CENTRAL FINITE-DIFFERENCE OF NUMERICAL SOLUTION FOR THREE-DIMENSIONAL ATMOSPHERIC TRANSPORT EQUATION

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Abstract. The energy transport equation is fundamental for the meteorological analysis; in this work, we analyze this equation in three dimensions using the methods of central finite differences; the analysis of convergence, consistency, and stability of the scheme shows a strong dependence of space and temporal variables. In conclusion, with the central finite differences was possible to predict the three-dimensional dynamics of the temperature.

Keywords: finite differences; atmospheric; fluid dynamics.

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1. INTRODUCTION

The energy transport equation is fundamental in applications as meteorology, aerodynamics, oceanography, hydrology, and engineering [1].

Taking as the object of study the tree-dimensional transport equation for meteorology, we made the numerical interpretation of atmospheric dynamics. The energy transport equation has peculiar characteristics that are difficult to find the solution for different methods, its a consequence

of advective term $v\dfrac{\partial \phi}{\partial x}$, because it explains the inertia of the model [7, 10].

Therefore, we use the finite difference method by the simplicity in the numerical implementation for small and large scale in the domain. The advective term of the equation will be discretized using central finite differences scheme [4, 5].

The model to describe the three-dimensional energy atmospheric transport is:

$$(1) \qquad \frac{\partial \theta}{\partial t} = -\overrightarrow{V}.\nabla \theta + \alpha \Delta \theta \ ,$$

where $\theta = \theta(x,y,z,t)$ is the temperature variable , $\mathbf{V} = (v_1, v_2, v_3)$ is the velocity field and α_i with $i = \overline{1,3}$ represent the thermal diffusion coefficient, those in that work will be constants[4]. The domain is a rectangular box $\Omega = [0,L_1] \times [0,L_2] \times [0,L_3]$ and time $t > 0$; the initial conditions is known $\theta(x,y,z,0) = f_0$, and the boundary condition $\theta(\xi,t)\,|_{\partial\Omega} = \mathscr{G}$ for $\xi \in \Omega$ and $t > 0$. The boundary condition $\partial\Omega$ are Dirichlet, in each face of the domain, the temperature $\mathscr{G}$ take values represented by f_i with $i = \overline{1,6}$.

The work is structured in different section as: section 2 we show the finite difference method for the model, section 3 we made the analysis of convergence, consistence and stability, section 4 we present the result and, some numerical experiment, section 5 and 6 we present the discussion and conclusion, and the references.

2. FINITE DIFFERENCE METHOD

The derivative of variables in the equation are replaced by finite differences for the dependent and independent variables, that process is called discretization and getting an algebraic system equation [8, 6].

The discretization of the model, consists of the discretization of the domain, variables, and the equation.

2.1. Discretization of the Domain.
The numerical solution of the partial differential equation of the energy, is necessary to have abounded domain, for this research the domain is restricted a rectangular box, expressed by $\Omega = [0,L_1] \times [0,L_2] \times [0,L_3]$, with $L_i > 0$, $\forall i = \overline{1,3}$, and Dirichlet boundary conditions $x = L_1$, $y = L_2$, $z = L_3$.

The domain in divided by number of finite number of rectangular sub-domains with dimension

given by $h_x > 0$, $h_y > 0$, $h_z > 0$, and $h_t > 0$ for the time $t > 0$.

We chose the nodes in the border of each sub-domain, and their coordinates are:

$$x_i = ih_x \; ; \; i = 0, 1, \cdots, m_1,$$

$$x_j = jh_y \; ; \; j = 0, 1, \cdots, m_2,$$

(2)

$$x_k = kh_z \; ; \; k = 0, 1, \cdots, m_3,$$

$$x_t = th_t \; ; \; t = 0, 1, \cdots, m_t,$$

the number of nodes are represented by : $m_1 + 1$, $m_2 + 1$, $m_3 + 1$ y $m_t + 1$, respectively.

The sub-domains in some cases have different longitude, and the mesh is defined by:

Definition 2.1. Given h_x, h_y, h_z, h_t positive numbers, a mesh is a set of points of the form $(x_i, y_j, z_k, t_n) = (ih_x, jh_y, kh_z, nh_t)$, called nodes, with i,j,k,n non negative integer numbers.

The solution is given at the nodes $\xi_{i,j,k} = (ih_x, jh_y, kh_z) \in \mathbb{R}^3$ of the discrete domain.

2.2. Discretization of variables. The variables θ of the problem are discretized using the definition 2.1, for each node (ih_x, jh_y, kh_z, nh_t) a value $\theta(ih_x, jh_y, kh_z, nh_t)$ is designed and represented by

(3) $$\theta_{i,j,k}^n = \theta(ih_x, jh_y, kh_z, nh_t).$$

Definition 2.2. The discrete function ϕ is defined over a mesh and each point (x_i, y_j, z_k, t_n) have a real number $\phi_{i,j,k}^n$.

The smooth function ϕ over $\Omega \times \mathbb{R}^+$ is discretized on the mesh defined at 2.1. Taking $\phi_{i,j,k}^n := \phi(x_i, y_j, z_k, t_n)$ in particular the solution θ of the problem (PVIC) given in (3) is discretized by $\theta_{i,j,k}^n = \theta(x_i, y_j, z_k, t_n)$.

As the solution θ is unknown, so the discrete solution $\theta_{i,j,k}^n$ is approximated by a discrete function $\phi_{i,j,k}^n$ such that $\theta_{i,j,k}^n \approx \phi_{i,j,k}^n$ in each node of the mesh.

The $\{\phi_{i,j,k}^n\}$ and $\{\phi_{i,j,k}^{n+1}\}$ denote a discrete function at level n and $n+1$ respectively, where :

(4) $$\phi_{i,j,k}^n \approx \theta(\xi_{i,j,k}, nh_t) \text{ and } \phi_{i,j,k}^{n+1} \approx \theta(\xi_{i,j,k}, nh_t + h_t),$$

also, the discretized initial condition is

$$\phi_{i,j,k}^0 = f_0(ih_x, jh_y, kh_z) \text{ with } i,j,k \in \mathbb{Z}^+,$$

and the discretized border conditions on the mesh $i,j,k \in \mathbb{Z}^+$ are designed by

(5)
$$\phi_{0,j,k}^n = f_1(jh_y, kh_z), \ \phi_{i,j,k}^n = f_2(jh_y, kh_z),$$
$$\phi_{i,0,k}^n = f_3(ih_x, kh_z), \ \phi_{i,j,k}^n = f_4(ih_x, kh_z),$$
$$\phi_{i,j,0}^n = f_5(ih_x, jh_y), \ \phi_{i,j,k}^n = f_6(ih_x, jh_y).$$

2.3. Discretization of the equation.

The approximation of the terms advection and diffusion of the equation (1) using Taylor's series truncated after the first and second term, we get the finite differences for the first ans second derivative at point $(i,j,k) \in \mathbb{Z}^+$, of the form

(6)
$$(\frac{\partial \phi}{\partial x})_{i,j,k} \approx \frac{\phi_{i+1,j,k} - \phi_{i-1,j,k}}{2h_x} \ ; \ (\frac{\partial^2 \phi}{\partial x^2})_{i,j,k} \approx \frac{\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k}}{h_x^2}$$
$$(\frac{\partial \phi}{\partial y})_{i,j,k} \approx \frac{\phi_{i,j+1,k} - \phi_{i,j-1,k}}{2h_y} \ ; \ (\frac{\partial^2 \phi}{\partial y^2})_{i,j,k} \approx \frac{\phi_{i,j+1,k} - 2\phi_{i,j,k} + \phi_{i,j-1,k}}{h_y^2}$$
$$(\frac{\partial \phi}{\partial z})_{i,j,k} \approx \frac{\phi_{i,j,k+1} - \phi_{i,j,k-1}}{2h_z} \ ; \ (\frac{\partial^2 \phi}{\partial z^2})_{i,j,k} \approx \frac{\phi_{i,j,k+1} - 2\phi_{i,j,k} + \phi_{i,j,k-1}}{h_z^2}.$$

The discretized equation with central differences given in (6) for $v_i > 0, i = \overline{1,3}$ is

(7)
$$\frac{\phi_{i,j,k}^{t+1} - \phi_{i,j,k}^t}{h_t} = -v_1 \frac{\phi_{i+1,j,k}^t - \phi_{i-1,j,k}^t}{2h_x} - v_2 \frac{\phi_{i,j-1,k}^t - \phi_{i,j-1,k}^t}{2h_y}$$
$$-v_3 \frac{\phi_{i,j,k+1}^t - \phi_{i,j,k-1}^t}{2h_z} + \alpha_1 \frac{\phi_{i+1,j,k}^t - 2\phi_{i,j,k}^t + \phi_{i-1,j,k}^t}{(h_x)^2}$$
$$+\alpha_2 \frac{\phi_{i,j+1,k}^t - 2\phi_{i,j,k}^t + \phi_{i,j-1,k}^t}{(h_y)^2} + \alpha_3 \frac{\phi_{i,j,k+1}^t - 2\phi_{i,j,k}^t + \phi_{i,j,k-1}^t}{(h_z)^2}.$$

To simplified the expression, we defined the discrete operators $S_{\ell\pm}$ for $\ell = 1,2,3$ as forward and backward displacement at $\ell = 1$ for x, $\ell = 2$ for y, and $\ell = 3$ for z. For example, at x direction the forward operator is S_{1+} and backward S_{1-}, those applied to discrete function $\phi_{i,j,k}^t$, we have:
$S_{1\pm}\phi_{i,j,k}^n := \phi_{i\pm1,j,k}^n$ such that $S_{1\pm}\phi = \{\phi_{i\pm1,j,k}^n\}$.
Given P differential and continuous operator and central finite differential scheme (7), we get

the discrete operator as:

$$(8) \qquad\qquad P_{h_x, h_y, h_k, h_t} \phi^n_{i,j,k} = 0$$

3. ANALYSIS OF CENTRAL FINITE DIFFERENCES OF THE EQUATION

3.1. Consistency.

Definition 3.1. $P_{h_x, h_y, h_k, h_t} \phi^n_{i,j,k}$ is consistent with $P\theta$ for a time $n > 0$ in ℓ^2-norm, if $\|P_{h_x, h_y, h_k, h_t} \phi^n_{i,j,k}\|_{\ell^2} = \|\kappa\tau(h_x, h_y, h_k, h_t)\|_{\ell^2}$; and $\tau(h_x, h_y, h_k, h_t) \to 0$ when $\kappa \to 0$. where $\tau(h_x, h_y, h_k, h_t)$ is a local truncated error at the time n_κ.

We proof the consistency of the discretized equation (6), for this, we denote A, B and P differential and continuous operators as Strikwerda [11] , defined by: $A = \dfrac{\partial}{\partial t} + u_1 \dfrac{\partial}{\partial x} + u_2 \dfrac{\partial}{\partial y} + u_2 \dfrac{\partial}{\partial z}$ y $B = -\alpha_1 \dfrac{\partial^2}{\partial x} - \alpha_2 \dfrac{\partial^2}{\partial y} - \alpha_2 \dfrac{\partial^2}{\partial z}$, applying those operators to a smooth function $\theta(\xi, t)$ for $\xi \in \mathbb{R}^3$, we have

$$(9) \qquad A\theta = \frac{\partial \theta}{\partial t} + u_1 \frac{\partial \theta}{\partial x} + u_2 \frac{\partial \theta}{\partial y} + u_3 \frac{\partial \theta}{\partial z}, B\theta = -\alpha_1 \frac{\partial^2 \theta}{\partial x^2} - \alpha_2 \frac{\partial^2 \theta}{\partial y^2} - \alpha_3 \frac{\partial^2 \theta}{\partial z^2},$$

and the equation (6) is written as

$$(10) \qquad\qquad P := A + B, \text{ and } P\theta = A\theta + B\theta = 0.$$

Then, with the progressive Taylor's formula at the time, we have:

$$(11)$$
$$\theta(\xi_{i,j,k}, t_n + h_t) = \theta(\xi_{i,j,k}, t_n) + \frac{1}{1!} h_t \theta_t(\xi_{i,j,k}, t_n) + \frac{1}{2!} h_t^2 \theta_{tt}(\xi_{i,j,k}, t_n) + \frac{1}{3!} h_t^3 \theta_{ttt}(\xi_{i,j,k}, t_n) + \cdots,$$

and replacing the notation (3) in the Taylor's expansion (11), we have

$$(12) \qquad\qquad \theta^{n+1}_{i,j,k} = \theta^n_{i,j,k} + \theta_t h_t + O(h_t^2).$$

The Taylor's formula with spatial variable is:

$$\theta(x_i \pm h_x, y_j, z_k, t_n) = \theta(\xi_{i,j,k}, t_n) \pm \frac{1}{1!} h_x \theta_x(\xi_{i,j,k}, t_n)$$

$$(13) \qquad +\frac{1}{2!}h_x^2 \theta_{xx}(\xi_{i,j,k}, t_n) \pm \frac{1}{3!}h_x^3 \theta_{xxx}(\xi_{i,j,k}, t_n) + \cdots,$$

using the reduced notation $\theta_{i\pm1,j,k}^n = \theta(x_i \pm h_x, y_j, z_k, t_n)$ in the Taylor's expansion (13), we have

$$(14) \qquad \theta_{i\pm1,j,k}^n = \theta_{i,j,k}^n \pm \theta_x h_x + O(h_x^2).$$

Using (12), (14) y (9) in the equation (6), the discrete operator (8) is

$$P_{h_x,h_y,h_k}\phi_{i,j,k}^n = \underbrace{\theta_t + u_1\theta_x + u_2\theta_y + u_3\theta_z}_{} \underbrace{- \alpha_1\theta_{xx} - \alpha_2\theta_{yy} - \alpha_3\theta_{zz}}_{}$$
$$+ o(h_x) + o(h_y) + o(h_z) + o(h_t) + o(h_x^2) + o(h_y^2) + o(h_z^2)$$

$$(15) \qquad P_{h_x,h_y,h_k,h_t}\phi_{i,j,k}^n = A\theta + B\theta + o(h_x) + o(h_y) + o(h_z) + o(h_t) + o(h_x^2) + o(h_y^2) + o(h_z^2),$$

of the equation (10) and (15), we have

$$(16) \qquad P\theta - P_{h_x,h_y,h_k,h_t}\phi_{i,j,k}^n = o(h_x) + o(h_y) + o(h_z) + o(h_t) + o(h_x^2) + o(h_y^2) + o(h_z^2).$$

This form, the scheme (6) is consistent with (10), on the right hand side of (16), we have that the truncate local error goes to zero, when $h_x, h_y, h_k, h_t \mapsto 0$ goes to zero, that is :

$$P\theta - P_{h_x,h_y,h_k,h_t}\phi_{i,j,k}^n \mapsto 0.$$

Therefore, we have contrast the definition 3.1.

3.2. Stability. To proof the stability of the scheme (6), the **Von Neumann** criteria should be satisfied, for that, we have the follow definitions

Definition 3.2. The central finite differences scheme (6) is **stable** with respect to a some norm $\|.\|$ if and only if the solution exist and is unique and it is dependent of initial conditions, that is, there are positives constants h_{x_0}, h_{y_0}, h_{z_0}, h_{t_0} and $C > 0, t > 0, \alpha > 0$ with $n \geq 0$, such that

$$\|\phi_{i,j,k}^{n+1}\| \leq Ce^{\alpha t}\|\phi_{i,j,k}^0\|,$$

for $0 \leq t = (n+1)h_t, 0 < h_x \leq h_{x_0}, 0 < h_y \leq h_{y_0}, 0 < h_z \leq h_{z_0}$ y $0 < h_t \leq h_{t_0}$.

Definition 3.3. Given $\phi(\xi, n)$ a discrete function defined on $\mathbb{Z}$, then the discrete Fourier transform $\phi(\xi, n)$ with $n \in \mathbb{Z}^+$, is denoted by $\widehat{\phi}(\xi, n)$ and defined by:

$$\widehat{\phi}(\xi, n) = \frac{1}{(2\pi)^{N/2}} \sum_{m \in \mathbb{Z}^N} e^{-\hat{\imath}mh.\xi} \phi_m^n h^N,$$

where $h\mathbb{Z}^N = \{hm : m \in \mathbb{Z}^N\}$ with $mh.\xi$ a inner product defined for $\xi \in [\frac{-\pi}{h}, \frac{\pi}{h}]^N$, and the inverse formula is

$$\phi_m^n = \frac{1}{(2\pi)^{N/2}} \int_{[\frac{-\pi}{h}, \frac{\pi}{h}]^N} e^{\hat{\imath}mh.\xi} \widehat{\phi}(\xi, n) d\xi.$$

We proceed to find the **amplification factor** that is necessary for the Von Neumann criteria, for that, we use the results of Strikwerda [11], and Rubio [5], there exist a biunivocal relation between discrete space $\ell^2(\mathbb{Z})$ and the space $L^2[-\frac{\pi}{h}, \frac{\pi}{h}]$ for $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$, it warranty that

$$(17) \qquad \|\phi_i^n\|_{\ell^2}^2 = h \sum_{m \in \mathbb{Z}} |\phi_m^n|^2 = \int_{[\frac{-\pi}{h}, \frac{\pi}{h}]} |\widehat{\phi}^n(\xi)|^2 d\xi = \|\phi\|_{L^2}^2,$$

this relation is called Parserval's relation [11],[12].

With the notation (4) and definition (3.3), taking $h_1 = h_x$, $h_2 = h_y$, $h_3 = h_z$, $h_\ell \mathbb{Z}^N = \{h_\ell m : m \in \mathbb{Z}^N\}$; we have that $mh_\ell \cdot \xi$ is the inner product for $\xi \in [\frac{-\pi}{h}, \frac{\pi}{h}]^N$ with $\ell = 1, 2, 3$. As $\phi = \{\phi_{i,j,k}^n\}$ is the discrete function, the discrete Furier Transform is $\phi(\xi, n)$ and expressed by

$$(18) \qquad \widehat{\phi}(\xi, n) = \frac{h_1 h_2 h_3}{(2\pi)^{3/2}} \sum_{i,j,k} \phi_{i,j,k}^n e^{-\hat{\imath}ih_1\xi_1} e^{-\hat{\imath}jh_2\xi_2} e^{-\hat{\imath}kh_3\xi_3},$$

where $\hat{\imath} \in \mathbb{C}$; $i, j, k \in \mathbb{Z}$ y $n \in \mathbb{Z}^+$.

Considering $\beta_1 = h_1\xi_1$, $\beta_2 = h_2\xi_2$, $\beta_3 = h_3\xi_3$, the discrete Fourier Transform of (18) is expressed as :

$$(19) \qquad \widehat{\phi}(\beta_1, \beta_2, \beta_3, n) = \frac{h_1 h_2 h_3}{(2\pi)^{3/2}} \sum_{i,j,k} \phi_{i,j,k}^n e^{-\hat{\imath}i\beta_1} e^{-\hat{\imath}j\beta_2} e^{-\hat{\imath}k\beta_3},$$

where $\hat{\imath} \in \mathbb{C}$; $i, j, k \in \mathbb{Z}$ y $n \in \mathbb{Z}^+$.

Applying the Discrete Fourier transform (19) to (6) we get

$$(20) \qquad \widehat{S_{\ell\pm}\phi}(\beta_1,\beta_2,\beta_3,n) = e^{\pm\hat{\imath}\beta_\ell}\,\widehat{\phi}(\beta_1,\beta_2,\beta_3,n),$$

where $\beta_\ell = h_\ell\xi$ for $\ell = 1,2,3$. Now, of the equation (19) and (20), we define

$$(21) \qquad \widehat{S_{\ell\pm}} := e^{\pm\hat{\imath}\beta_\ell} \text{ for } \ell = 1,2,3.$$

We write a equation (7) as:

$$\phi_{i,j,k}^{t+1} = \phi_{i,j,k}^t - \frac{h_t v_1}{2h_x}(\phi_{i+1,j,k}^t - \phi_{i-1,j,k}^t) - \frac{h_t v_2}{2h_y}(\phi_{i,j+1,k}^t - \phi_{i,j-1,k}^t)$$

$$- \frac{h_t v_3}{2h_z}(\phi_{i,j,k+1}^t - \phi_{i,j,k-1}^t) + \frac{h_t\alpha_1}{(h_x)^2}(\phi_{i+1,j,k}^t - 2\phi_{i,j,k}^t + \phi_{i-1,j,k}^t)$$

$$(22) \qquad + \frac{h_t\alpha_2}{(h_y)^2}(\phi_{i,j+1,k}^t - 2\phi_{i,j,k}^t + \phi_{i,j-1,k}^t) + \frac{h_t\alpha_3}{(h_z)^2}(\phi_{i,j,k+1}^t - 2\phi_{i,j,k}^t + \phi_{i,j,k-1}^t)$$

using the notation :

$$(23) \qquad \lambda_1 = \frac{h_t v_1}{h_x}, \; \lambda_2 = \frac{h_t v_2}{h_y}, \; \lambda_3 = \frac{h_t v_3}{h_z},$$

$$(24) \qquad \mu_1 = \frac{h_t\alpha_1}{(h_x)^2}, \; \mu_2 = \frac{h_t\alpha_2}{(h_y)^2}, \mu_3 = \frac{h_t\alpha_3}{(h_z)^2},$$

where $\lambda_1,\lambda_2,\lambda_3$ are the Courant's numbers.

With the notation (23) and (24), the equation (22) is written as

(25)

$$\phi_{i,j,k}^{t+1} = (1 - 2\mu_1 - 2\mu_2 - 2\mu_3)\phi_{i,j,k}^t + (\frac{-\lambda_1}{2} + \mu_1)S_{1+}\phi_{i,j,k}^t + (\frac{\lambda_1}{2} + \mu_1)S_{1-}\phi_{i,j,k}^t$$

$$+ (\frac{-\lambda_2}{2} + \mu_2)S_{2+}\phi_{i,j,k}^t + (\frac{\lambda_2}{2} + \mu_2)S_{2-}\phi_{i,j,k}^t + (\frac{-\lambda_3}{2} + \mu_3)S_{3+}\phi_{i,j,k}^t + (\frac{\lambda_3}{2} + \mu_3)S_{3-}\phi_{i,j,k}^t.$$

Finally, the equation (25) is expressed as

$$(26) \qquad \phi_{i,j,k}^{t+1} = Q\phi_{i,j,k}^t$$

where $Q = Q(S_{1+},S_{1-},S_{2+},S_{2-},S_{3+},S_{3-})$ is a polynomial of the form

$$(27) \quad \begin{aligned} Q &= (1 - 2\mu_1 - 2\mu_2 - 2\mu_3) + (\frac{-\lambda_1}{2} + \mu_1)S_{1+} + (\frac{\lambda_1}{2} + \mu_1)S_{1-} \\ &+ (\frac{-\lambda_2}{2} + \mu_2)S_{2+} + (\frac{\lambda_1}{2} + \mu_2)S_{2-} + (\frac{-\lambda_3}{2} + \mu_3)S_{3+} + (\frac{\lambda_3}{2} + \mu_3)S_{3-}. \end{aligned}$$

Replacing (21), (23) , (24) in the polynomial (27) and applying the inversion Fourier formula given in definition (3.3) [11], the explicit scheme (26) is expressed by

$$(28) \qquad \widehat{\phi}_{i,j,k}^{n+1} = \rho \widehat{\phi}_{i,j,k}^{n} \text{ where } \widehat{Q} = \rho(\beta_1, \beta_2, \beta_3, h_x, h_y, h_z, h_t),$$

where the spectral radio $\rho(\beta_1, \beta_2, \beta_3, h_x, h_y, h_z, h_t) = \widehat{Q}(e^{\pm \hat{i}h_1\xi_1}, e^{\pm \hat{i}h_2\xi_2}, e^{\pm \hat{i}h_3\xi_3})$ is written of the form

$$(29) \quad \rho = 1 - 4\mu_1 sen^2\frac{\beta_1}{2} - 4\mu_2 sen^2\frac{\beta_2}{2} - 4\mu_3 sen^2\frac{\beta_3}{2} - \hat{i}(\lambda_1 sen\beta_1 + \lambda_2 sen\beta_2 + \lambda_3 sen\beta_3).$$

This equation is called **amplification factor** and shows the amplitude of the general solution for central finite differences scheme.

Von Neumman criteria. Using Fourier analysis, we have the necessary and sufficient conditions for the stability of the finite differences scheme, this is called Von Neumann analysis. For this analysis we have the following theorems

Remark. The central finite differences (6) satisfy the Von Neumann criteria as in Gary [13] , if exist a constant $C > 0$ independent of $h_x, h_y, h_z, h_t, \beta_1, \beta_2, \beta_3, \kappa$, such that

$$(30) \qquad |\rho(\beta_1, \beta_2, \beta_3, h_x, h_y, h_z, h_t)| \le 1 + C\kappa.$$

Where $\kappa > 0$ is the step of the time and $\rho(\beta_1, \beta_2, \beta_3, h_x, h_y, h_z, h_t)$ denote a spectral radio of amplification factor (29).

If $\rho(\beta_1, \beta_2, \beta_3, h_x, h_y, h_z, h_t)$ is independent of h_x, h_y, h_z, h_t, the stability condition (30) is replaced by a stability condition of the form

$$(31) \qquad |\rho(\beta_1, \beta_2, \beta_3)| \le 1.$$

In the next theorem, we use the Von Neumann criteria above to proof the stability of the central finite differences (6).

Theorem 3.1. *The central finite differences (6) is stable with the norm ℓ^2, if and only if, it satisfy the criteria (3.2) of **Von Neumann**.*

The theorem (3.1) shows that only is necessary the amplification factor $\rho(\beta_1, \beta_2, \beta_3, h_x, h_y, h_z, h_t)$ to determine the stability of central finite differences scheme (6).

Proof

$\Leftarrow$) If the **Von Neumann** criteria (3.2) is satisfied, the central finite differences scheme (6) is stable with the norm ℓ^2.

Applying Fourier transform to the explicit scheme (6), we get

$$(32) \qquad \widehat{\phi}_{i,j,k}^{n+1} = \rho \, \widehat{\phi}_{i,j,k}^{n},$$

simplifying this expression, we write as

$$(33) \qquad \widehat{\phi}_{i,j,k}^{n} = \rho \, \widehat{\phi}_{i,j,k}^{n-1} = \rho^2 \widehat{\phi}_{i,j,k}^{n-2} = \ldots = \rho^n \widehat{\phi}_{i,j,k}^{0}.$$

Using the Parseval's relation (17) of greater dimension, we have

$$(34) \quad \|\phi_{i,j,k}^{n}\|_{\ell^2}^2 = h_1 h_2 h_3 \sum_{i,j,k} (\phi_{i,j,k}^{n})^2 = \int_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \int_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int_{-\frac{\pi}{h_3}}^{\frac{\pi}{h_3}} |\widehat{\phi}^{n}(\beta_1, \beta_2, \beta_3)|^2 d\beta_1 d\beta_2 d\beta_3,$$

in effect

$$(35) \qquad \|\phi_{i,j,k}^{n}\|_{\ell^2}^2 = \int_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \int_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int_{-\frac{\pi}{h_3}}^{\frac{\pi}{h_3}} |\widehat{\phi}^{n}(\beta_1, \beta_2, \beta_3)|^2 d\beta_1 d\beta_2 d\beta_3$$

replacing (33) in the Parselval's (35), we get

$$
(36) \qquad \|\phi_{i,j,k}^n\|_{\ell^2}^2 = \int_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \int_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int_{-\frac{\pi}{h_3}}^{\frac{\pi}{h_3}} |\rho(\beta_1,\beta_2,\beta_3,h_x,h_y,h_z,h_t)|^{2n} |\widehat{\phi}^0(\beta_1,\beta_2,\beta_3)|^2 d\beta_1 d\beta_2 d\beta_3.
$$

Applying the criteria (3.2), such that $|\rho(\beta_1,\beta_2,\beta_3,h_x,h_y,h_z,h_t)| \le 1+C\kappa$ con $C>0$ y $\kappa>0$, the equation (36) is wrote as

$$
\|\phi_{i,j,k}^n\|_{\ell^2}^2 \le (1+C\kappa)^{2n} \int_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \int_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int_{-\frac{\pi}{h_3}}^{\frac{\pi}{h_3}} |\widehat{\phi}^0(\beta_1,\beta_2,\beta_3)|^2 d\beta_1 d\beta_2 d\beta_3,
$$

whit the result 34, we have

$$
\|\phi_{i,j,k}^n\|_{\ell^2}^2 \le (1+C\kappa)^{2n} \sum_{i,j,k} (\phi_{i,j,k}^0)^2 h_1 h_2 h_3,
$$

therefore, we have that

$$
(37) \qquad \|\phi_{i,j,k}^n\|_{\ell^2}^2 \le (1+C\kappa)^{2n} \|\phi_{i,j,k}^0\|_{\ell^2}^2.
$$

Given $T>0$ sufficient greater such that $n\kappa \le T$, then $n \le \dfrac{T}{\kappa}$, and $1+C\kappa \le e^{C\kappa}$ with $C>0$ and $\kappa > 0$, we have $(1+C\kappa)^{2n} \le (1+C\kappa)^{2\frac{T}{\kappa}} \le e^{2CT}$, and replacing in the inequality (37), we have

$$
(38) \qquad \|\phi_{i,j,k}^n\|_{\ell^2} \le e^{CT} \|\phi_{i,j,k}^0\|_{\ell^2}.
$$

Of the (37) and (38) the central finite differences (6) is stable .

This result contrast the definition 3.2 in the discrete form.

$\Rightarrow$) If the criteria (3.2) of Von Neumann is not satisfy, then the central finite different (6) is unstable. To proof that, only we need to proof the unidimensional case.

For the continuity of $\rho(\beta)$, for some $C>0$ there is a $\beta_C \in [\beta_1,\beta_2]$ such that $|\rho(\beta_C,h_x,h_t)| > 1+C\kappa$, see figure 1.
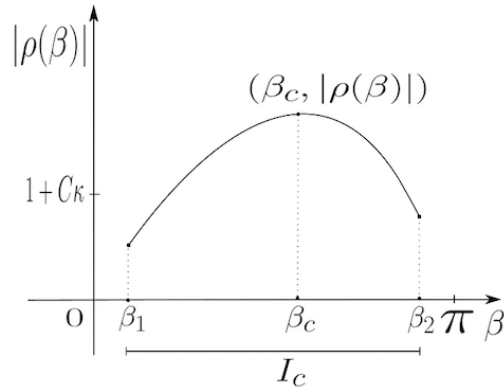
FIGURE 1. Given $\beta_C \in I_C = [\beta_1, \beta_2]$ with $|\rho(\beta_c)| > 1 + Ck$ for some value of $h\xi = \beta_c$.

Taking the initial condition ϕ_i^0 and building a function such that

(39)
$$\widehat{\phi}^0(\xi) = \begin{cases} 0 & \text{if } \xi \notin [\dfrac{\beta_1}{h_x}, \dfrac{\beta_2}{h_x}] \\ \\ \sqrt{\dfrac{h}{\beta_2 - \beta_1}} & \text{if } \xi \in [\dfrac{\beta_1}{h_x}, \dfrac{\beta_2}{h_x}]. \end{cases}$$

Observe, that $\|\widehat{\phi}^0\| = 1$.

Applying the Fourier transform to the scheme (6) for the unidimensional case, and knowing that $C > 0$ and $\beta_c \in [-\pi, \pi]$, we have that $\widehat{\phi}^0(\beta_c) \neq 0$, then for $|\rho(\beta_c)| > 1 + C\kappa$ have

(40)
$$\widehat{\phi}^n = \rho(\beta_c)\widehat{\phi}^{n-1}$$
$$= \rho^2(\beta_c)\widehat{\phi}^{n-2} = \cdots = \rho^n(\beta_c)\widehat{\phi}^0 > (1 + C\kappa)^n\widehat{\phi}^0.$$

Using the Parseval's relation (17), have

$$\|\phi_i^n\|_{\ell^2}^2 = h_x\sum_i(\phi_i^n)^2 = \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} |\widehat{\phi}^n(\beta_c)|^2 d\beta_c,$$

in effect

$$(41) \qquad \|\phi_i^n\|_{\ell^2}^2 = \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} |\widehat{\phi}^n(\beta_c)|^2 \, d\beta_c,$$

replacing (40) in the Parselval's (41), have that

$$\|\phi_i^n\|_{\ell^2}^2 = \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} |\rho(\beta_c)|^{2n} |\widehat{\phi}^0(\beta_c)|^2 \, d\beta_c,$$

or equivalent

$$(42) \qquad \|\phi_i^n\|_{\ell^2}^2 = \int_{\frac{\beta_1}{h_x}}^{\frac{\beta_2}{h_x}} |\rho(h_x\xi, h_x, h_t)|^{2n} |\widehat{\phi}^0(\xi)|^2 \, d\xi.$$

By hypothesis, we have that $|\rho(\beta_c, h_x, h_t)| > 1 + C\kappa$ with $C > 0$ and $\kappa > 0$, and replacing in (42) have

$$(43) \qquad \|\phi_i^n\|_{\ell^2}^2 > (1 + C\kappa)^{2n} \int_{\frac{\beta_1}{h_x}}^{\frac{\beta_2}{h_x}} |\widehat{\phi}^0(\xi)|^2 \, d\xi,$$

from the inequality (43) and the function (39), we have that

$$(44) \qquad \|\phi_i^n\|_{\ell^2}^2 > (1 + C\kappa)^{2n},$$

choosing $C > 0$ such that exist $T > 0$ and $\kappa > 0$ with $\kappa n \sim T$ such that satisfy $2 + 2CT \geq e^{2CT} \sim (1 + \kappa C)^{2\frac{T}{\kappa}}$, the inequality (44) is written as

$$\|\phi_i^n\|_{\ell^2}^2 \geq \frac{1}{2}(1 + C\kappa)^{2\frac{T}{\kappa}} \sim \frac{1}{2}e^{2CT} \cdot 1,$$

we can conclude that

$$(45) \qquad \|\phi_i^n\|_{\ell^2}^2 \geq \frac{1}{2}e^{2CT} \|\phi_i^0\|_{\ell^2}^2,$$

this shows that central finite differences scheme (6) is unlimited $\forall C > 0$, as a consequence is unstable.

From inequality (38) and (45) the theorem (3.1) is proved $\square$.

Remark. Note that of the result (38) that is a discrete representation of the definition 3.2, we conclude that: The scheme (6) is called stable is exist a constant $\kappa > 0$, $C > 0$, $T > 0$ and a norm ℓ^2 such that

$$\|\phi_{i,j,k}^n\|_{\ell^2} = \|Q^n \phi_{i,j,k}^0\|_{\ell^2} \leq e^{CT} \|\phi_{i,j,k}^0\|_{\ell^2},$$

where $n\kappa \leq T$; κ and C independent of $h_1, h_2, h_3, h_t, \beta_1, \beta_2, \beta_3$, with $n > 0$ and $\beta_\ell = h_\ell \xi$ with $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]^N$ for $\ell = 1, 2, 3$.

Note that, from criteria (3.2), if $\rho(\beta_1, \beta_2, \beta_3, h_x, h_y, h_z, h_t) = \rho(\beta_1, \beta_2, \beta_3)$ then, the Von Neumann criteria is replaced by $|\rho(\beta_1, \beta_2, \beta_3)| \leq 1$.

In effect, applying this result in (29) have that

$$(46) \qquad |1 - 4\mu_1 sen^2\frac{\beta_1}{2} - 4\mu_2 sen^2\frac{\beta_2}{2} - 4\mu_3 sen^2\frac{\beta_3}{2}| + |\lambda_1 sen\beta_1 + \lambda_2 sen\beta_2 + \lambda_3 sen\beta_3| \leq 1,$$

Observe that, considering $h_x = h_y = h_z = h_t$ and the equation (46), have that :

$$(47) \qquad |1 - 4(\frac{\alpha_1}{h_x} + \frac{\alpha_2}{h_y} + \frac{\alpha_3}{h_z})|_{max} + |u_1 + u_2 + u_3|_{max} \leq 1.$$

3.3. Convergence.

Definition 3.4. The central finite differences scheme of the equation (1) is convergent with some norm $\| \cdot \|$ if the partial differential solution $\theta(\xi, t)$, and the solution of the finite differences scheme $\phi_{i,j,k}^n$, such that $\phi_{i,j,k}^0$ converge to $\theta_0(\xi)$ when ih_x, jh_y, kh_z, nh_t converge to x, y, z, t respectively, then $\phi_{i,j,k}^n$ converge to $\theta(\xi, t)$ when (ih_x, jh_y, kh_z, nh_t) converge to (x, y, z, t) when h_x, h_y, h_z, h_t converge to 0; with $\xi \in \mathbb{R}^3$ y $t > 0$.

Proposition 3.2. The solution of central finite differences scheme (6) is convergent with the norm ℓ^2, with the solution of partial differential equation (1), represented by $\theta(\xi, t)$, for $\xi \in \mathbf{R}^3$ if $\|\theta(\xi, t) - \phi_{i,j,k}^n\|_{\ell^2} \to 0$ when $h_x, h_y, h_z, h_t \to 0$.

Proof

Given $h_x = h_y = h_z = h_t = h$ the central finite differences (6) is written as

(48)
$$\theta_{i,j,k}^{n+1} = Q\theta_{i,j,k}^n + 4o(h) + 3o(h^2).$$

Given $\theta(\xi,t)$ a solution of equation (10), as the central differences is consistent with order of precision $(1,2)$, of the result (16), and considering $h_x = h_y = h_z = h_t = h$ we have that

(49)
$$P\theta - P_{h_x,h_y,h_k,h_t}\phi_{i,j,k}^n = 4o(h) + 3o(h^2),$$

that is

(50)
$$\theta_{i,j,k}^n = Q\theta_{i,j,k}^{n-1} + 4o(h) + 3o(h^2).$$

(51)
$$\text{and} \quad \varphi_{i,j,k}^n = \theta_{i,j,k}^n - \phi_{i,j,k}^n \text{ the error at n-th time step,}$$

(52)
$$\text{such that} \quad \varphi_{i,j,k}^0 = \max_{i,j,k}|\theta_{i,j,k}^0 - \phi_{i,j,k}^0| = 0.$$

The equation (50) and the equation (51) is expressed by

$$\varphi_{i,j,k}^n = Q\varphi_{i,j,k}^{n-1} + 4o(h) + 3o(h^2)$$
$$= Q^2\varphi_{i,j,k}^{n-2} + Q[4o(h) + 3o(h^2)] + 4o(h) + 3o(h^2)$$
$$= \dots$$
$$= Q^n\varphi_{i,j,k}^0 + \sum_{j=0}^{n-1} Q^j[4o(h) + 3o(h^2)],$$

using the equation (52), we have that

$$\varphi_{i,j,k}^n = \sum_{j=0}^{n-1} Q^j[4o(h) + 3o(h^2)]$$
$$\|\varphi_{i,j,k}^n\|_{\ell^2} \le \sum_{j=0}^{n-1} \|Q^j\|_{\ell^2}[4o(h) + 3o(h^2)],$$

and with the conclusion 3.2 of stability that inequality is written by

(53)
$$\|\varphi_{i,j,k}^n\|_{\ell^2} \le \sum_{j=0}^{n-1} e^{CT_j}[4o(h) + 3o(h^2)].$$

Replacing the equation (51) in the inequality (53) have

$$(54) \qquad \|\theta_{i,j,k}^{n} - \phi_{i,j,k}^{n}\|_{\ell^2} \leq \sum_{j=0}^{n-1} e^{CT_j}[4o(h) + 3o(h^2)],$$

when $T_j \sim t_j = jh$, we have that

$$(55) \qquad \|\theta_{i,j,k}^{n} - \phi_{i,j,k}^{n}\|_{\ell^2} = o(h) + o(h^2)$$

by the result of (55) the central finite differences is convergent with order (1,2).

We conclude that the criteria of consistence, convergence and stability are established.

4. RESULT

4.1. Application 1. For $j = k = 0$ y $\alpha_l = v_n = 0$ with $l = 2,3$ y $n = 1,2,3$, the equation (7) is written by

$$(56) \qquad \phi_i^{t+1} = \phi_i^t + \frac{h_t \alpha_1}{(h_x)^2}(\phi_{i+1}^t - 2\phi_i^t + \phi_{i-1}^t)$$

As $\rho(\xi_1, \xi_2, \xi_3, h_x, h_y, h_z, t) = \rho(\xi_1, \xi_2, \xi_3)$, the CVN given in the definition (3.2) is replaced by:

$$(57) \qquad |\rho(\xi_1, \xi_2, \xi_3)| \leq 1.$$

For the case 1D, of the equation (29) and the inequality (57), we have that:
$-1 \leq \rho(\xi_1) = 1 - 4\mu_1 Sin^2\frac{\xi_1}{2}$; $\mu_1 Sin^2\frac{\xi_1}{2} \leq \frac{1}{2}$, taking the grater value of $0 \leq \mu_1 \leq \frac{1}{2}$, therefore:

$$0 \leq \frac{h_t \alpha_1}{h_x^2} \leq \frac{1}{2}$$

Using the equation (56) of 1D with $h_x = h_t = 0.1$, we compared with results of Fletcher [15] for the initial condition $\phi(x,0) = 0$ and the boundary condition $\phi(0,t) = \phi(1,t) = 100$ at 3000 seconds, with θ the exact solution [15] of the equation (56) showed in the table 2.

The table 1 and 2 represent the changes of temperature at 500s, 1000s, 1500s, 2000s, 2500s until 3000s. The condition $\phi(0,0) = \phi(1,0) = 50$ permit to get the best approximation of exact solution [15], see figure 3, table 1 and table 2.

					x						scheme
0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	Central
100.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00	500s
100.00	50.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	50.00	100.00	1000s
100.00	50.00	25.00	0.00	0.00	0.00	0.00	0.00	25.00	50.00	100.00	1500s
100.00	62.50	25.00	12.50	0.00	0.00	0.00	12.50	25.00	62.50	100.00	2000s
100.00	62.50	37.50	12.50	6.25	0.00	6.25	12.50	37.50	62.50	100.00	2500s
100.00	68.75	37.50	21.87	6.25	6.25	6.25	21.87	37.50	68.75	100.00	3000s
100.00	68.75	41.41	21.88	14.06	6.25	14.06	21.88	41.41	68.75	100.00	3500s
100 .00	68.33	41.53	22.49	11.68	8.25	11.68	22.49	41.53	68.33	100.00	θ

TABLE 1. Changes of temperature

					x						scheme
0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	Central
50.00	25.00	.00	.00	.00	.00	.00	.00	.00	25.00	50.00	500s
100.00	50.00	12.50	.00	.00	.00	.00	.00	12.50	50.00	100.00	1000s
100.00	56.25	25.00	6.25	.00	.00	.00	6.25	25.00	56.25	100.00	1500s
100.00	62.50	31.25	12.50	3.13	.00	3.13	12.50	31.25	62.50	100.00	2000s
100.00	65.63	37.50	17.19	6.25	3.13	6.25	17.19	37.50	65.63	100.00	2500s
100.00	68.75	41.41	21.88	10.16	6.49	10.16	21.88	41.41	68.75	100.00	3000s
100.00	68.75	41.41	21.88	10.16	6.49	10.16	21.88	41.41	68.75	100.00	Fletcher[15]
100 .00	68.33	41.53	22.49	11.68	8.25	11.68	22.49	41.53	68.33	100.00	θ

TABLE 2. Changes of temperature

Grid						Error		
$m_x \times m_t$	μ_1	t_{max}	x	ϕ	θ	$\|e\|_2$	ρ	
10×10	0.5	3000s	0.9	68.75	68.33	0.9418		Fletcher [15]
10×10	0.5	3000s	0.9	68.75	68.33	0.94195	0.9995	Present work

TABLE 3. Temperature at time of 3000 seconds
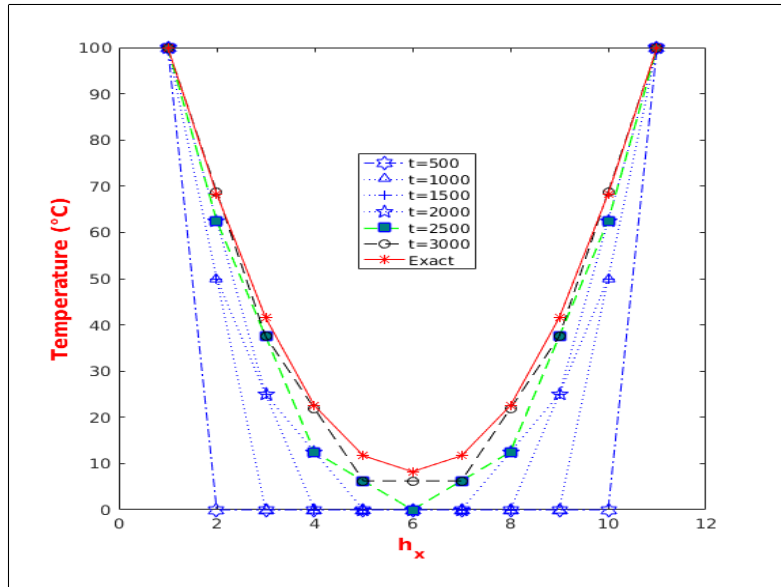
FIGURE 2. Change of temperature for the central scheme

FIGURE 3. Change of temperature adding boundary condition $\phi(0,0) = \phi(1,0) = 50$.

4.2. Application 2. For $k = 0$ and $\alpha_l = v_n = 0$, $l = 3$; $n = \overline{1,3}$ the equation (22) is written as:

$$(58) \qquad \phi_{i,j}^{t+1} = \phi_{i,j}^t + \frac{h_t \alpha_1}{(h_x)^2}(\phi_{i+1,j}^t - 2\phi_{i,j}^t + \phi_{i-1,j}^t) + \frac{h_t \alpha_2}{(h_y)^2}(\phi_{i,j+1}^t - 2\phi_{i,j}^t + \phi_{i,j-1}^t).$$

For the case 2D, the equation (29) and the inequality (57) have that : $-1 \leq \rho(\xi_1, \xi_2) = 1 -$
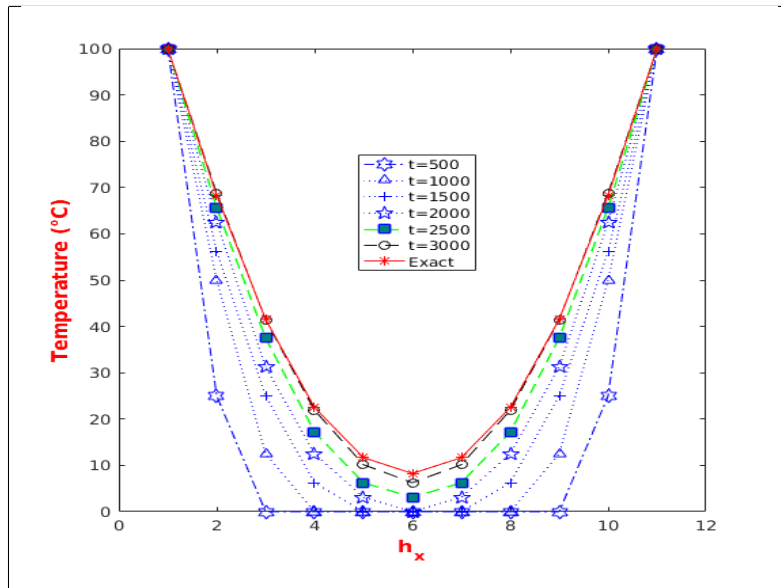$4\mu_1 Sin^2 \frac{\xi_1}{2} - 4\mu_2 Sin^2 \frac{\xi_2}{2}$, taking the greater values $0 \leq \mu_1 + \mu_2 \leq \frac{1}{2}$, therefore:

$$0 \leq \frac{h_t \alpha_1}{h_x^2} + \frac{h_t \alpha_2}{h_y^2} \leq \frac{1}{2}.$$

Considering the equation (58) with $h_x = h_y = h_t = 0.1$ and $\mu_1 = \mu_2 = 0.25$, $\alpha_1 = \alpha_2 = 0.025$ we compare with the results of Gary [13] with respect to exact solution by variable separable method and is written as $\theta(x,y,t) = e^{-2\alpha_1 \pi^2 t} Sin(\pi x) Sin(\pi y)$, with initial condition $\phi(x,y,0) = Sin(\pi x) Sin(\pi y)$ and boundary condition $\phi(0,y,t) = \phi(1,y,t) = 0$ and $\phi(x,0,t) = \phi(x,1,t) = 0$. However, for $\alpha_1 = \alpha_2 = 0.160$ and $\mu_1 = \mu_2 = 0.16$, we observe that the greater approximation of exact solution θ, see table 4.

The table 3 shows the correspond ℓ_2 - error and the amplification factor ρ at the time $t = 1$ s.

					x=y=t						
0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Gary [13]
0.0000	0.0128	0.0244	0.0336	0.0395	0.0415	0.0395	0.0336	0.0244	0.0128	0.0000	
0.0000	0.0244	0.0464	0.0639	0.0751	0.0790	0.0751	0.0639	0.0464	0.0244	0.0000	
0.0000	0.0336	0.0639	0.0880	0.1034	0.1087	0.1034	0.0880	0.0639	0.0336	0.0000	
0.0000	0.0395	0.0751	0.1034	0.1216	0.1278	0.1216	0.1034	0.0751	0.0395	0.0000	
0.0000	0.0415	0.0790	0.1087	0.1278	0.1344	0.1278	0.1087	0.0790	0.0415	0.0000	
0.0000	0.0395	0.0751	0.1034	0.1216	0.1278	0.1216	0.1034	0.0751	0.0395	0.0000	
0.0000	0.0336	0.0639	0.0880	0.1034	0.1087	0.1034	0.0880	0.0639	0.0336	0.0000	
0.0000	0.0244	0.0464	0.0639	0.0751	0.0790	0.0751	0.0639	0.0464	0.0244	0.0000	
0.0000	0.0128	0.0244	0.0336	0.0395	0.0415	0.0395	0.0336	0.0244	0.0128	0.0000	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Present
0.0000	0.0128	0.0244	0.0336	0.0395	0.0415	0.0395	0.0336	0.0244	0.0128	0.0000	work
0.0000	0.0244	0.0464	0.0639	0.0751	0.0790	0.0751	0.0639	0.0464	0.0244	0.0000	$\alpha_1 = 0.025$
0.0000	0.0336	0.0639	0.0880	0.1034	0.1087	0.1034	0.0880	0.0639	0.0336	0.0000	$\alpha_2 = 0.01$
0.0000	0.0395	0.0751	0.1034	0.1216	0.1278	0.1216	0.1034	0.0751	0.0395	0.0000	
0.0000	0.0415	0.0790	0.1087	0.1278	0.1344	0.1278	0.1087	0.0790	0.0415	0.0000	
0.0000	0.0395	0.0751	0.1034	0.1216	0.1278	0.1216	0.1034	0.0751	0.0395	0.0000	
0.0000	0.0336	0.0639	0.0880	0.1034	0.1087	0.1034	0.0880	0.0639	0.0336	0.0000	
0.0000	0.0244	0.0464	0.0639	0.0751	0.0790	0.0751	0.0639	0.0464	0.0244	0.0000	
0.0000	0.0128	0.0244	0.0336	0.0395	0.0415	0.0395	0.0336	0.0244	0.0128	0.0000	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Present
0.0000	0.0132	0.0252	0.0346	0.0407	0.0428	0.0407	0.0346	0.0252	0.0132	0.0000	work
0.0000	0.0252	0.0479	0.0659	0.0775	0.0814	0.0775	0.0659	0.0479	0.0252	0.0000	$\alpha_1 = 0.160$
0.0000	0.0346	0.0659	0.0907	0.1066	0.1121	0.1066	0.0907	0.0659	0.0346	0.0000	$\alpha_2 = 0.160$
0.0000	0.0407	0.0775	0.1066	0.1253	0.1318	0.1253	0.1066	0.0775	0.0407	0.0000	
0.0000	0.0428	0.0814	0.1121	0.1318	0.1386	0.1318	0.1121	0.0814	0.0428	0.0000	
0.0000	0.0407	0.0775	0.1066	0.1253	0.1318	0.1253	0.1066	0.0775	0.0407	0.0000	
0.0000	0.0346	0.0659	0.0907	0.1066	0.1121	0.1066	0.0907	0.0659	0.0346	0.0000	
0.0000	0.0252	0.0479	0.0659	0.0775	0.0814	0.0775	0.0659	0.0479	0.0252	0.0000	
0.0000	0.0132	0.0252	0.0346	0.0407	0.0428	0.0407	0.0346	0.0252	0.0132	0.0000	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0000	solution
0.0000	0.0133	0.0252	0.0347	0.0408	0.0429	0.0408	0.0347	0.0252	0.0133	-0.0000	exact
0.0000	0.0252	0.0480	0.0661	0.0777	0.0816	0.0777	0.0661	0.0480	0.0252	-0.0000	
0.0000	0.0347	0.0661	0.0909	0.1069	0.1124	0.1069	0.0909	0.0661	0.0347	-0.0000	
0.0000	0.0408	0.0777	0.1069	0.1256	0.1321	0.1256	0.1069	0.0777	0.0408	-0.0000	
0.0000	0.0429	0.0816	0.1124	0.1321	0.1389	0.1321	0.1124	0.0816	0.0429	-0.0000	
0.0000	0.0408	0.0777	0.1069	0.1256	0.1321	0.1256	0.1069	0.0777	0.0408	-0.0000	
0.0000	0.0347	0.0661	0.0909	0.1069	0.1124	0.1069	0.0909	0.0661	0.0347	-0.0000	
0.0000	0.0252	0.0480	0.0661	0.0777	0.0816	0.0777	0.0661	0.0480	0.0252	-0.0000	
0.0000	0.0133	0.0252	0.0347	0.0408	0.0429	0.0408	0.0347	0.0252	0.0133	-0.0000	
-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	0.0000	

TABLE 4. Change of temperature

4.3. Application 3. The equation (7), for $j = k = 0$ and $\alpha_l = v_n = 0$ with $l = 2, 3$ y $n = 2, 3$; is written as:

$$(59) \qquad \phi_i^{t+1} = \phi_i^t - 0.5v_1 \frac{h_t}{h_x}(\phi_{i+1}^t - \phi_{i-1}^t) + \frac{h_t \alpha_1}{(h_x)^2}(\phi_{i+1}^t - 2\phi_i^t + \phi_{i-1}^t).$$

For the **case 1D** of the equation (29) and the inequality (57), we have:

$\rho(\beta_1) = 1 - 4\mu_1 Sin^2\frac{\beta_1}{2} - \hat{\imath}\lambda_1 Sin(\beta_1)$, and taking (46), we have

$$(60) \qquad\qquad |1 - 4\mu_1 sen^2\frac{\beta_1}{2}| + |\lambda_1 sen\beta_1| \leq 1.$$

The transport equation (59) for the case 1D with $h_x = 0.2$, $h_t = 0.1$, allow us compare the central scheme getting for Fletcher [15] with boundary condition $\phi(-2,t) = 1$ and $\phi(-2,t) = 0$ for all $t \in [0,1]$ and initial condition

$$(61) \qquad\qquad \phi(x,0) = \begin{cases} 1 & \text{if} \quad -2 \leq x \leq 0, \\ 0 & \text{if} \quad 0 < x \leq 2, \end{cases}$$

under these conditions, an exact solution with methods of separation of variables, is

$$(62) \qquad \theta(x,t) = 0.5 - \frac{2}{\pi}\sum_{k=1}^{N} sin\left[\frac{(2k-1)\pi}{L_1}(x - ut)\right]\frac{e^{-\alpha[\frac{(2k-1)\pi}{L_1}]^2 t}}{2k-1},$$

where θ is the exact solution [15] of the equations (59) showed in the 5 and 6 and represent the change of temperature at 0s, 0.5s to 1s.

Considering $h_t = 0.05$, $h_x = 0.2$ and the conditions $v_1 = 0.5$, $\lambda_1 = 0.125$, $\alpha_1 = 0.1$, $\mu_1 = 0.125$ with n=m=20 and Raynold's number $(Re = v_1\frac{h_x}{\alpha_1})$ $Re = 1$, with these conditions, we have gotten a best approximation of the exact solution [15], this implied a decreasing ℓ_2-error wrote as $\|e\|_{\ell^2}$, such that

$$(63) \qquad\qquad \|e\|_{\ell^2} = \sqrt{\frac{\sum_{i,t=0}^{m_x,m_t} \phi(i,t) - \theta(i,t)}{m_x - 1}},$$

considering at 1s, see figure 4, and the table 7.

-2.0	-1.8	-1.6	-1.4	-1.2	-1	-0.8	-0.6	-0.4	-0.2	0.0	schema Central
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0s
1.0	1.0	1.0	1.0	1.0	0.99972	0.99858	0.99421	0.98070	0.94678	0.87713	1s
1.0	1.0	1.0	1.0	1.0	1.0	0.999	0.994	0.981	0.947	0.877	1s-Fletcher[15]
1.0	1.0	1.0	1.0	1.0	1.0	0.998	0.993	0.978	0.941	0.868	θ

TABLE 5. Changes of temperature

0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	scheme Central
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0s
0.75979	0.59759	0.41415	0.24546	0.12064	0.04747	0.01429	0.00308	0.00042	0.00	1s
0.760	0.598	0.414	0.245	0.121	0.047	0.014	0.003	0.000	0.00	1s-Fletcher[15]
0.749	0.558	0.412	0.251	0.132	0.059	0.022	0.007	0.002	0.00	θ

TABLE 6. Changes of temperature

$m_x \times m_t$	μ_1	t_{max}	x	ϕ	θ	$\|e\|_2$	$\rho(\beta)$
10×10	0.25	1s	-0.2	0.94678205	1.0	0.00603304	0.62986082
20×20	0.125	1s	-0.2	0.94622105	1.0	0.00285694	0.81493044

TABLE 7. Temperature at 1s

4.4. Application 4. Given the exact solution (7) for $v_1 = v_2 = v_3 = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.

$$(64) \qquad \theta(x,y,z,t) = e^{-3\alpha\pi^2 t} Sin(\pi x) Sin(\pi y) Sin(\pi z)$$

where the equation (65) is the initial condition and the equation (66) is the boundary condition.

$$(65) \qquad \phi(x,y,z,0) = Sin(\pi x) Sin(\pi y) Sin(\pi z).$$

$$\phi(0,y,z,t) = \phi(1,y,z,t) = \phi(x,0,z,t) = 0,$$

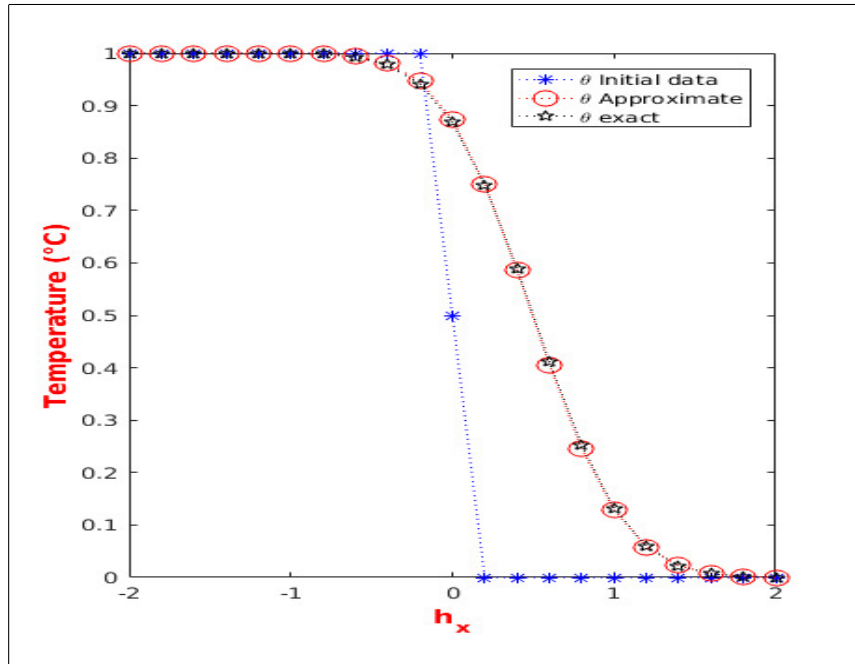$$(66) \qquad \phi(x,1,z,t) = \phi(x,y,0,t) = \phi(x,y,1,t) = 0.$$

FIGURE 4. Changes of temperature for $h_t = 0.05$

4.4.1. *Comparison of central finite differences.* For $v_j = 0$, $j = \overline{1,3}$; $\alpha_i = 0.333333 = \alpha$, $i = 1, 2, 3$, in the equation (7) with $h_x = h_y = h_z = h_t = h = 0.1$ and $\mu_1 = \mu_2 = \mu_3 = \mu$, is possible to compare the central finite differences with the results of Ortigoza [9], with initial condition (65) and Dirichlet boundary condition (66).

To visualize was necessary to do three section with plane in $x = 0.2$, $x = 0.5$ and $x = 0.8$; in the domain the temperature decreases with increasing time. The results by Ortigoza present instability after time $t = 0.15$, because the value $\alpha_i = 0.333333$, $i = 1, 2, 3$ is outside of the domain of stability of the inequality (67), see figure(5) case (d).
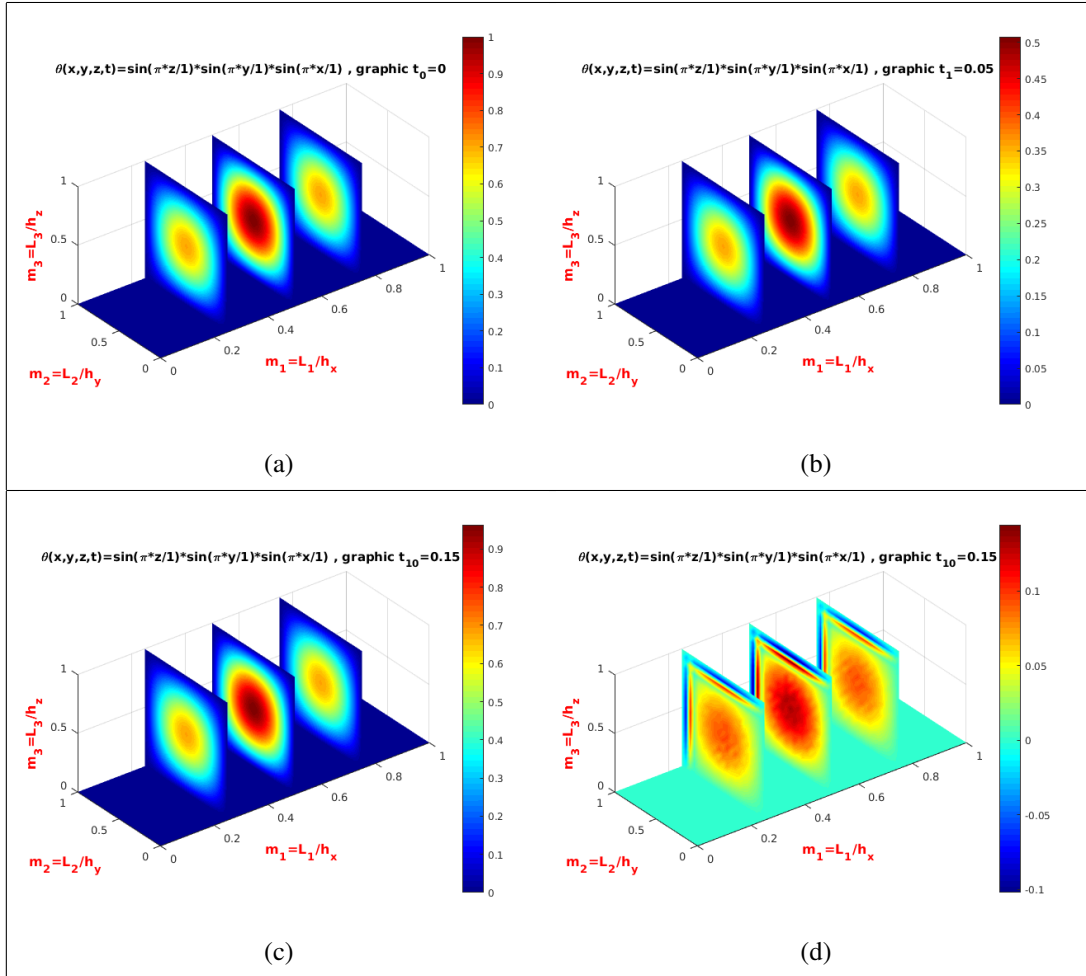
FIGURE 5. The temperature (a),(b),(c) stable y (d) unstable.

For the case 3D, the equation (29) and inequality (57), we have that:

$-1 \le \rho(\xi_1, \xi_2, \xi_3) = 1 - 4\mu_1 Sin^2 \dfrac{\xi_1}{2} - 4\mu_2 Sin^2 \dfrac{\xi_2}{2} - 4\mu_3 Sin^2 \dfrac{\xi_3}{2}$, them getting the grater value of

$0 \le \mu_1 + \mu_2 + \mu_3 \le \dfrac{1}{2}$ therefore:

$$(67) \qquad\qquad 0 \le \frac{h_t \alpha_1}{h_x^2} + \frac{h_t \alpha_2}{h_y^2} + \frac{h_t \alpha_3}{h_z^2} \le \frac{1}{2}.$$

From the equation (67) and $h_x = h_y = h_z = h_t = h$, $v_j = 0$, $j = \overline{1,3}$ with $\mu_1 = \mu_2 = \mu_3 = \mu$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, we have that $\alpha_i \le \dfrac{1}{6}$; $i = \overline{1,3}$, it permit compared the central finite difference scheme of the equation (64), initial condition (65) and Dirichlet boundary condition (66) used by Ortigoza [9], the results are stable , see figure (5) case (a), (b) and (c).

The figure (6) for $h = 0.05$, $\alpha = 0.00014$ and $\mu = 0.165$ in the equation (22) shows the temperature in the domain at time $t_{20} = 1$s and for visualizing we have three section in $x = 0.2$, 0.5 y 0.8, Thar results shows us, that the temperature decrease and keep the stability.
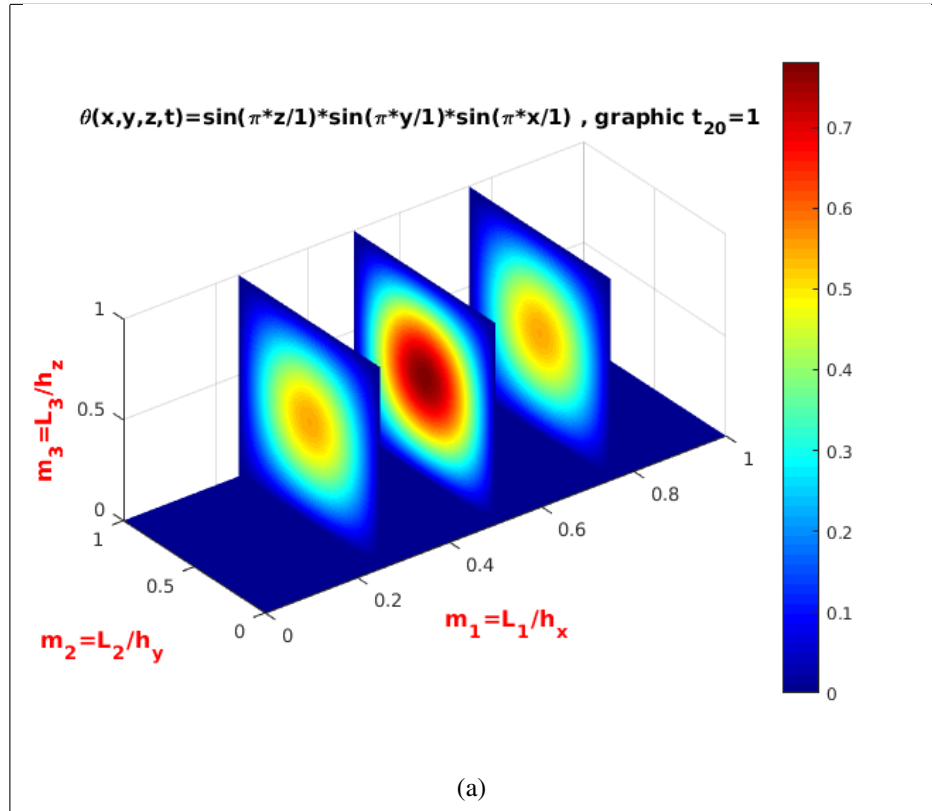

(a)

FIGURE 6. Temperature (a) stable in time $t_{20} = 1$.

5. DISCUSSION

This research present the analysis of convergence, consistence and stability of central finite differences of atmospheric transport equation; with the computational implementation of the equation (1) was possible to make the simulation in 1D, 2D and 3D of the atmospheric transport equation and compare the result with others works presented in the applications 4.1, 4.2 and 4.3. The method presented in the present work is more stable than the works cited in the tables (1 and 2).

The equation (7) is stable if the constants of velocity are bounded by the equation of stability (46) in particular by (47).

The amplification factor $\rho(\xi_1, \xi_2, \xi_3)$ obtained from equation (29) satisfy the Von Neumann

condition, if exist a constant $C > 0$ and $t > 0$, such that $|\rho(\xi_1, \xi_2, \xi_3)| \leq 1 + Ct$.

Changing the coefficient of thermal diffusion α_i with $i = \overline{1,3}$ and applying to the inequality (67) and adding the source S_θ (see [4]) in the equation (1), is possible improve the simulation of physical problems.

We recommend work the mathematical modeling as a system, considering the conservation mass laws, heat, water and aerosol.

6. CONCLUSION

The results show that the criteria of convergence, consistence and stability of the central finite differences scheme, it has a strong dependence between the spatial and temporal variables, that is the Von Neumann condition is satisfied.

The amplification factor from equation (29) is very important to get the Von Neumann condition. On the other hand, we conclude that the equation (46) determine the stability of the equation (7) without it, the convergence from approximate solution to exact solution is impossible.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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