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ON FRACTIONAL VECTOR ANALYSIS

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Abstract. In this paper, we introduce fractional vector operators: the gradient, the divergence and the curl operators. Further we discuss fractional line integral and prove a fractional form of Green's Theorem, using conformable derivative.

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1. Introduction

Fractional differential equations are very important in applied sciences, that is why there is so much work on fractional calculus and fractional differential equations.

The subject of fractional derivative is as old as calculus. In 1695, L'Hopital asked if the expression $\frac{d^{0.5}}{dx^{0.5}}f$ has any meanings. Since then, many researchers have been trying to generalize the concept of the usual derivative to fractional derivatives. These days, many definitions for the fractional derivative are available. Most of these definitions use an integral form. The most popular definitions are:

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(i) Riemann - Liouville Definition: If n is a positive integer and $\alpha \in [n-1,n)$, the α^{th} derivative of f is given by

$$T_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

(ii) Caputo Definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$T_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Now, all definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property.

In [5], a new definition called $\alpha-$conformable fractional derivative was introduced as follows:

Let $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to R$. For $x \in E$, let: $D^{\alpha} f(x) = \lim\limits_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon}$. If the limit exists then it is called the $\alpha-$ conformable fractional derivative of f at x. For $x = 0$, $D^{\alpha} f(0) = \lim\limits_{x \to 0} D^{\alpha} f(0)$ if such limit exists.

The new definition satisfies:

1.$T_{\alpha}(af+bg) = aT_{\alpha}(f)+bT_{\alpha}(g)$, for all $a,b \in \mathbb{R}$.

2.$T_{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0,1]$ and and f,g be $\alpha-$differentiable at a point t, with $g(t) \neq 0$, we have:

3. $T_{\alpha}(fg) = fT_{\alpha}(g)+gT_{\alpha}(f)$.

4. $T_{\alpha}(\frac{f}{g}) = \frac{gT_{\alpha}(f)-fT_{\alpha}(g)}{g^2}$

We list here the fractional derivatives of certain functions,

(1) 1.$T_{\alpha}(t^p) = p\, t^{p-\alpha}$.

(2) $T_{\alpha}(\sin \frac{1}{\alpha} t^{\alpha}) = \cos \frac{1}{\alpha} t^{\alpha}$.

(3) $T_{\alpha}(\cos \frac{1}{\alpha} t^{\alpha}) = -\sin \frac{1}{\alpha} t^{\alpha}$.

(4) $T_{\alpha}(e^{\frac{1}{\alpha} t^{\alpha}}) = e^{\frac{1}{\alpha} t^{\alpha}}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(t) = 1$. Hence $T_{\frac{1}{2}}(f)(0) = 1$, but $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

Many good generalizations to conformable derivatives and applications were written by many authors. We refer to [1], [2], [4], [5] and [6] for more on that direction. For more on fractional calculus and its applications we refer to [1]-[16].

2. Main Definitions

The concept of line integral and surface integral and the associated theorems like Greens Theorem, and stokes Theorem and Divergence Theorem are very important concepts in physics. The main goal of this paper is to try to put some of such concepts in the fractional form. In this section we introduce the fractional gradient, divergence and curl.

Throughout this paper we write: $D_x^\alpha u$ to denote the conformable $\alpha-$derivative of u with respect to the variable x, where u is a function of several variables with domain $\{(x,y,z) : x,y,z>0\}$. Such space of functions will be denoted by SV. The space of vector fields $F : R^3 \to R^3$, with domain $\{(x,y,z) : x,y,z>0\}$ will be denoted by VF.

Definition 2.1. (i). For $f \in SV$. Then

$$\nabla^\alpha f = D_x^\alpha fi + D_y^\alpha fj + D_z^\alpha fk.$$

This is the fractional gradient. So,

$$\nabla^{\frac{1}{2}}(xyz) = \sqrt{x}yzi + \sqrt{y}xzj + \sqrt{z}xyk.$$

The following Lemma is immediate:

Lemma2.1. (i) ∇^α is linear. (ii) $\nabla^\alpha f = 0$ if f is constant.

Elements of the domain of ∇^α will be called $\alpha-$potentials.

If the second mixed fractional derivatives of f exist and continuos, then we call $\nabla^\alpha f$ to be $\alpha-$conservative force.

Definition 2.2. Let $F = Pi + Qj + Rk \in VF$. Then

$$\nabla^\alpha .F = D_x^\alpha P + D_y^\alpha Q + D_z^\alpha R$$

This is the fractional divergence. So,

$$\nabla^{\frac{1}{2}}.(2\sqrt{x}i + 2\sqrt{y}j + 2\sqrt{z}k = 3.$$

Again, the following Lemma is immediate:

Lemma 2.2. (i).∇^α. is linear. (ii). Let $f \in SV$. Then

$$\nabla^\alpha.(\nabla^\alpha f) = D_x^{2\alpha} f + D_y^{2\alpha} f + D_z^{2\alpha} f.$$

This will be called the fractional Laplacian of f. We write $\nabla^{2\alpha} f$ for $\nabla^\alpha.(\nabla^\alpha f)$.

Functions that satisfy $\nabla^{2\alpha} f = 0$, will be called $\alpha-$ harmonic functions. Clearly $\sqrt{x}+\sqrt{y}+\sqrt{z}$ is $\frac{1}{2}$ - harmonic function.

Definition 2.3. Let $F = Pi + Qj + Rk \in VF$. Then

$$\nabla^{2\alpha} \wedge F = (D_y^\alpha R - D_z^\alpha Q)i - -(D_x^\alpha R - D_z^\alpha P)j + (D_x^\alpha Q - D_y^\alpha P)k.$$

This will be called the fractional curl of F. One can see easily that the curl is linear.

Theorem 2.1. Let $r(\alpha) = \frac{x^\alpha}{\alpha} i + \frac{y^\alpha}{\alpha} j + \frac{z^\alpha}{\alpha} k$. Then

1. $\nabla^\alpha f(\|r(\alpha)\|) = f'(\|r(\alpha)\|)\frac{r(\alpha)}{\|r(\alpha)\|}$
2. $\nabla^\alpha \wedge (\nabla^\alpha f) = 0$ for all nice $f \in SV$
3. $\nabla^\alpha.(\nabla^\alpha \wedge F) = 0$ for all nice $F \in VF$
4. $\nabla^\alpha.(fF) = F.\nabla^\alpha f + f\nabla^\alpha.F$

The proof follows from the nice properties of the conformable derivative.

3. Fractional Line Integral

Let $F = Pi + Qj$, where $P, Q : R^2 \to R$. In this section we want to introduce the fractional line integral of F and study its basic properties.

Let $r(\alpha) = \frac{x^\alpha}{\alpha} i + \frac{y^\alpha}{\alpha} j$. Now:

$$dr(\alpha) = \frac{dx}{x^{1-\alpha}} i + \frac{dy}{y^{1-\alpha}} j.$$

Hence we have:

$$F.dr(\alpha) = \frac{P}{x^{1-\alpha}} dx + \frac{Q}{y^{1-\alpha}} dy.$$

Definition 3.1. Let γ be a piecewise smooth curve in the domain of F which lies in $E = \{(x,y) : x, y > 0\}$. Then:

$$\int_\gamma F.dr(\alpha) = \int_\gamma P dx^\alpha + Q dy^\alpha.$$

where we mean by $\frac{dx^\alpha}{\alpha}$ just $\frac{dx}{x^{1-\alpha}}$. Similarly for $\frac{dy^\alpha}{\alpha}$.

Theorem 3.1. Let Ω be a simply connected open set in $E \subset R^2$. Assume $F : \Omega \to R^2$ be an $\alpha-$conservative force. Then $\int_\gamma F.dr(\alpha)$ is path independent.

Proof. Since F is $\alpha-$conservative, then

$$F = \nabla^\alpha f = D_x^\alpha f i + D_y^\alpha f j$$

for some $f : \Omega \to R$.

Now, if $F = Pi + Qj$, then $P = D_x^\alpha f$, and $Q = D_y^\alpha f$. Thus, using properties of conformable derivative in [8], we get

$$\begin{aligned} F.dr(\alpha) &= D_x^\alpha f \frac{dx}{x^{1-\alpha}} + D_y^\alpha f \frac{dy}{y^{1-\alpha}} \\ &= x^{1-\alpha} f_x \frac{dx}{x^{1-\alpha}} + y^{1-\alpha} f_y \frac{dy}{y^{1-\alpha}} \\ &= f_x dx + f_y dy = df. \end{aligned}$$

Consequently,

$$\int_\gamma F.dr(\alpha) = \int_\gamma df = f(x_2, y_2) - f(x_1, y_1),$$

where γ is joining (x_1, y_1) to (x_2, y_2). Hence the integral is independent of γ.This ends the proof.

Now, we prove:

Theorem 3.2. Fractional Green's Theorem.

Let $F : G \subset R^2 \to R^2$, with $F = Pi + Qj$. Assume that

(i) P, Q are continuos functions of 2 variables

(ii) P, Q have continuos $\alpha-$partial derivatives

(iii) G is simply connected.

Then, for any closed piece wise smooth curve γ in G, we have

$$\int_\gamma F.dr(\alpha) = \int_\gamma P\frac{dx^\alpha}{\alpha} + Q\frac{dy^\alpha}{\alpha} = \int\int_W (D_x^\alpha Q - D_y^\alpha P)\frac{dx^\alpha}{\alpha}\frac{dy^\alpha}{\alpha}$$

Proof. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be any two points on the curve γ. Then $\gamma = \gamma_1 \cup \gamma_2$, where γ_1 joins z_1 to z_2, and γ_2 joins z_2 to z_1.

Now,

$$\begin{aligned}
\int_\gamma F.dr(\alpha) &= \int_\gamma P\frac{dx^\alpha}{\alpha} + Q\frac{dy^\alpha}{\alpha} \\
&= (\int_{\gamma_1} P\frac{dx^\alpha}{\alpha} + Q\frac{dy^\alpha}{\alpha}) - (\int_{\gamma_2} P\frac{dx^\alpha}{\alpha} + Q\frac{dy^\alpha}{\alpha})
\end{aligned}$$

On γ_1, let $x = t$, and $y = g_1(t)$, $a \le t \le b$. Similarly, on γ_2, put $x = t$ and $y = g_2(t)$, $a \le t \le b$.

Consider:

$$\begin{aligned}
\int_{\gamma_1} P\frac{dx^\alpha}{\alpha} - \int_{\gamma_2} P\frac{dx^\alpha}{\alpha} &= \int_a^b P(t, g_1(t))\frac{dt^\alpha}{\alpha} - \int_a^b P(t, g_2(t))\frac{dt^\alpha}{\alpha} \\
&= \int_a^b P(t, g(t)]_{g_2(t)}^{g_1(t)}\frac{dt^\alpha}{\alpha} \\
&= \int_a^b (\int_a^b D_y^\alpha P\frac{dy^\alpha}{\alpha})\frac{dx^\alpha}{\alpha}
\end{aligned}$$

Similarly, we use the same technique for the other piece of the line integral and we get the result. This ends the proof.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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