Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 6, 2296-2306
https://doi.org/10.28919/jmcs/4852
ISSN: 1927-5307

e-ADJACENCY MATRIX AND e-LAPLACIAN MATRIX OF SEMIGRAPH

AMBIKA BIRADAR*

Department of Mathematics, Dr. D.Y. Patil Institute of Technology, Pimpri, Pune, India

Abstract: In this paper we define e-Adjacency matrix $A_e(S)$ and e-Laplacian matrix of a Semigraph $L_e(S)$. Also discuss some results of eigenvalues of these matrices. We define e-Energy of Semigraph $E_e(S)$ using eigenvalues of its e-adjacency matrix and e-Laplacian energy of Semigraph $LE_e(S)$ using eigenvalues of its e-Laplacian matrix. We investigate relation between e-energy$E_e(S)$ and e-Laplacian energy $LE_e(S)$ for regular Semigraphs.

Keywords: e-adjacency matrix; e-Laplacian matrix; eigenvalue of a matrix; e- energy of semigraph $E_e(S)$; e-Laplacian energy of semigraph $LE_e(S)$.

2010 AMS Subject Classification: 15A15, 15A18.

1. INTRODUCTION

The concept of Semigraph was first introduced by E Sampathkumar [1] in 2000 as a generalization of a graph where an edge can have more than two vertices. In last two decades lot of research has been done in this area of Semigraph. Many concepts of graphs have been generalized to Semigraph. Domination in Semigraph has been extensively studied by S S Kamath, R S Bhat, V. Swaminathan and S Gomathi. Adjacency matrix of Semigraph has been defined by Y. S. Gaidhani,

*Corresponding author
E-mail address: ambikabiradar12@gmail.com
Received July 17, 2020

C.S. Deshpande and B. Y. Bam (see [3]). Energy of Semigraph has been studied by Y.S. Gaidhani, C.M. Deshpande, S. Pirzada (see [4]).

2. PRELIMINARIES

Definition 2.1. Semigraph is a pair $S(V,E)$, where V is a nonempty set of elements called vertices and E is the set of ordered n-tupples of distinct vertices called edges for $n \geq 2$ where these edges satisfy the following two conditions

1. any of the two edges have at most one vertex in common.
2. any two edges $u = (u_1, u_2, u_3, \dots \dots . u_r)$ and $v = (v_1, v_2, v_3, \dots \dots . v_s)$ are equal if
   a) $r = s$ and
   b) $u_i = v_i$ for i=1,2,3,……..r or $u_i = v_{r-i+1}$

This implies that $u = (u_1, u_2, u_3, \dots \dots . u_r)$ is same as

$u = (u_r, u_{r-1}, u_{r-2}, \dots \dots . u_1)$ where u_1 and u_r are called the end vertices of an edge, while $u_2, u_3, \dots \dots . . u_{r-1}$ are called the middle vertices of edge u.

Example.2.2 Let $S(V,E)$ be a semigraph where

$V = (1,2,3,4,5,6)$ and $E = (1,2,5), (3,4,6), (2,4,5), (1,6)$

In $S(V,E)$, 1,3,5,6 are end vertices, 2 is middle end vertex and 4 is middle vertex.

Cardinality of edge: It is the number of vertices in an edge and it is denoted as $|u|$.

Adjacent vertices: Two vertices are said to be adjacent if they belong to same edge and they are said to be consecutively adjacent if they belong to same edge and are consecutive in order.

Adjacent edges: Two edges are said to be adjacent if there is a vertex in common.

Special Types of Semigraphs [2]

Regular Semigraph: A semigraph S(V,E) is said to be regular if all its vertices have the same degree of a particular type of degree.

r-uniform Semigraph: A semigraph S(V,E) is said to be r-uniform if cardinality of each edge in S is r.

Complete Semigraph: If any two vertices in a semigraph S(V,E) are adjacent then the semigraph

S(V,E) is said to be complete.

Following are some of the families of semigraph which are complete.

1) E_r^c a semigraph consisting of a single s-edge of cardinality r.

2)) T_{r-1}^1 a semigraph consisting of an s-edge of cardinality r-1 and one vertex joined with each of the r-1 vertices by an edge of cardinality two.

Incidence matrix and Adjacency matrix of a semigraph has been studied byY S Gaidhani and C M Deshpande see [3],[4].

For a vertex v in a semigraph $S(V,E)$various types of degrees are defined.

1) $d(v)$: Degree of vertex v is the number of edges having v as an end vertex.
2) $d_e(v)$: Edge degree of a vertex v is the number of edges containing vertex v.
3) $d_a(v)$: Adjacent degree of a vertex v is the number of vertices adjacent to v.
4) $d_{ca}(v)$ Consecutive adjacent degree of a vertex v is the number of vertices which are consecutively adjacent to v.

3. MAIN RESULTS

In first section of this paper we define e-Adjacent degree of a vertex v $(d_{ae}(v))$. Also we define an e-Adjacency matrix, e-Degree matrix and e-Laplacian matrix of a Semigraph and discuss some results of eigenvalues of these matrices. In the second section of this paper we define e- energy of Semigraph using eigenvalues of its e-adjacency matrix and e-Laplacian energy of Semigraph using eigenvalues of its e-Laplacian matrix.

4. E-ADJACENCY DEGREE OF VERTEX IN SEMIGRAPH S(V, E).

Definition 4.1: Let $S(V,E)$ be a Semigraph where $V$is a non empty set of elements called vertices &E is the set of ordered n-tupples of distinct vertices called edges for $n \geq 2$ .The e-Adjacent degree of a vertex v ($d_{ae}(v)$) is defined as number of vertices adjacent to v only from those edges in which v is end vertex.

Definition 4.2: e-Regular Semigraph: A semigraph S(V, E) is said to be e-Regular if all its end vertices have the same degree of a particular type of degree.

We observe that Adjacent degree of a vertex v $d_a(v)$ is same as e-Adjacent degree of a vertex v $d_{ae}(v)$ if v is an end vertex

i) $d_{ae}(v) = d_a(v)$, if v is an end vertex

ii) $d_{ae}(v) < d_a(v)$, if v is middle end vertex

iii) $d_{ae}(v) = 0$, if v is middle vertex

iv) $d_{ae}(v) = \sum_{\text{v is end vertex of } e_j}(|e_j| - 1)$

Example. 2: Let $S(V, E)$ be a semigraph where

$V = (1,2,3,4,5,6)$ and $E = (1,2,5), (3,4,6), (2,4,5), (1,6)$

In $S(V, E)$, 1,3,5,6 are end vertices, 2 is middle end vertex and 4 is middle vertex

$d(1) = 2$	$d_e(1) = 2$	$d_a(1) = 3$	$d_{ae}(1)$=3
$d(3) = 1$	$d_e(3) = 1$	$d_a(3) = 2$	$d_{ae}(3)$=2
$d(5) = 2$	$d_e(5) = 2$	$d_a(5) = 4$	$d_{ae}(5)$=4
$d(6) = 2$	$d_e(6) = 2$	$d_a(6) = 3$	$d_{ae}(6)$=3
$d(2) = 1$	$d_e(2) = 2$	$d_a(2) = 4$	$d_{ae}(2)$=2
$d(4) = 0$	$d_e(4) = 2$	$d_a(4) = 4$	$d_{ae}(4)$=0

5. E-ADJACENCY MATRIX OF A SEMIGRAPH S(V, E)

Definition 5.1: Let $S(V, E)$ be a semigraph with vertex set $V = (1,2,3,4, \dots \dots p)$ and edge set $E = (e_1, e_2, e_3, \dots \dots . e_q)$ then e-Adjacency matrix of a Semigraph is defined as

$$A_e(S) = [a_{ij}] \quad \text{where } a_{ij} = |e_r| - 1, \quad \text{i, j are end vertices of edge } e_r$$
$$= 0, \quad \text{otherwise}$$

5.2 Properties of e-Adjacency matrix of a Semigraph S(V, E).

Let $S(V, E)$ be a semigraph with vertex set $V = (1,2,3,4, \dots \dots p)$ and edge set $E = (e_1, e_2, e_3, \dots \dots . e_q)$ then

a) $a_{ij} \in \{0,1,2, \dots \dots \dots . . (p - 1)\} \quad \forall i, j$

b) i is a middle vertex or an isolated vertex if all entries of i^{th} row and i^{th} column are zeroes

c) If i is an end vertex or middle end vertex $then\ \sum_i a_{ij} = \sum_j a_{ij}$

d) e-Adjacency matrix is symmetric and thus its eigenvalues are real

Example. 5.3 Let $S(V,E)$ be a semigraph where $V = (1,2,3,4,5,6)$ and

$E = (1,2,5), (3,4,6), (2,4,5), (1,6)$

$$A_e(S) = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

6. E-DEGREE MATRIX OF A SEMIGRAPH S(V, E)

Definition 6.1: Let $S(V,E)$ be a semigraph with vertex set $V = (1,2,3,4, \dots \dots p)$ and edge set $E = (e_1, e_2, e_3, \dots \dots . e_q)$ then degree matrix of semigraph is defined as

$$D_e(S) = [a_{ij}] \text{ where } a_{ij} = d_{ae}(i), \quad i=j$$
$$= 0, \quad i \neq j$$

where $d_{ae}(i)$ is an e-adjacent degree of vertex i. $d_{ae}(i) = \sum_{\text{i is end vertex of } e_j} (|e_j| - 1)$

Example. 6.2 Let $S(V,E)$ be a semigraph where $V = (1,2,3,4,5,6)$ and

$E = (1,2,5), (3,4,6), (2,4,5), (1,6)$

$$D_e(S) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

7. E-LAPLACIAN MATRIX OF SEMIGRAPH

Definition 7.1: Let $S(V,E)$ be a semigraph with vertex set $V = (1,2,3,4, \dots \dots p)$ and edge set $E = (e_1, e_2, e_3, \dots \dots . e_q)$ then e-Laplacian matrix of semigraph is defined as

$L_e(S) = D_e(S) - A_e(S)$ where $A_e(S)$ is e − adjacency matrix of a semigraph

$D_e(S)$ is e − degree matrix of a semigraph

Example. 7.2 Let $S(V,E)$ be a semigraph where $V = (1,2,3,4,5,6)$ and

$E = (1,2,5),(3,4,6),(2,4,5),(1,6)$

$$L_e(S) = \begin{bmatrix} 3 & 0 & 0 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 4 & 0 \\ -1 & 0 & -2 & 0 & 0 & 3 \end{bmatrix}$$

7.3 Properties of e-Laplacian matrix of a semigraph S(V,E).

a) Let $S(V,E)$ be a semigraph with vertex set $V = (1,2,3,4,\ldots\ldots p)$ and edge set $E = (e_1, e_2, e_3, \ldots\ldots. e_q)$ then $a_{ij} \in \{0, \pm1, \pm2, \ldots\ldots\ldots.. \pm(p-1)\}$ $\forall i,j$

b) i is a middle vertex or an isolated vertex if all entries of i^{th} row and i^{th} column are zeroes.

c) i is an end vertex or middle end vertex if sum of all entries of i^{th} row is zero and sum of all entries of i^{th} column is zero.

d) e-Laplacian matrix is symmetric and thus its eigenvalues are real.

8. Eigenvalues of e-Adjacency Matrix and e-Laplacian Matrix of a Semigraph S(V,E)

Definition (8.1): The polynomial det(A_e(S)-λI)=0 is called characteristics polynomial of e-Adjacency matrix A_e(S) and roots of characteristic polynomial are called eigenvalues of e-Adjacency matrix A_e(S). Characteristic Polynomial of A_e(S) is denoted by $\chi(A_e(S), \lambda)$ and the corresponding eigenvalues are denoted by $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots \lambda_n$

Definition (8.2): The polynomial det(L_e(S)-λI)=0 is called characteristics polynomial of e-Laplacian matrix $L_e(S)$ and roots of characteristic polynomial are called eigenvalues of e-Laplacian matrix $L_e(S)$.

Characteristic Polynomial of $L_e(S)$ is denoted by $\chi(L_e(S), \lambda)$ and the corresponding eigenvalues are denoted by $\mu_1, \mu_2, \mu_3, \ldots\ldots \mu_n$.

Example. 8.3 Let $S(V, E)$ be a semigraph where $V = (1,2,3,4,5,6)$ and $E = (1,2,5), (3,4,6), (2,4,5), (1,6)$

Eigenvalues of e-Adjacency matrix of S(V,E) are 0,0,3,-3,2,-2

Eigenvalues of e-Laplacian matrix of S(V,E) are 0 , 0, 0.5483, 2.4805, 4.6600 , 6.3112

Example. 8.4 Let T_4^1 a semigraph consisting of an s-edge of cardinality 4 and one vertex joined with each of the 4 vertices by an edge.

Eigenvalues of e-Adjacency matrix of T_4^1 are 0 with multiplicity 1, eigenvalue 1 with multiplicity 1, eigenvalue -3 with multiplicity 1, eigenvalue -1.6458 with multiplicity 1 and eigenvalue 3.6458 with multiplicity 1

Eigenvalues of e-Laplacian matrix of T_4^1 are 0 with multiplicity 1, eigenvalue 1 with multiplicity 2 eigenvalue 5 with multiplicity 1 and eigenvalue 7 with multiplicity 1.

Example. 8.5 Let E_n^c be a semigraph with n vertices consisting of a single s-edge of cardinality n.

Eigenvalues of e-Adjacency matrix of E_n^c are 0 with multiplicity n-2, n-1 with multiplicity 1 and –(n-1) with multiplicity 1

Eigenvalues of e-Laplacian matrix of E_n^c are 0 with multiplicity n-1, eigenvalue 2n-2 with multiplicity 1.

9. Some Results of Eigenvalues of e-Adjacency Matrix and e-Laplacian Matrix of Semigraph S(V,E)

Let G(V,E) be a graph with n vertices and m edges and let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n$, be eigenvalues of adjacency matrix of graph G(V,E) and $\mu_1, \mu_2, \mu_3, \ldots\ldots \mu_n$ be Laplacian eigenvalues of Laplacian matrix of graph G(V,E).

These eigenvalues satisfy following relations see[5,6,7,8]

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \sum_{i=1}^{n} \lambda_i^2 = 2m$$

$$\sum_{i=1}^{n} \mu_i = 2m \qquad \sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + 2m$$

Result 1. If the graph G is regular, then LE(G) = E(G)

Result 2. If the graph G consists of (disconnected) components G_1 and G_2, and if G_1 and G_2 have equal average vertex degrees, then LE(G) = LE(G_1) + LE(G_2). see [10]

The above results of eigenvalues are generalized to Semigraph S(V, E).

a) Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ldots\ldots \lambda_n$ be the eigenvalues of the e-adjacency matrix of semigraph S(V,E) then $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 2\sum_{e_j \in E}\left(|e_j| - 1\right)^2$

b) Let $\mu_1, \mu_2, \mu_3, \mu_4 \ldots\ldots \mu_n$ be the eigenvalues of the e-adjacency matrix of semigraph S(V,E) then $\sum_{i=1}^{n} \mu_i = 2m'$ where $m' = \sum_{e_{j \in E}}(\lceil e_j \rceil - 1)$.

$\sum_{i=1}^{n} \mu_i^2 = \sum_{u \in V}(d_{ae}(u))^2 + 2\sum_{e_j \in E}\left(|e_j| - 1\right)^2$ where u is end vertex or middle end vertex of semigraph S(V,E).

Matrix Energy and Polynomial energy of a Semigraph has been defined by Y S Gaidhani and CM Deshpande considering adjacency matrix of Semigraph see [6].

Here we define e-Energy of Semigraph with respect to e- Adjacency matrix. Also we define e-Laplacian energy of Semigraph S(V,E).

10. E-ENERGY OF SEMIGRAPH S(V,E)

Definition 10.1: Let S(V,E) be a semigraph with n vertices and m edges**.** Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ldots\ldots \lambda_n$ be the eigenvalues of the e-adjacency matrix of semigraph S(V,E) then $E_e(S) = \sum_{i=1}^{n} |\lambda_i|$

Result 1. $E_e(S) \geq 0$ equality is attained if and only if m = 0.

Result 2. If Semigraph S(V,E) consists of (disconnected) components $S_1(V_1,E_1)$ and $S_2(V_2,E_2)$, then

$$E_e(S) = E_e(S_1) + E_e(S_2).$$

11. E-LAPLACIAN ENERGY OF SEMIGRAPH S(V,E)

Definition 11.1: Let S(V,E) be a semigraph with n vertices and m edges**.** Let $\mu_1, \mu_2, \mu_3, \mu_4 \ldots\ldots\ldots\ldots \mu_n$ be the eigenvalues of the e-Laplacian matrix of semigraph S(V,E) then $LE_e(S) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m'}{n'}\right|$, where $m' = \sum_{e_j \in E}(\lceil e_j \rceil - 1)$and n' =n - no of middle vertices.

Theorem 10.1. Let S(V,E) be an e-Regular semigraph with degree r .[here S(V,E) is e-Regular with respect to e-adjacent degree of end vertex v].Then $LE_e(S) = E_e(S) + pr$ where p is number of middle vertices of S(V,E).

Proof: Since S(V,E) is an e-Regular semigraph with degree r and p is number of middle vertices then eigenvalues of e-adjacency matrix of S(V,E) are . $0,0,0,\ldots..(p\ times), \lambda_1, \lambda_{2,}, \lambda_{3,}, \lambda_{4,}\ldots\ldots\ldots \lambda_{n-p,}$ Thus e-Energy of Semigraph S(V,E) is $E_e(S) = \sum_{i=1}^{n-p}|\lambda_i|$.

Now e-Laplacian matrix of semigraph is defined as $\mathrm{L_e(S) = D_e(S) - A_e(S)}$ where $\mathrm{A_e(S)}$ is $\mathrm{e-adjacency\ matrix\ of\ a\ semigraph,}$

$\mathrm{D_e(S)}\ is\ \mathrm{e-degree\ \ matrix\ of\ a\ semigraph}$ thus e-Laplacian eigenvalues are

$\mathrm{0,0,0,\ldots..(ptimes), r - \lambda_1, r - \lambda_{2,}, r - \lambda_{3,}, \ldots\ldots\ldots\ldots\ldots r - \lambda_{n-p}}$.

Now we know if $\mu_1, \mu_2, \mu_3, \mu_4 \ldots\ldots\ldots\ldots \mu_n$ are the eigenvalues of the e-Laplacian matrix of semigraph S(V,E) then $LE_e(S) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m'}{n'}\right|$ where $m' = \sum_{e_j \in E}(\lceil e_j \rceil - 1)$and

n' =n - number of middle vertices=$n - p$. Here $\frac{2m'}{n''} = r$

$$LE_e(S) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m'}{n'}\right|$$

$$= \sum_{i=1}^{n-p}\left|r - \lambda_i - \frac{2m'}{n'}\right| + \sum_{i=1}^{p}\left|0 - \frac{2m'}{n'}\right|$$

$$LE_e(S) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m'}{n'}\right| = \sum_{i=1}^{n-p}|r - \lambda_i - r| + \sum_{i=1}^{p}|0 - r|$$

$$= \sum_{i=1}^{n-p}|-\lambda_i| + \sum_{i=1}^{p}|0 - r|$$

$$= \sum_{i=1}^{n-p} |\lambda_i| + \sum_{i=1}^{p} |r|$$

$$= E_e(S) + pr.$$

Theorem 10.2. If a Semigraph S(V,E) consists of (disconnected) components $S_1(V_1,E_1)$ and $S_2(V_2,E_2)$, such that $S_1(V_1,E_1)$ and $S_2(V_2,E_2)$ have equal average vertex degrees, then $LE_e(S(V,E)) = LE_e(S_1(V_1,E_1)) + LE_e(S_2(V_2,E_2))$

Proof: Let S(V,E) be a semigraph with n vertices and m edges consisting of two components $S_1(V_1,E_1)$ and $S_2(V_2,E_2)$.Here $S_1(V_1,E_1)$ has $n_1$vertices and m_1 edges with average degree$\frac{2m_1'}{n_1'}$ where $m_1' = \sum_{e_j \in E_1}(|e_j| - 1)$ and $n_1' = n_1 -$ number of middle vertices of $S_1(V_1,E_1)$ and $S_2(V_2,E_2)$ has n_2 vertices and m_2 edges with average degree $\frac{2m_2'}{n_2'}$ where $m_2' = \sum_{e_j \in E_2}(|e_j| - 1)$ and $n_2' = n_2 - n$umber of middle vertices of $S_2(V_2,E_2)$

Since both have equal average degree $\frac{2m_1'}{n_1'} = \frac{2m_1'}{n_1'} = \frac{2(m_1'+m_2')}{n_1'+n_2'} = \frac{2m'}{n'}$

$$LE_e(S) = \sum_{i=1}^{n=n_1+n_2} \left|\mu_i - \frac{2m'}{n'}\right| = \sum_{i=1}^{n_1} \left|\mu_i - \frac{2m_1'}{n_1'}\right| + \sum_{i=n_{1+2}}^{n_2} \left|\mu_i - \frac{2m_2'}{n_2'}\right|$$

$$= LE_e\big(S_1(V_{1,}E_1)\big) + LE_e\big(S_2(V_{2,}E_2)\big)$$

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

REFERENCES

[1] E. Sampathkumar, Semigraphs and their Applications, Report on the DST project, 2000.

[2] B.Y. Bam, N.S. Bhave, On Some problems of Graph Theory in Semigraphs, Ph.D Thesis, University of Pune, 2005.

[3] Y.S. Gaidhani, C.M. Deshpande, B.P. Athawale, Adjacency Matrix of a Semigraph, Electron. Notes Discr. Math. 63 (2017), 399–406.

[4] G.Y.S., D. C.M., P. S., Energy of a semigraph, AKCE Int. J. Graphs Comb. 16 (2019), 41–49.

[5] C.M. Deshpande, Y. Gaidhani, B.P. Athawale, "Incidence matrix of a semigraph, Br. J. Math. Comput. Sci. 9(2015), 12-20.

[6] R. Grone, R. Merris, The Laplacian Spectrum of a Graph II, SIAM J. Discr. Math. 7 (1994), 221-229.

[7] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218–238.

[8] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994), 143–176.

[9] R. Merris, A survey of graph Laplacians, Linear Multilinear Algebra 39 (1995), 19–31.

[10] I. Gutman, B. Zhou, Laplacian energy of a graph, Linear Algebra Appl. 414 (2006), 29–37.