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ON GRAPH CLIQUISH FUNCTIONS

PIYALI MALLICK*
Department of Mathematics, Government General Degree College, Kharagpur -II, West Bengal, India

Abstract. In the present paper we introduce a new notion of graph cliquish functions from a topological space to a metric space and study its relation with other types of generalized continuity. We also give a characterization of that new notion of generalized continuity on a dense set of points.

Keywords: graph continuity; graph quasi-continuity; quasi-continuity; cliquish functions; graph cliquish functions..
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1. INTRODUCTION AND BASIC NOTATIONS

In 1977 Z. Grande [2] introduced the notion of F- continuity for functions from $[0,1]$ to $\mathbb{R}$. Lately A. Zaharescu [11] called this type generalized continuity appropriately the graph continuity. K. Sakalava [8],[9] gave a relationship between graph continuity and quasi-continuity. A. Mikuka [3] in 2003 introduced the notion of graph quasi-continuity.

In what follows X is a topological space and Y is a metric space with metric d. For a subset $A \subseteq X$, $f|_A$ denotes the restriction of a function $f: X \to Y$ on A. If $G(f)$ denotes the graph of $f: X \to Y$ then the symbol $cl(G(f))$ denotes the closure of $G(f)$ in the product topology of $X \times Y$. By $C(f)$

*Corresponding author
E-mail address: piyali.mallick1@gmail.com
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we denote the set of all points at which $f: X \to Y$ is continuous. The letters $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ stand for the set of all reals, rationals and integers respectively and $S(x, r)$ denotes the open sphere with centre x and radius r.

A function $f: X \to Y$ is said to be

- graph continuous if there exists a continuous function $g: X \to Y$ such that $G(g) \subseteq cl(G(f))$ [7].

-graph quasi-continuous if there exists a quasi-continuous function $g: X \to Y$ such that $G(g) \subseteq cl(G(f))$[3].

-quasi-continuous at a point $x_0 \in X$ if for each $\epsilon > 0$ and each open neighbourhood U of x_0, there exists a non-empty open set $G \subseteq U$ such that $d(f(x), f(x_0)) < \epsilon$ for each $x \in G$ [4].

-cliquish at a point $x_0 \in X$ if for each $\epsilon > 0$ and each open neighbourhood U of x_0, there exists a non-empty open set $G \subseteq U$ such that $d(f(x), f(y)) < \epsilon$ whenever $x, y \in G$ [10].

f is called quasi-continuous (cliquish) if it has this property at each point.

Definition 1.1: A function $f: X \to Y$ is said to be graph cliquish if there exists a cliquish function $g: X \to Y$ such that $G(g) \subseteq cl(G(f))$.

Evidently every cliquish function is graph cliquish. Also, it follows that

Remark 1.1: If a function $f: X \to Y$ is graph cliquish with closed graph then f is cliquish.

2. THE GRAPH CLIQUISH AND OTHER CONTINUITY TYPES

The following implications follow from the above definitions:

$$\begin{array}{ccccc} \text{Continuity} & \Longrightarrow & \text{quasi-continuity} & \Rightarrow & \text{cliquish} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Graph continuity} & \Rightarrow & \text{graph quasi-continuity} & \Rightarrow & \text{graph cliquish} \end{array}$$

And all of these are not invertible.

Example 2.1: Consider the real line $\mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$. Here f is not cliquish but graph continuous.

Example 2.2: Consider the real line $\mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & x \in \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Z}) \\ 2, & \text{otherwise} \end{cases}$$

Here f is graph cliquish but not cliquish. Also f is not graph quasi-continuous.

3. RESULTS

The following results are known:

Result 3.1: If $f: X \to Y$ is cliquish then $X \setminus C(f)$ is of first category [5]. Also, we know that

Result 3.2: In a Baire space the complement of every set of first category is dense [6].

Using these two results it easily follows that

Result 3.3: If X is a Baire space and if $f: X \to Y$ is cliquish then $C(f)$ is dense in X.

Now we can formulate the following properties of a graph cliquish function.

Theorem 3.1: Let $f: X \to Y$ be graph cliquish. Then for any $\varepsilon > 0$ the set $A(f, g, \varepsilon) = \{x \in X : \ d(f(x), g(x)) < \varepsilon\}$ is dense in X, for any cliquish function $g: X \to Y$ with $G(g) \subseteq cl(G(f))$.

Proof: Let $\varepsilon > 0$ and U be a non-empty open set in X. Let $x_0 \in U$. Since g is cliquish at x_0, there exists a non-empty open set $U_1 \subseteq U$ such that $d(g(x), g(y)) < \frac{\varepsilon}{2}$ whenever $x, y \in U_1$.

Let $x_1 \in U_1$. Then $(x_1, g(x_1)) \in cl(G(f))$. So, $\left[U_1 \times S(g(x_1), \frac{\varepsilon}{2})\right] \cap G(f) \neq \varphi$.

Choose $x_2 \in U_1$ such that $d(f(x_2), g(x_1)) < \frac{\varepsilon}{2}$.

Now, $d(f(x_2), g(x_2)) \leq d(f(x_2), g(x_1)) + d(g(x_1), g(x_2)) < \varepsilon$

So, $x_2 \in A(f, g, \varepsilon)$.

Hence $A(f, g, \varepsilon)$ is dense in X.

Remark 3.1: Let $f: X \to Y$ be given and $g: X \to Y$ be a cliquish function such that for any $\varepsilon > 0$, the set $A(f, g, \varepsilon)$ is dense in X. Then it is not necessarily true that $G(g) \subseteq cl(G(f))$.

Example 3.1: Consider $\mathbb{R}$ with the topology $\tau = \{A \subseteq \mathbb{R}: 0 \in A\} \cup \{\varphi\}$ and $\mathbb{R}$ with the usual metric d.

The functions $f, g: (\mathbb{R}, \tau) \to (\mathbb{R}, d)$ are defined as

$$f(x) = 0;\ \forall\, x \in \mathbb{R} \text{ and } g(x) = \begin{cases} 0, & x = 0 \\ 1, & \text{otherwise} \end{cases}$$

g is cliquish. Now, $A(f, g, \varepsilon)$=$\begin{cases} \{0\}, & 0 < \varepsilon \leq 1 \\ \mathbb{R}, & \varepsilon > 1 \end{cases}$

$A(f, g, \varepsilon)$ is dense in $(\mathbb{R}, \tau)$ for any $\varepsilon > 0$. But, $G(g) \nsubseteq cl(G(f))$.

Remark 3.2: In example 3.1, $C(g) = \{0\}$ and $G(g|_{c(g)}) \subseteq cl(G(f|_{c(g)})$

Result 3.4: Let $A(\subseteq X)$ be dense in X. If $f: X \to Y$ is cliquish then $f|_A$ is also cliquish.

Proof: Let $x_0 \in A$, U be an open neighbourhood of x_0 in A and $\varepsilon > 0$.

Now, $U = A \cap U_1$, U_1 is open in X.

Since f is cliquish at x_0, $\exists$ a non-empty open set $G(\subseteq U_1)$ in X such that $d(f(x), f(y)) < \epsilon$ whenever $x, y \in G$.

Since A is dense in X, $A \cap G \neq \varphi$. Also $A \cap G$ is open in A and $d\big((f|_A)(x), (f|_A)(y)\big) < \varepsilon$ whenever $x, y \in A \cap G$.

So, $f|_A$ is cliquish.

Now we can formulate the following characterization of graph cliquish function on a dense set.

Theorem 3.2: Let X be a Baire space and $f: X \to Y$ be given. For a cliquish function $g: X \to Y$ the following conditions are equivalent:

a) $G(g|_{C(g)} \subseteq cl(G(f|_{C(g)})$

b) For any $\varepsilon > 0$, $A(f|_{C(g)},\ g|_{C(g)}, \varepsilon)$ is dense in X.

Proof:

a)$\Rightarrow$ $\boldsymbol{b}$):

It follows from the Result 3.4 and Theorem 3.1.

b)$\Rightarrow$ $\boldsymbol{a}$):

Let $x_0 \in C(g)$, U be an open neighbourhood of x_0 and $\varepsilon > 0$. It is sufficient to show that $[U \times S(g(x_0), \varepsilon)] \cap G(f|_{C(g)}) \neq \varphi$.

Since g is continuous at x_0, there exists an open neighbourhood U_1 of x_0 such that $U_1 \subseteq U$ and $g(U_1) \subseteq S(g(x_0), \frac{\varepsilon}{2})$.

Now $A\left(f|_{C(g)}, g|_{C(g)}, \frac{\varepsilon}{2}\right) = \left\{x \in C(g): d\big(f(x), g(x)\big) < \frac{\varepsilon}{2}\right\}$ is dense in X.

So, $U_1 \cap A\left(f|_{C(g)}, g|_{C(g)}, \frac{\varepsilon}{2}\right) \neq \varphi$.

Choose $x_1 \in U_1 \cap C(g)$ such that $d(f(x_1), g(x_1)) < \frac{\varepsilon}{2}$.

Now, $d\big(f(x_1), g(x_0)\big) \leq d\big(f(x_1), g(x_1)\big) + d(g(x_1), g(x_0)) < \varepsilon$.

So, $(x_1, f(x_1)) \in [U \times S(g(x_0), \varepsilon)] \cap G(f|_{C(g)})$.

Theorem 3.3: Let $f: X \to Y$ be cliquish. Then for any $\varepsilon > 0$ the set $B(f, g, \varepsilon) = \left\{x \in X: d\big(f(x), g(x)\big) \geq \varepsilon\right\}$ is nowhere dense in X for any cliquish function $g: X \to Y$ with $G(g) \subseteq cl(G(f))$.

Proof: Let $\varepsilon > 0$ and U be a non-empty open set in X.

Let $x_0 \in U$. Since g is cliquish at x_0, $\exists$ a non-empty open set $U_1 \subseteq U$ such that $d(g(x), g(y)) < \frac{\epsilon}{3}$ whenever $x, y \in U_1$.

Let $x_1 \in U_1$. Since f is cliquish at x_1, $\exists$ a non-empty open set $U_2 \subseteq U_1$ such that $d(f(x), f(y)) < \frac{\epsilon}{3}$ whenever $x, y \in U_2$.

By Theorem 3.1, $U_2 \cap A(f, g, \frac{\varepsilon}{3}) \neq \varphi$.

Choose $x_2 \in U_2$ such that $d(f(x_2), g(x_2)) < \frac{\epsilon}{3}$.

Let $x_3 \in U_2$.

Then, $d\big(f(x_3), g(x_3)\big) \leq d\big(f(x_3), f(x_2)\big) + d\big(f(x_2), g(x_2)\big) + d\big(g(x_2), g(x_3)\big)$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \varepsilon$$

So, $x_3 \in X \backslash B(f, g, \varepsilon)$.

Hence, $U_2 \cap B(f, g, \varepsilon) = \varphi$.

Thus, $B(f, g, \varepsilon)$ is nowhere dense in X.

Corollary 3.1: If $f: X \to Y$, $g: X \to Y$ are cliquish functions such that $G(g) \subseteq cl(G(f))$ then $A(f, g, \varepsilon)$ is semi-open for any $\varepsilon > 0$.

It follows from the result that the complement of a no-where dense set is semi-open [1].

Theorem 3.4:

Let $f: X \to Y$ be such that the set $B(f, g, \varepsilon)$ is nowhere dense for any $\varepsilon > 0$ and for any cliquish function $g: X \to Y$. Then f is cliquish on X.

Proof: Let $x_0 \in X$, U be an open neighbourhood of x_0 and $\varepsilon > 0$.

Since $g: X \to Y$ is cliquish at x_0, there exists a non-empty open set $U_1 \subseteq U$ such that $d(g(x), g(y)) < \frac{\epsilon}{3}$ for $x, y \in U_1$.

As, $B(f, g, \frac{\varepsilon}{3})$ is nowhere dense, we can find a non-empty open set $U_2 \subseteq U_1$ such that

$$U_2 \cap B\left(f, g, \frac{\varepsilon}{3}\right) = \varphi$$

Then $d(f(x), g(x)) < \frac{\epsilon}{3}$ for $x \in U_2$.

Let $x_1, x_2 \in U_2$.

Then $d\big(f(x_1), f(x_2)\big) \leq d\big(f(x_1), g(x_1)\big) + d\big(g(x_1), g(x_2)\big) + d\big(g(x_2), f(x_2)\big) < \varepsilon$.

Then f is cliquish.

Theorem 3.5: Let $f: X \to Y$ and $g: X \to Y$ be two cliquish functions such that $G(g) \subseteq cl(G(f))$. Then the set $\{x \in X: f(x) \neq g(x)\}$ is of first category.

Proof:

Now, $\{x \in X: f(x) \neq g(x)\}=\bigcup_{n=1}^{\infty} B(f, g, \frac{1}{n})$. The sets $B(f, g, \frac{1}{n})$ is nowhere dense by Theorem 3.3 and so the proof is completed.

Corollary 3.2: Let X be a Baire space. If $f: X \to Y$ and $g: X \to Y$ are cliquish functions such that $G(g) \subseteq cl(G(f))$ then the set $\{x \in X: f(x) = g(x)\}$ is dense in X.

Now, $W = \{x \in X: f(x) = g(x)\} = \mathrm{X} \setminus \{x \in X: f(x) \neq g(x)\}$ is residual. Since X is a Baire space, W is dense in X.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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