d}(G_g) = 5$. Therefore $5 \leq Z_{2d}(G_g)$. Now assume that we have a 2-distance forcing set consisting of $n+1$-black vertices. Since $5 \leq Z_{2d}(G_g)$, we can construct a 2 distance forcing set starting with five black vertices. If we start to construct a 2 distance forcing set consisting of 5 black vertices, then two cases arises for applying the color change rule.

Case -1. Assume that the five black vertices are distributed among the outer cycle of the graph G_g and they are connected. One can easily verify that the central vertex u is forced to black. Now to start the further forcing, we have to include at least one vertex from the cycle into the 2-distance forcing set. Then we can force a maximum of one more white vertex to black. This process continues and at each step it can be noted that we have to add at least one vertex from the cycle into the 2-distance forcing set. If $n = 3$, then one can verify that we need at least 5 black vertices to form a 2-distance forcing set. If $n = 4$, then we need at least 6 black vertices to form a 2-distance forcing set. Similarly if $n = 5$, then we need at least 7 black vertices to form a 2-distance forcing set. Therefore we need at least $n+2$ black vertices to form a 2-distance forcing set for G_g. Hence

$$Z_{2d}(G_g) \geq n+2 \tag{2}$$

Case -2. Assume that four black vertices are distributed among the outer cycle of the graph G_g and they are connected and the vertex u is black. One can easily verify that with these five black vertices we can force a maximum of one more vertex to black. Now to start the further forcing we have to include at least one vertex from the cycle into the 2-distance forcing set. Then we can force a maximum of one more white vertex to black. This process continues and at each step it can be noted that we have to add at least one vertex from the cycle into the 2-distance forcing set. If $n = 3$, then one can verify that we need at least 5 black vertices to form a 2-distance forcing set. If $n = 4$, then we need at least 6 black vertices to form a 2-distance forcing set. Similarly if $n = 5$, then we need at least 7 black vertices to form a 2-distance forcing set. Therefore, we need at least $n+2$ black vertices to form a 2-distance forcing set of the graph G_g. Hence

$$Z_{2d}(G_g) \geq n+2 \tag{3}$$

Therefore from (1), (2) and (3), the result follows.

□

Definition (See [2]). Jelly fish graph, denoted by $J(m,n)$, is the graph obtained from a 4-cycle $wxyzw$ by joining the vertex w and the vertex y by an edge and attaching the central vertex of $K_{1,m}$ to x and attaching the central vertex of $K_{1,n}$ to z.

Theorem 11. *Let $J(m,n)$ denotes the Jelly fish graph. Then, $Z_{2d}(G) = m+n+1$.*

Proof. Our aim is to construct a 2-distance forcing set consisting of $m+n+1$ black vertices. For, we proceed as follows.

Color all the m vertices $v_1, v_2, \ldots, v_m$ of $K_{1,m}$ as black. Also we color the vertices w and y to black. Then the vertex $v_i, (i = 1, 2, \ldots, m)$ forces the vertex x to black. Now consider the black vertex x. The vertex x has only one 2-distance white vertex z. So $x \rightarrow z$ to black. Again, the 2-distance white vertices of z are $u_1, u_2, \ldots, u_n$. If we color the vertices $u_1, u_2, \ldots, u_{n-1}$ to black, then $z \rightarrow u_n$ to black. There fore, the set $Z_{2d} = \{v_1, v_2, \ldots, v_m, w, y, u_1, u_2, \ldots, u_{n-1}\}$ forms a 2-distance forcing set for G. The cardinality of the set Z is $m+n+1$. Hence

$$Z_{2d}(G) \leq m+n+1. \tag{4}$$

To establish the reverse inequality, we proceed as follows:

It can be noted that in any 2-distance forcing set of G, it is customary to include $m-1$-vertices from the sub graph $K_{1,m}$ and $n-1$-vertices from $K_{1,n}$. Now we have a set from G consisting of $m+n-2$ black vertices and these vertices are from the sub graphs $K_{1,m}$ and $K_{1,n}$ of G. Now we can choose the vertices x,y,z and w to form a 2-distance forcing set of G. We claim that we need to choose at least 3 more vertices from x,y,z and w, otherwise we arrive a contradiction as follows:

If we choose combinations of two vertices x,y,z and w as black, then we can observe that these vertices will have at least three 2-distance white neighbors. Therefore, it is not possible to form a 2-distance forcing set of G with $m+n-2+2=m+n$ black vertices. Hence

$$Z_{2d}(G) \geq m+n+1 \tag{5}$$

Now from (4) and (5), the result follows.

□

Definition (See [2]). The Helm graph, denoted by H_n, is the graph constructed from the wheel graph W_n by appending a pendant edge to each vertex of the outer cycle.

Theorem 12. *If G is the Helm graph, then $Z_{2d}(G) \leq n+1$.*

Proof. Let $u,u_1,u_2,\ldots,u_n$ be the vertices of the wheel graph, where u is the central vertex. Let $v_1,v_2,\ldots,v_n$ be the pendant vertices of the graph G. We claim that the set $Z_{2d} = \{v_1,v_2,\ldots,v_{n-2},u_1,u_2,u_n\}$ forms a 2-distance forcing set for G. For, we start with the black vertex v_1. Clearly $v_1 \to u$ to black. Then the vertex $v_2 \to u_3$ to black. Again, consider the black vertex v_3. The vertex $v_3 \to u_4$ to black and the process continues. Now consider the black vertex v_{n-2}. We can see that the vertex $v_{n-2} \to u_{n-1}$ to black. Now $u_{n-2} \to v_{n-1}$ to black. Thus with the set Z_{2d}, we can force all the vertices of G to black. The cardinality of Z_{2d} is $n+1$. Therefore, $Z_{2d}(G) \leq n+1$. □

Definition [2]. The sunflower graph, denoted by SF_n, is the graph obtained by taking a wheel graph W_n with central vertex u and the outer n-cycle $u_1, u_2, \ldots, u_n$ and additional vertices $v_1, v_2, \ldots, v_n$, where v_i is joined by edges to u_i,u_{i+1}, where $i+1$ is taken modulo n.

Theorem 13. *Let G be the Sunflower graph SF_n. Then*

$$Z_{2d}(G) \begin{cases} = n+3 \;\; if \;\; 3 \leq n \leq 5 \\ \leq n+2 \;\; if \;\; n > 5 \end{cases} .$$

where n is the number of vertices on the outer cycle of the wheel graph.

Proof. Let $A = \{u\}$, $B = \{u_1, u_2, \ldots, u_n\}$ and $C = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of the Sunflower graph SF_n, where u is the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the outer n-cycle and $v_1, v_2, \ldots, v_n$ are the additional vertices.

Case-1. Assume that $n = 3$. Then G is a graph with diameter 2. We have from Theorem 3, $Z_{2d}(G) =$ the number of vertices in G-1 $= 7 - 1 = 6$.

Case-2. Assume that $n = 4$. Let $A = \{u, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ be the vertex set of the sunflower graph SF_4, where u is the central vertex and v_1, v_2, v_3, v_4 be the additional vertices. Now consider the set $Z_{2d} = \{v_1, v_2, v_4, u_1, u_2, u_3, u_4\}$ as the set of black vertices in SF_4. Clearly, the vertex $v_1 \to u$. Then $v_2 \to v_3$ to black and hence the set Z_{2d} forms a 2-distance forcing set of SF_4. Therefore,

$$Z_{2d}(G) \leq 7 \tag{6}$$

Now suppose that there exists a zero forcing set consisting of 6 black vertices. Then there will be 3 white vertices remains as non colored in G. At least two of these white vertices will be there in the 2 distance neighborhood of each of these black vertices. Since $\mid N_{2d}[u_i] \mid = 9, 1 \leq i \leq 4$, $\mid N_{2d}[u] \mid = 9$ and $\mid N_{2d}[v_i] \mid = 8, 1 \leq i \leq 4$. Therefore further forcing is not possible. Clearly ,

$$Z_{2d}(G) \geq 7 \tag{7}$$

Hence from (6) and (7), the result follows.

Case-3. Suppose that $n = 5$. Consider the set $Z_{2d} = \{v_1, v_2, v_4, v_5, u_1, u_2, u_3, u_5\}$ of black vertices. Then clearly the black vertex $v_1 \to u$ to black. Then the black vertex $v_5 \to u_4$ to black. Consequently the vertex v_3 will be colored black. Thus the set Z_{2d} generates a 2-distance forcing set for G. Cardinality of Z_{2d} is $8 = n+3$. There fore,

$$Z_{2d}(G) \leq 8 = n+3. \tag{8}$$

On the contrary, we assume that there exists a 2-distance forcing set for G consisting of 7 black vertices. Then we have the following cases.

Sub Case 3.1. Assume that the vertex u is black.

Sub Case 3.1.1. Take 5 black vertices from the set B and one black vertex from the set C. In this case, further forcing is not possible since each black vertex has at least two 2-distance white neighbors. A contradiction to our assumption.

Sub case 3.1.2. Consider one black vertex from the set B and five black vertices from the set C. Here also further forcing is not possible, a contradiction.

sub Case 3.1.3. Select four black vertices form the set B and two black vertices from the set C. In this case, we can force a maximum of one more vertex to black, not all. There fore color change rule is not possible, a contradiction.

Sub Case 3.1.4. Select two black vertices from the set B and four black vertices from the set C. In this we cannot form a derived coloring for G, a contradiction to our assumption.

Sub Case 3.1.5. Select three black vertices from both the sets B and C. In this case also we can force one more vertex to black, not all. A contradiction.

Sub Case 3.2. Assume that the vertex u is not black. Then we have the following sub cases.

Sub Case 3.2.1. Select five black vertices from the set B and two black vertices from the set C. Again a contradiction to our assumption, since the forcing is not possible because each black vertex has at least two 2-distance white neighbors.

Sub Case 3.2.2. Consider two black vertices from the set B and five black vertices from the set C. In this case also further forcing is not possible, a contradiction.

Sub Case 3.2.3. Select four black vertices from the set B and three black vertices from the set C. Here also derived coloring is not possible, because in this case we can force only one more vertex to black, not all. Again a contradiction.

Sub Case 3.2.4. Three black vertices from the set B and four black vertices from the set C. One can easily observe that color change rule is not possible in this case. A contradiction to our assumption.

Hence from the above cases, we can conclude that

$$Z_{2d}(G) \geq 8 = n+3. \tag{9}$$

This completes the proof of Case 3.

Case- 4. In this case we assume that $n > 5$. Consider the set

$$Z_{2d} = \{u_1, u_2, \ldots, u_{n-2}, u_n, v_1, v_2, v_n\}$$

of black vertices. Now clearly the black vertex $v_1 \rightarrow u$ to black. Then the vertex $v_2 \rightarrow v_3$ to black, since v_3 is the only 2-distance white vertex of v_2. Again, consider the black vertex v_3. Then $v_3 \rightarrow v_4$ to black, $v_4 \rightarrow v_5$ to black, $v_5 \rightarrow v_6$ to black and so on. Now the black vertex $v_{n-4} \rightarrow v_{n-3}$ to black. The white vertex u_{n-1} can be colored to black by any one of the vertices $u_2, u_3, \ldots, u_{n-4}$. Then the black vertex $v_{n-3} \rightarrow v_{n-2}$ to black, the black vertex $v_{n-2} \rightarrow v_{n-1}$ to black. Thus the set, $Z_{2d} = \{u_1, u_2, \ldots, u_{n-2}, u_n, v_1, v_2, v_n\}$ forms a 2-distance forcing set for G. Cardinality of the set Z_{2d} is $n+2$. So $Z_{2d}(G) \leq n+2$. Hence the proof. □

4. 2-DISTANCE FORCING NUMBER OF SOME GRAPHS WITH LARGE DIAMETER

In this section, we compute the 2-distance forcing number of some special graphs with large diameter. We start with the comet graph considered in [2].

Definition (See [2]). A comet graph is the graph obtained by appending m pendant edges to one end of the path graph $P_n (n \geq 3)$. The order of a comet graph is $m+n$ and the diameter is n, where n is the number of vertices of the path P_n.

Theorem 14. *Let G be the comet graph. Then $Z_{2d}(G) = m+1$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path graph P_n and $v_1, v_2, \ldots v_m$ be the m vertices appended to the vertex u_n of the path P_n. By coloring the vertices u_1 and u_2 to black, we can color the remaining white vertices of the path P_n to black. Since all the m pendant vertices are the 2-distance vertices of u_n, by coloring $m-1$ of these vertices, say $v_1, v_2, \ldots, v_{m-1}$, to black we can obtain a derived coloring for G. Hence with the set $Z_{2d} = \{u_1, u_2, v_1, v_2, \ldots, v_{m-1}\}$ of black vertices, we can generate a 2-distance forcing set for G and the cardinality of the set Z_{2d} is $m+1$. Therefore,

$$Z_{2d}(G) \leq m+1. \tag{10}$$

To establish the reverse inequality, we claim that with m black vertices we cannot generate a 2-distance forcing set for G. For, we consider the following cases.

Case 1. Start the coloring process with any one of the $v_i, 1 \leq i \leq m$. With out loss of generality, color the vertex v_1 to black. Since the vertex v_1 has $m-1+2 = m+1$ 2-distance white neighbors, to apply color change rule we have to assign at least m of these 2-distance non-colored vertices to black. So we must have at least $m+1$ black vertices required to start forcing from v_1. Therefore, we cannot form a 2-distance forcing set for G with m black vertices if we start from any one of the vertices $v_i, 1 \leq i \leq m$.

Case 2. Consider the vertices $u_1, u_2 \ldots, u_{n-2}$, where $N_{2d}(u_i) = 4, i = 3, 4, \ldots, n-2$. Without loss of generality, consider the vertex u_3. Color the vertex u_3 to black. To proceed further we have to color, say, u_1, u_2 and u_4 to black. Then the black vertex $u_3 \to u_5$ to black. Now the black vertex $u_4 \to u_6$ to black, $u_5 \to u_7, \ldots, u_{n-3} \to u_{n-1}, u_{n-2} \to u_n$ to black. Again, to start the color change rule from the vertex u_n, we have to color at least $m-1$ pendant vertices, say, $v_1, v_2, \ldots, v_{m-1}$ to black. Thus we have a set $Z_{2d} = \{u_1, u_2, u_3, u_4, v_1, v_2, \ldots, v_{m-1}\}$ of black vertices. Cardinality of the set Z_{2d} is $m+3$.

Case 3. It can be noted that color change rule is not possible with m black vertices if we start the process either from u_{n-1} or from u_n, Since $N_{2d}(u_{n-1}) = m+3$ and $N_{2d}(u_n) = m+2$.

From the above three cases, we can conclude that with m black vertices, we cannot form a 2-distance forcing set for G. Hence

$$Z_{2d}(G) \geq m+1. \tag{11}$$

This completes the proof. □

Definition(See [2]). The n-pan graph is the graph formed by joining the cycle graph C_n to singleton graph K_1 by a bridge. The order of the n-pan graph is $n+1$ and the diameter is $\lfloor \frac{n}{2} \rfloor + 1$.

Theorem 15. *If G is the n-pan graph, then $Z_{2d}(G) = 4$.*

Proof. Represent the vertices of the cycle graph C_n in G by $u_1, u_2, \ldots, u_n$, $n \geq 6$ and the singleton graph K_1 by u. Let u be joined to the vertex u_1 of C_n in G. Color any four adjacent vertices of C_n to black, except the vertex u_1. Clearly these four vertices give a derived coloring for G. Consequently the singleton graph K_1(that is the vertex u) will be colored black. Also we note that three black vertices never give a derived coloring for G. There fore, $Z_{2d}(G) = 4$. □

5. CONCLUSION

In this article, we introduced the notion of 2-distance forcing number of a graph. In Section 2, we found exact values of $Z_{2d}(G)$ for paths, cycles, wheel graphs, Petersen graph, star graph, friendship graph, pineapple graph and the fan graph. Also we proved that if G is a connected graph of order $n \geq 3$ and if $diam(G) = 2$, then $Z_{2d}(G) = n-1$.

In Section 3, we focused on some classes of graphs with diameter lies between 2 and 5 and their 2-distance forcing number is determined. Finding an exact formula for the 2-distance forcing number of Helm graph and the sunflower graph is open.

Finally, in section 4, we found the 2-distance forcing number of some graphs with large diameter such as the comet graph and the n-pan graph.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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