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$H(.,.)$-ϕ-η-MIXED ACCRETIVE MAPPING WITH AN APPLICATION

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Abstract. The motive of this article to introduce the notion called $H(.,.)$-ϕ-η-mixed accretive mappings in Banach spaces. We generalized the idea of proximal-point mappings related with generalized m-accretive mappings to the $H(.,.)$-ϕ-η-mixed accretive mappings and perusal its aspects single-valued property as well as Lipschitz continuity. Since proximal point mapping plays an important role to solve variational inclusion problems. Therefore, we design an iterative algorithm involving introduced proximal point mapping to solve variational inclusion problem. In last, we discuss its convergence with considerable assumptions.

Keywords: $H(.,.)$-ϕ-η-mixed accretive; iterative algorithm; proximal point mapping; variational-like inclusion.

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1. Introduction

It is a satisfactory way that variational inequality theory is a very effectual and strong tool to perusal a broad area of problems come to light in mechanics, differential equations, optimization, optimal control and operation research problems, etc. The study of mixed type variational inequality involving nonlinear operators is very essential. The projection method cannot be utilized because of the nonlinear term to perusal the existence and uniqueness of solutions for

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the generalized mixed type variational inequalities. These crucial points impelled Hassouni and Moudafi [5] to put forward the perturbed approach independent from the projection technique. In this continuation, they investigated and studied the mixed type of variational inequalities, called variational inclusions. The proximal-point mapping technique is a very efficient tool to study variational inclusions and their generalization.

Initially, Huang and Fang [4] introduced generalized m-accretive mapping and give its proximal-point mapping in Banach spaces. Since then many heuristics give the various classes of generalized m-accretive mappings, see for examples [3, 13, 14, 15]. Sun et al. [16] presented a new class of M-monotone mapping in Hilbert spaces. Several such schemes can be found in [1, 12, 18], and references therein.

Recently Husain and Gupta [7] introduced $H(.,.)$-mixed accretive mappings in Banach spaces, a natural extension of m-accretive mapping and focussed on variational inclusions involving discussed mappings.

The present work is impelled by the noble research works discussed above. We look into the notion $H(.,.)$-ϕ-η-mixed accretive mappings and define its proximal-point mapping. Next, we will study its characteristics single-valued property as well as Lipschitz continuity and to show its application we attempt to find the solution of generalized set-valued variational inclusions in real q-uniformly smooth Banach spaces. We construct an iterative algorithm involving introduced proximal-point mapping and prove its convergence with appropriate assumptions. Our work is the extension and refinement of the existing results, e.g. see [1, 7, 8, 9, 10, 11, 18].

2. Preliminaries

Let Y be a real Banach space. Its norm and topological dual space is given by $\|.\|$ and Y^*, respectively. The inner product $\langle .,. \rangle$ signify the dual pair among Y and Y^*.

Definition 2.1. [17] *Let $J_q : Y \multimap Y^*$ be a multi-valued mapping and $q > 1$. Then, J_q is generalized duality mapping, if*

$$J_q(y) \; = \; \{y^* \in Y^* : \; \langle y, y^* \rangle = \|y\|^q, \; \|y^*\| = \|y\|^{q-1}\}, \; \forall \, y \in Y.$$

If $\jmath_2$ is the usual normalized duality mapping on Y, then

$$\jmath_q(y) = \|y\|^{q-1}\jmath_2(y)\ \forall\ y(\neq 0)\ \in Y.$$

If $Y \equiv X$, a real Hilbert space, then $\jmath_2$ reduced to identity mapping on X.

Definition 2.2. [17] *A Banach space Y is called smooth if for every $y \in Y$ with $\|y\| = 1$, there exists a unique $h \in Y^*$ such that $\|h\| = h(y) = 1$.*

Definition 2.3. *Let $\varrho_Y : [0, \infty) \to [0, \infty)$ be a function. Then modulus of smoothness of Y defined as*

$$\varrho_Y(s) = \sup\left\{\frac{\|y + y'\| + \|y - y'\|}{2} - 1 :\ \|y\| \leq 1,\ \|y'\| \leq s\right\}.$$

Definition 2.4. [17] A Banach space Y is called *uniformly smooth* if $\lim_{s\to 0} \varrho_Y(s)/s = 0$ *and q-uniformly smooth for $q > 1$, if there exists $c > 0$ such that $\varrho_Y(s) \leq c\ s^q,\ s \in [0, \infty)$.*
Note that, if Y is uniformly smooth then $\jmath$ is single-valued.

Lemma 2.5. [17] *The real uniformly smooth Banach space Y is q-uniformly smooth if and only if there exists a non-negative constant $c_q > 0$ such that, for every $y, y' \in Y$,*

$$\|y + y'\|^q \leq \|y\|^q + q\langle y', \jmath_q(y)\rangle + c_q\|y'\|^q.$$

In order to proceed our next step, we write basic important concepts and definitions, which will be used in this work.

Definition 2.6. Let single-valued mappings $H,\ \eta : Y \times Y \to Y$, and $Q, R : Y \to Y$, then

(i) $H(Q, .)$ is *η-cocoercive* in regards Q with non-negative constant μ, if

$$\langle H(Qu, x) - H(Qu', x),\ \jmath_q(\eta(u, u'))\rangle \geq \mu\ \|Qu - Qu'\|^q,\ \forall x,\ u,\ u' \in Y;$$

(ii) $H(., R)$ is γ-relaxed$-$*η-accretive* in regards R with non-negative constant γ, if

$$\langle H(x, Ru) - H(x, Ru'),\ \jmath_q(\eta(u, u'))\rangle \geq -\gamma\ \|u - u'\|^q,\ \forall x,\ u,\ u' \in Y;$$

(iii) $H(Q, .)$ is said to be κ_1-*Lipschitz continuous* in regards Q with non-negative constant κ_1, if

$$\|H(Qu, x) - H(Qu', x)\| \leq \kappa_1\ \|u - u'\|,\ \forall x,\ u, u' \in Y;$$

(iv) $H(.,R)$ is κ_2-*Lipschitz continuous* in regards R with non-negative constant κ_2, if

$$\|H(x,Ru)-H(x,Ru')\| \ \leq \ \kappa_2 \ \|u-u'\|, \ \forall x, \ u, \ u' \in Y;$$

(v) η is be τ-*Lipschitz continuous* with $\tau > 0$, if

$$\|\eta(u,u')\| \ \leq \ \tau \ \|u-u'\|, \ \forall u, \ u' \in Y;$$

(vi) Q is α-*expansive* with non-negative constant α, if

$$\|Q(u)-Q(u')\| \ \geq \ \alpha \ \|u-u'\|, \ \forall u, \ u' \in Y.$$

Mapping Q becomes *expansive* when α equal to 1.

Definition 2.7. Let $F, \eta : Y \times Y \rightarrow Y$ be the mappings and $M : Y \times Y \multimap Y$ be the multi-valued mapping. Then

(i) M is m-relaxed η-*accretive* if

$\langle u-u', \ J_q(\eta(x,x'))\rangle \geq -m\|x-x'\|^q, \ \forall x, x' \in Y, u \in M(x,t), u' \in M(x',t),$ for each fixed t $\in$ Y;

(ii) F is ν-relaxed η-*accretive* in regards first component with non-negative constant ν if

$$\langle F(x,u)-F(x',u), \ J_q(\eta(x,x'))\rangle \ \geq \ -\nu\|x-x'\|^q, \ \forall x, \ x', \ u \in Y;$$

(iii) $F(.,.)$ is ε_1-*Lipschitz continuous* in regards first component with non-negative constant ϵ_1, if

$$\|F(x,u)-F(x',u)\| \ \leq \ \epsilon_1 \ \|x-x'\|, \ \forall x, \ x', \ u \in Y;$$

(iv) $F(.,.)$ is ε_2-*Lipschitz continuous* in regards second component with non-negative constant ϵ_2, if

$$\|F(u,x)-F(u,x')\| \ \leq \ \epsilon_2 \ \|x-x'\|, \ \forall x, \ x', \ u \in Y.$$

Definition 2.8. A multi-valued mapping $S : Y \multimap CB(Y)$ is called *D-Lipschitz continuous* with constant $l > 0$, if

$$D(Su,Sv) \leq \ l \ \|u-v\|, \ \forall u, \ v \in Y.$$

3. $H(.,.)$-ϕ-η-Mixed Accretive Mappings

First, we define $H(.,.)$-ϕ-η-mixed accretive mappings and make some assumptions which are needed in subsequent part of the section. Next, we will focus on its properties.

Let assume that $\eta, H : Y \times Y \to Y$, and $\phi, Q, R : Y \to Y$ be single-valued mappings and $M : Y \times Y \multimap Y$ be a multi-valued mapping.

Definition 3.1. Let $H(.,.)$ is η-cocoercive in regards Q with non-negative constant μ and η-relaxed accretive in regards R with non-negative constant γ, then M is called $H(.,.)$-ϕ-η-*mixed accretive* in regards Q and R if

(i) for each fixed t, $\phi o M(.,t)$ is m-relaxed η-accretive in regards first argument ;

(ii) $(H(.,.) + \phi o M(.,t))(Y) = Y$.

Remark 3.2. *If $\phi(u) = \rho u$, $\forall\ u \in Y$ and $\rho > 0$, $M(.,.) = M$ and $\eta(u,u') = u - u'$. Then $H(.,.)$-ϕ-η-mixed accretive becomes $H(.,.)$-mixed accretive mapping, see* [7].

Let us consider the following

Assumption $\mathbf{M_1}$: Let H is η-cocoercive in regards Q with non-negative constant μ and η-relaxed accretive in regards R with non-negative constant γ with $\mu > \gamma$.

Assumption $\mathbf{M_2}$: Let Q is α-expansive.

Assumption $\mathbf{M_3}$: Let η is τ-Lipschitz continuous.

Assumption $\mathbf{M_4}$: Let M is $H(.,.)$-ϕ-η-mixed accretive mapping in regards Q and R for each fixed $t \in Y$

Theorem 3.3. *Let assumptions M_1, M_2 and M_4 hold good with $\ell = \mu\alpha^q - \gamma > m$, then $(H(Q,R) + \phi o M(.,t))^{-1}$ is single-valued.*

Proof. Let $y, z \in (H(Q,R) + \phi o M(.,t))^{-1}(x)$ for any given $x \in Y$. It is obvious that

$$\begin{cases} -H(Qy, Ry) + x \in\ \phi o M(y,t), \\ -H(Qz, Rz) + x \in\ \phi o M(z,t). \end{cases}$$

Since $\phi oM(.,t)$ is m-relaxed η-accretive in the first argument, we have

$$\begin{aligned}
-m\|y-z\|^q &\leq \langle -H(Qy,Ry)+x-(-H(Qz,Rz)+x),\ j_q(\eta(y,z))\rangle \\
&= -\langle H(Qy,Ry)-H(Qz,Rz),\ j_q(\eta(y,z))\rangle \\
&= -\langle H(Qy,Ry)-H(Qz,Ry),\ j_q(\eta(y,z))\rangle \\
&\quad -\langle H(Qz,Ry)-H(Qz,Rz),\ j_q(\eta(y,z))\rangle.
\end{aligned}$$

(3.1)

Since assumption M_1 holds, we have

$$-m\|y-z\|^q \leq -\mu\|Qy-Qz\|^q+\gamma\|y-z\|^q. \tag{3.2}$$

Since assumption M_2 holds, we have

$$\begin{aligned}
-m\|y-z\|^q &\leq -\mu\alpha^q\|y-z\|^q+\gamma\|y-z\|^q \\
&= -(\mu\alpha^q-\gamma)\ \|y-z\|^q \\
0 \leq -(\ell-m)\ \|y-z\|^q &\leq 0, \text{where } \ell = \mu\alpha^q-\gamma.
\end{aligned}$$

Since $\mu > \gamma,\ \alpha > 0$, it follows that $\|y-z\| \leq 0$. We get $y = z$, therefore $(H(Q,R)+\phi oM(.,t))^{-1}$ is single-valued.

Definition 3.4. Let assumptions M_1, M_2 and M_4 hold good with $\ell = \mu\alpha^q-\gamma > m$ then the *proximal-point mapping* $R^{H(.,.)-\phi-\eta}_{M(.,t)} : Y \to Y$ is given as

$$R^{H(.,.)-\phi-\eta}_{M(.,t)}(u) = (H(Q,R)+\phi oM(.,t))^{-1}(u),\ \forall\, u \in Y. \tag{3.3}$$

The next attempt is to prove the Lipschitz continuity of the proximal-point mapping defined by (3.3).

Theorem 3.5. *Let assumptions M_1-M_4 hold good with $\ell = \mu\alpha^q-\gamma > m$ and η is τ-Lipschitz then $R^{H(.,.)-\phi-\eta}_{M(.,t)} : Y \to Y$ is $\frac{\tau^{q-1}}{\ell-m}$-Lipschitz continuous, that is,*

$$\|R^{H(.,.)-\phi-\eta}_{M(.,t)}(y)-R^{H(.,.)-\phi-\eta}_{M(.,t)}(z)\| \leq \frac{\tau^q-1}{\ell-m}\|y-z\|,\ \forall\ y,z \in Y,\ \textit{and fixed } t \in Y.$$

Proof. For given points $y, z \in Y$, It proceed from equation (3.3) that

$$R^{H(.,.)-\phi-\eta}_{M(.,t)}(y) = (H(Q,R) + \phi o M(.,t))^{-1}(y),$$
$$R^{H(.,.)-\phi-\eta}_{M(.,t)}(z) = (H(Q,R) + \phi o M(.,t))^{-1}(z).$$

Let $u_0 = R^{H(.,.)-\phi-\eta}_{M(.,t)}(y)$ and $u_1 = R^{H(.,.)-\phi-\eta}_{M(.,t)}(z)$.

$$\begin{cases} y - H\Big(Q(u_0), R(u_0)\Big) \in \phi o M\ (u_0, t) \\ z - H\Big(Q(u_1), R(u_1)\Big) \in \phi o M\ (u_1, t). \end{cases}$$

Since M is m-relaxed η-accretive in the first arguments, we have

$$\begin{cases} \langle (y - H(Q(u_0), R(u_0))) - (z - H(Q(u_1), R(u_1))), j_q(\eta(u_0, u_1))\rangle \\ \geq -m\ \|u_0 - u_1\|^q, \\ \langle (y - z - H(Q(u_0), R(u_0)) + H(Q(u_1), R(u_1)), j_q(\eta(u_0, u_1))\rangle \\ \geq -m\ \|u_0 - u_1\|^q, \end{cases}$$

which implies

$$\begin{aligned} \langle y - z, j_q(\eta(u_0, u_1))\rangle &\geq \langle H(Q(u_0), R(u_0)) - H(Q(u_1), R(u_1)), j_q(\eta(u_0, u_1))\rangle \\ &\geq -m\|u_0 - u_1\|^q. \end{aligned}$$

Now, we have

$$\begin{aligned} &\|y - z\|\ \|\eta(u_0, u_1)\|^{q-1} \geq \langle y - z,\ \eta(u_0, u_1)\rangle \\ &\geq\ \langle H(Q(u_0), R(u_0)) - H(Q(u_1), R(u_1)), j_q(\eta(u_0, u_1))\rangle - m\ \|u_0 - u_1\|^q \\ &=\ \langle H(Q(u_0), R(u_0)) - H(Q(u_1), R(u_0)),\ j_q(\eta(u_0, u_1))\rangle \\ &\quad + \langle H(Q(u_1), R(u_0)) - H(Q(u_1), R(u_1)), j_q(\eta(u_0, u_1))\rangle - m\ \|u_0 - u_1\|^q. \end{aligned}$$

Since assumption M_1 holds, we have

$$\|y - z\|\ \|u_0 - u_1\|^{q-1} \geq\ \mu\|Q(u_0) - Q(u_1)\|^q - \gamma\|u_0 - u_1\|^q - m\ \|u_0 - u_1\|^q.$$

Since assumptions $M_2,\ M_3$ hold and η is τ-Lipschitz continuous, we have

$$\begin{aligned} \|y - z\|\ \tau^{q-1}\ \|u_0 - u_1\|^{q-1} &\geq (\mu\alpha^q - \gamma)\ \|u_0 - u_1\|^q - m\ \|u_0 - u_1\|^q \\ &\geq\ (\ell - m)\ \|u_0 - u_1\|^q, \end{aligned}$$

where $\ell = (\mu\alpha^q - \gamma)$.

Hence,

$$\|y-z\|\ \tau^{q-1}\|u_0-u_1\|^{q-1} \ \geq \ (\ell - m)\ \|u_0-u_1\|^q, \text{ i.e.}$$
$$\|R^{H(.,.)-\phi-\eta}_{M(.,t)}(y) - R^{H(.,.)-\phi-\eta}_{M(.,t)}(z)\| \leq \frac{\tau^{q-1}}{\ell - m}\ \|y-z\|,\ \forall\, y,z \in Y.$$

Hence, we get the required result.

4. An Application of $H(.,.)$-ϕ-η-Mixed Accretive Mapping

Here we attempt to show that $H(.,.)$-ϕ-η-mixed accretive mapping under acceptable assumptions can be used as a powerful tool to solve variational inclusion problems in Banach space.

Let $S,T,G : Y \multimap CB(Y)$ be the multi-valued mappings, and let $Q,R,\phi : Y \to Y$, $F : Y\times Y \to Y$ and η, $H : Y\times Y \to Y$ be single-valued mappings. Suppose that multi-valued mapping $M : Y \times Y \multimap Y$ be a $H(.,.)$-ϕ-η-mixed accretive mapping in regards Q, R. We consider the following generalized set-valued variational like inclusion problem to find $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$ such that

$$0 \in\ F(v,w) + M(u,t). \tag{4.1}$$

If Y is real Hilbert space and $M(.,t)$ is maximal monotone operator, then the similar problem to (4.1) studied by Huang et al. [6].

If $G \equiv T \equiv 0$, S is identity mapping and $M(.,.) = M(.)$, $F(.,.) = F(.)$, then the problem (4.1) reduced to find $u \in Y$ such that

$$0 \in\ F(u) + M(u), \tag{4.2}$$

considered by Bi et al. [2]. Now, It is understood the appropriate choice of mapping M included in problem (4.1), gives the various variational inclusion problems which have been studied in the recent past, for example, see [14, 15].

Lemma 4.1. *Let mapping $\phi : Y \to Y$ satisfying the properties $\phi(u+v) = \phi(u) + \phi(v)$ and $Ker(\phi) = \{0\}$, where $Ker(\phi) = \{u \in Y :\ \phi(u) = 0\}$. If (u,v,w,t), where $u \in Y$, $v \in S(u)$,*

$w \in T(u)$ and $t \in G(u)$ is a solution of problem (4.1) if and only if (u, v, w, t) satisfies the following relation:

$$u = R^{H(.,.)-\phi-\eta}_{,M(.,t)} [H(Qu, Ru) - \phi o F(v, w)]. \tag{4.3}$$

Proof. Assume that $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$ satisfies the equation (4.3):

$$u = R^{H(.,.)-\phi-\eta}_{,M(.,.)} [H(Qu, Ru) - \phi o F(v, w)].$$

By definition (3.1), we have

$$\begin{aligned}
&u = [H(Q, R) + \phi o M(., t)]^{-1} [H(Qu, Ru) - \phi o F(v, w)] \\
&\Leftrightarrow [H(Qu, Ru) - \phi o F(v, w)] \in [H(Qu, Ru) + \phi o M(u, t)] \\
&\Leftrightarrow 0 \in \phi o F(v, w) + \phi o M(u, t) \\
&\Leftrightarrow \phi^{-1}(0) \in (F(v, w) + M(u, t)) \\
&\Leftrightarrow 0 \in (F(v, w) + M(u, t)).
\end{aligned}$$

Algorithm 4.2. For any given $z_0 \in Y$, we can choose $u_0 \in Y$, $v_0 \in S(u_0)$, $w_0 \in T(u_0)$, $t_0 \in G(u_0)$ and $0 < \epsilon < 1$ such that sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{t_n\}$ satisfy

$$\begin{cases}
u_{n+1} = R^{H(.,.)-\phi-\eta}_{M(.,t)}(z_n), \\
v_n \in S(u_n), \ \|v_n - v_{n+1}\| \leq D(S(u_n), S(u_{n+1})) + \epsilon^{n+1} \|u_n - u_{n+1}\|, \\
w_n \in T(u_n), \ \|w_n - w_{n+1}\| \leq D(T(u_n), T(u_{n+1})) + \epsilon^{n+1} \|u_n - u_{n+1}\|, \\
t_n \in G(u_n), \ \|t_n - t_{n+1}\| \leq D(G(u_n), G(u_{n+1})) + \epsilon^{n+1} \|u_n - u_{n+1}\|, \\
z_{n+1} = H(Qu_n, Ru_n) - \phi o F(v_n, w_n),
\end{cases}$$

where $n \geq 0$, and $D(.,.)$ is the Hausdorff metric on $\mathrm{CB}(Y)$.

Next, we find the convergence of the iterative algorithm for generalized set-valued variational inclusion (4.1).

Theorem 4.3. *Let us consider the problem (4.1) with assumptions M_1-M_3 hold good and $\phi(u + v) = \phi(u) + \phi(v)$ and $Ker(\phi) = \{0\}$. Let assume that*

(i) *S, T and G are l_1, l_2 and l_3 D-Lipschitz continuous, respectively;*

(ii) *$H(Q, R)$ is κ_1, κ_2-Lipschitz continuous in regards A and B, respectively;*

(iii) *$\phi o F$ is is ν-relaxed η-accretive in regards first component;*

(iv) *$\phi o F$ is ϵ_1, ϵ_2-Lipschitz continuous in regards first and second component, respectively;*

(v) $0 < \sqrt[q]{\left[(\kappa_1+\kappa_2)^q + q\nu l_1^q + q\epsilon_1 l_1\left[(\kappa_1+\kappa_2)^{q-1} + \tau^{q-1} l_1^{q-1}\right] + c_q \epsilon_1^q l_1^q\right]} < \frac{(\ell-m)(1-\xi l_3)}{\tau^{q-1}} - \epsilon_2 l_2;$

(vi) $\|R^{H(.,.)-\phi-\eta}_{M(.,z_n)}(u) - R^{H(.,.)-\phi-\eta}_{M(.,z_{n-1})}(u)\| \leq \xi\|z_n - z_{n-1}\|,\ \forall\ z_n, z_{n-1} \in Y, \xi > 0;$

Then problem (4.1) has a solution (u, v, w, t), where $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$, and the iterative sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{t_n\}$, generated by Algorithms 4.2 converges strongly to $(u,\ v,\ w,\ t)$.

Proof. Using Algorithms 4.2 and D-Lipschitz continuity of S, T and G, we have

$$\begin{aligned}\|v_n - v_{n-1}\| &\leq\ D(S(u_n), S(u_{n-1})) + \epsilon^n\|u_n - u_{n-1}\| \\ &\leq\ l_1\ \|u_{n+1} - u_n\| + \epsilon^n\|u_n - u_{n-1}\|.\end{aligned}$$

Similarly we have

$$\|v_n - v_{n-1}\| \leq\ (l_1 + \epsilon^n)\ \|u_n - u_{n-1}\| \tag{4.4}$$

$$\|w_n - w_{n-1}\| \leq\ (l_2 + \epsilon^n)\ \|u_n - u_{n-1}\| \tag{4.5}$$

$$\|t_n - t_{n-1}\| \leq\ (l_3 + \epsilon^n)\ \|u_n - u_{n-1}\|, \tag{4.6}$$

where $n = 1,\ 2,\ .....$

By Lipschitz continuity of proximal point mapping and condition (vi), we have

$$\begin{aligned}\|u_{n+1} - u_n\| \leq\ & \|R^{H(.,.)-\phi-\eta}_{M(.,t_n)}[H(Qu_n, Ru_n) - \phi o F(v_n, w_n)] \\ & -R^{H(.,.)-\phi-\eta}_{M(.,t_{n-1})}[H(Qu_{n-1}, Ru_{n-1}) - \phi o F(v_{n-1}, w_{n-1})]\| \\ \leq\ & \|R^{H(.,.)-\phi-\eta}_{M(.,t_n)}[H(Qu_n, Ru_n) - \phi o F(v_n, w_n)] \\ & -R^{H(.,.)-\phi-\eta}_{M(.,t_n)}[H(Qu_{n-1}, Ru_{n-1}) - \phi o F(v_{n-1}, w_{n-1})]\| \\ & +\|R^{H(.,.)-\phi-\eta}_{M(.,t_n)}[H(Qu_{n-1}, Ru_{n-1}) - \phi o F(v_{n-1}, w_{n-1})]\| \\ & -R^{H(.,.)-\phi-\eta}_{M(.,t_{n-1})}[H(Qu_{n-1}, Ru_{n-1}) - \phi o F(v_{n-1}, w_{n-1})]\|\end{aligned}$$

$$\leq \frac{\tau^{q-1}}{\ell - m}\|H(Qu_n, Ru_n) - \phi oF(v_n, w_n) - (H(Qu_{n-1}, Ru_{n-1}) - \phi oF(v_{n-1}, w_{n-1}))\|$$
$$+\xi\|z_n - z_{n-1}\|$$
$$\leq \frac{\tau^{q-1}}{\ell - m}\|H(Qu_n, Ru_n) - H(Qu_{n-1}, Qu_{n-1}) - (\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n))\|$$

$$+\frac{\tau^{q-1}}{\ell - m}\|\phi oF(v_{n-1}, w_n) - \phi oF(v_{n-1}, w_{n-1})\| + \xi\|z_n - z_{n-1}\|. \tag{4.7}$$

Now, we compute

$$\|H(Qu_n, Ru_n) - H(Qu_{n-1}, Ru_{n-1}) - (\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n))\|^q$$
$$\leq \|H(Qu_n, Ru_n) - H(Qu_{n-1}, Ru_{n-1})\|^q$$
$$-q\langle \phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n), j_q(\eta(v_n, v_{n-1}))\rangle$$
$$-q\langle \phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n),$$
$$j_q[H(Qu_n, Ru_n) - H(Qu_{n-1}, Qu_{n-1})] - j_q(\eta(v_n, v_{n-1}))\rangle$$
$$+c_q\|\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n)\|^q$$
$$\leq \|H(Qu_n, Ru_n) - H(Qu_{n-1}, Ru_{n-1})\|^q$$
$$-q\langle \phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n), j_q(\eta(v_n, v_{n-1}))\rangle$$
$$+q\|\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n)\|$$
$$\times\Big[\|H(Qu_n, Ru_n) - H(Qu_{n-1}, Ru_{n-1})\|^{q-1} + \|\eta(v_n, v_{n-1})\|^{q-1}\Big]$$

$$+c_q\|\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n)\|^q. \tag{4.8}$$

Since $H(Q, R)$ is κ_1, κ_2-Lipschitz continuous in regards Q, R, respectively, We have

$$\|H(Qu_n, Ru_n) - H(Qu_{n-1}, Ru_{n-1})\|^q \leq (\kappa_1 + \kappa_2)^q\|u_n - u_{n-1}\|^q. \tag{4.9}$$

Since $\phi oF(.,.)$ is ν-relaxed η-accretive, then we have

$$\langle \phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n), j_q(\eta(v_n, v_{n-1}))\rangle \geq -\nu\|v_n - v_{n-1}\|$$

$$\geq -\nu(l_1 + \epsilon^n)\|u_n - u_{n-1}\|. \tag{4.10}$$

As $\phi oF(.,.)$ is ϵ_1-Lipschitz continuous in the first argument and using (4.4), we have

$$\|\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n)\| \leq \epsilon_1 \|v_n - v_{n-1}\| \leq \epsilon_1 (l_1 + \epsilon^n)\|u_n - u_{n-1}\|. \tag{4.11}$$

Similarly we have

$$\|\phi oF(v_{n-1}, w_n) - \phi oF(v_{n-1}, w_{n-1})\| \leq \epsilon_2 (l_2 + \epsilon^n)\|u_n - u_{n-1}\|. \tag{4.12}$$

Using Equation (4.9),(4.11) and assumption (M_3), we have

$$\begin{aligned}
&\|\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n)\| \\
&\qquad \times \Big[\|H(Qu_n, Ru_n) - H(Qu_{n-1}, Ru_{n-1})\|^{q-1} + \|\eta(v_n, v_{n-1})\|^{q-1}\Big] \\
&\leq \epsilon_1 (l_1 + \epsilon^n)\|u_n - u_{n-1}\| \times \Big[(\kappa_1 + \kappa_2)^{q-1}\|u_n - u_{n-1}\|^{q-1} + \tau^{q-1}\|v_n - v_{n-1}\|^{q-1}\Big] \\
&\leq \epsilon_1 (l_1 + \epsilon^n)\|u_n - u_{n-1}\| \times \Big[(\kappa_1 + \kappa_2)^{q-1} + \tau^{q-1}(l_1 + \epsilon^n)^{q-1}\Big]\|u_n - u_{n-1}\|^{q-1}
\end{aligned}$$

$$= \epsilon_1 (l_1 + \epsilon^n)\Big[(\kappa_1 + \kappa_2)^{q-1} + \tau^{q-1}(l_1 + \epsilon^n)^{q-1}\Big]\|u_n - u_{n-1}\|^q. \tag{4.13}$$

By using (4.9)-(4.11), (4.13) in equation (4.8), we have

$$\begin{aligned}
&\|H(Qu_n, Ru_n - H(Qu_{n-1}, Ru_{n-1}) - (\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n))\|^q \\
&\qquad \leq (\kappa_1 + \kappa_2)^q \|u_n - u_{n-1}\|^q + q\nu (l_1 + \epsilon^n)^q \|u_n - u_{n-1}\|^q \\
&\qquad\quad + q\epsilon_1 (l_1 + \epsilon^n)\Big[(\kappa_1 + \kappa_2)^{q-1} + \tau^{q-1}(l_1 + \epsilon^n)^{q-1}\Big]\|u_n - u_{n-1}\|^q \\
&\qquad\quad + c_q \epsilon_1^q (l_1 + \epsilon^n)^q \|u_n - u_{n-1}\|^q
\end{aligned}$$

$$\begin{aligned}
&\|H(Qu_n, Ru_n - H(Qu_{n-1}, Ru_{n-1}) - (\phi oF(v_n, w_n) - \phi oF(v_{n-1}, w_n))\| \\
&\quad \leq \Big[(\kappa_1 + \kappa_2)^q + q\nu (l_1 + \epsilon^n)^q + q\epsilon_1 (l_1 + \epsilon^n)\Big[(\kappa_1 + \kappa_2)^{q-1} + \tau^{q-1}(l_1 + \epsilon^n)^{q-1}\Big] \\
&\qquad + c_q \epsilon_1^q (l_1 + \epsilon^n)^q\Big]^{\frac{1}{q}}\|u_n - u_{n-1}\|.
\end{aligned} \tag{4.14}$$

Using condition (vi) and (4.5),(4.6),(4.14) in equation (4.7) becomes

$$\|u_{n+1} - u_n\| \leq \Big[\frac{\tau^{q-1}}{\ell - m} \Big[(\kappa_1 + \kappa_2)^q + q\nu(l_1 + \epsilon^n)^q + q\epsilon_1(l_1 + \epsilon^n)\Big[(\kappa_1 + \kappa_2)^{q-1} + \tau^{q-1}(l_1 + \epsilon^n)^{q-1}\Big] + c_q\epsilon_1^q(l_1 + \epsilon^n)^q\Big]^{\frac{1}{q}} + \epsilon_2(l_2 + \epsilon^n)\Big] + \xi(l_3 + \epsilon^n)\Big]\|u_n - u_{n-1}\|.$$

We can rewrite,

$$\|u_{n+1} - u_n\| \;\leq\; \Theta(\epsilon^n)\;\|u_n - u_{n-1}\|, \text{ where} \tag{4.15}$$

$$\Theta(\epsilon^n) \;=\; \Big[\frac{\tau^{q-1}}{\ell - m} \Big[\sqrt[q]{\Big[(\kappa_1 + \kappa_2)^q + q\nu(l_1 + \epsilon^n)^q + q\epsilon_1(l_1 + \epsilon^n)\Big[(\kappa_1 + \kappa_2)^{q-1} + \tau^{q-1}(l_1 + \epsilon^n)^{q-1}\Big] + c_q\epsilon_1^q(l_1 + \epsilon^n)^q\Big]} + \epsilon_2(l_2 + \epsilon^n)\Big] + \xi(l_3 + \epsilon^n)\Big].$$

Since $0 < \epsilon < 1$, this implies that $\Theta(\epsilon^n) \to \Theta$ as $n \to \infty$, where

$$\Theta \;=\; \Big[\frac{\tau^{q-1}}{\ell - m}\Big[\sqrt[q]{\Big[(\kappa_1 + \kappa_2)^q + q\nu l_1^q + q\epsilon_1 l_1\Big[(\kappa_1 + \kappa_2)^{q-1} + \tau^{q-1} l_1^{q-1}\Big] + c_q\epsilon_1^q l_1^q\Big]} + \epsilon_2 l_2)\Big] + \xi l_3\Big].$$

It is given that $0 < \Theta < 1$, then $\{u_n\}$ is a Cauchy sequence in Y. As Y is a Banach space then $u_n \to u$ as $n \to \infty$.

From equation (4.4)-(4.6) and Algorithm 4.2, the sequences $\{v_n\}$, $\{w_n\}$ and $\{t_n\}$ are also Cauchy sequences in Y. Thus, there exist v, w and t such that $v_n \to v$, $w_n \to w$ and $t_n \to t$ as $n \to \infty$. In the sequel, we will show that $v \in S(u)$. Since $v_n \in S(u_n)$, then

$$\begin{aligned} d(v,\ S(u)) \;&\leq\; \|v - v_n\| \;+\; d(v_n,\ S(u)) \\ &\leq\; \|v - v_n\| \;+\; D(S(u_n),\ S(u)) \\ &\leq\; \|v - v_n\| \;+\; {}_1\,\|u_n - u\| \to\; 0, \text{ as } n \to \infty, \end{aligned}$$

which implies that $d(v,\ S(u)) = 0$. Due to $S(u) \in CB(Y)$, we have $v \in S(u)$. In the same manner, we easily show that $w \in T(u)$ and $t \in G(u)$.

By the continuity of $R_{M(.,t)}^{H(.,.)-\phi-\eta}$, $Q,\ R,\ S,\ T\ G,\ \phi oF,\ \eta$ and M and Algorithms 4.2, we know that $u,\ v,\ w$and t satisfy

$$u \ = \ R_{M(.,t)}^{H(.,.)-\phi-\eta}\ [H(Qu, Ru) - \phi oF(v, w)].$$

Now using Lemma 4.1, (u, v, w, t) is a solution of the problem (4.1). This completes the proof.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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