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GOMPERTZIAN LAW OF GROWTH FOR RESERVED AREA FISHERY MODEL WITH NON-SELECTIVE HARVESTING IN UNRESERVED AREA

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Abstract: This paper deals with the combined harvesting of two species from a prey-predator fishery model in an unreserved area of a marine aquatic eco-system where in primary phase of life the prey species is in marine protected areas (reserved areas). In a reserved area no harvesting and no predation can occur So prey species can grow smoothly and after that they migrate to unreserved open access areas where harvesting and predation are permitted. These kinds of fisheries are a good process to maintain the prey-predator biomass level up to a good size in the long run. They can also prevent species extinction. Here the prey follows the Gompertz law of growth in both reserved and unreserved areas. Initially the dynamic behaviour of the system was studied under a deterministic case with local stability and bionomic equilibrium. The optimal harvesting policy is studied using Pontryagian's maximum principle. In second part of the work, we investigated the stability of the model under stochastic arena using Gaussian white noise. Finally, a comparison has been made. Moreover, the model has also been discussed through numerical example.

Keywords: prey-predator; Gompertz law of growth; reserved area; optimal harvesting.

2010 AMS Subject Classification: 91B76.

1. INTRODUCTION

The pioneering work for the mathematical modelling of a prey-predator relation in an ecological system was first introduced by Lotka [1] and Volterra [2] and since then there have been numerous studies considering the prey-predator interaction for fisheries and other renewable resources such

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as Nicholson et al.[3], Gurtin and Maccamy [4], De Angelis[5], Dekker[6], Landhal and Hansen[7], Kapur [8] and Maynard & Smith [9,10] etc. Also in last few years researchers have investigated several papers regarding fisheries [11-21] etc. But in the course of time with the rapid changes of environment, the normal eco-system is hugely affected and for this reason, to fulfil the need of society with renewable resources, the researchers have to rethink the management and modelling of the eco-system with the help of modern day technology. As the fishes are good source of low budget protein specially in the developing countries, so unrestricted harvesting is continuously going on in adjacent coastal area. As a result the fisheries eco-system is largely damaged. Besides this man made situation there are also some cause due to environmental uncertainty. For these reasons, to prevent the extinction of some fish species nowadays Reserved-Unreserved area fisheries is a good choice for marine fisheries management. In which from the time of spawning to a certain size the species will be in reserved area which is no predation and no harvesting zone. After that the species will migrate to unreserved area where all kinds of harvesting and predation may occur. It has been observed that this kind of fisheries management is a very good technique to maintain the biomass of the species for long run. Also since the harvesting is a very important part for fishery in modern day life there is also need for some restriction to use the natural resources for a long period of time, for which in some of the places harvesting is prohibited for few months of a year, especially when the species are in juvenile stage.

Considering the harvesting of natural resources Clark [22-30] has done pioneering work on optimal harvesting policy, also Chaudhuri[11,12], Mesterton-Gibbons[31, 32], Ganguli and Chaudhuri[33] etc. in recent times have developed some papers on harvesting.

Till now in mathematical ecology there are only very few research works [34-39] that have been done considering migration. But in these investigations except Kar and Mishra [35] and Sadhukhan *et. al.* [39] others have only considered single species populations. Also there is no such investigation considering Gompertzian law [40] of growth for prey. In our work we have considered Gompertz growth law for prey and Holling type-I response function in unreserved area with combined harvesting of both prey and predator. We also assumed that almost no migration occurred from unreserved zone to reserved zone. We have discussed the local stability, global stability, bionomic equilibrium for the system and after that we have investigated the local stability with time delay and finally justification of the system has been checked with numerical example and computer simulation using MATLAB.

2. MODEL FORMULATION

Consider a fishery habitat, in an aquatic ecosystem, with reserved and unreserved areas. In reserved area, it is considered that no harvesting and predation will take place while the unreserved area is the harvesting and predation zone. Let x, y and z be the respective population size of the prey in unreserved and reserved zone and let be the biomass densities of the predator at time. Let r_1 and r_2 are intrinsic growth rate of prey in unreserved and reserved area $k_1,\ k_2$are the carrying capacity of prey in unreserved and reserved area and r_3 is the birth rate of predator. $\alpha > 0$ and $\beta > 0$ are respectively Predation coefficient and conversion parameter. Let the prey sub-population of unreserved area migrate into reserved area at a rate σ_1 and prey sub population of reserved area migrate into unreserved area at a rate σ_2. Also let $'E'$ be the combined harvesting effort for the fish population in unreserved areas and q_2, q_3 are catchability co-efficient of prey and predator in unreserved area. Again, we assume that in each area prey population follows Gompertzian law of growth. Therefore, with this condition in view the dynamics of the prey-predator system may be written in the form of a system of differential equation as:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x \, ln\frac{k_1}{x} - \sigma_1 x + \sigma_2 y \\ \frac{dy}{dt} &= r_2 y \, ln\frac{k_2}{y} + \sigma_1 x - \sigma_2 y - \alpha yz - Eq_2 y \\ \frac{dz}{dt} &= r_3 z + \beta yz - Eq_3 z \end{aligned} \tag{1}$$

In our model we consider that almost no migration will take place unreserved zone to reserved zone, so $\sigma_2 \approx 0$. With this assumption model takes the form.

$$\begin{aligned} \frac{dx}{dt} &= r_1 x \, ln\frac{k_1}{x} - \sigma_1 x \\ \frac{dy}{dt} &= r_2 y \, ln\frac{k_2}{y} + \sigma_1 x - \alpha yz - Eq_2 y \\ \frac{dz}{dt} &= r_3 z + \beta yz - Eq_3 z \end{aligned} \tag{2}$$

Now the problem is to investigate the system (2) in which all the parameters are positive and system is to be analyzed along with the initial conditions $0 \leq x \leq k_1, 0 < y \leq k_2$ and $z > 0$.

3. EXISTENCE OF STEADY STATES

The equilibrium point of the system (2) is obtained by solving, $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$. The possible solutions of the above system of algebraic equations may be considered as $P_0(0,0,0)$,

$P_1(0, y_{21}, z_{31})$, $P_2(x_{12}, 0, z_{32})$, $P_3(x_{13}, y_{23}, 0)$, $P_4(0, 0, z_{34})$, $P_5(0, y_{25}, 0)$, $P_6(x_{16}, 0, 0)$ and $P_7(x^*, y^*, z^*)$.

From the ecological point of view and for the co-existence of all the species of reserved and unreserved area we are focusing our investigation only on the equilibrium point $P_7(x^*, y^*, z^*)$.

Where P_7 is the non-trivial solution of the algebraic equations:

$$r_1 x \ln\frac{k_1}{x} - \sigma_1 x = 0 \tag{3}$$

$$r_2 y \ln\frac{k_2}{y} + \sigma_1 x - \alpha yz - Eq_2 y = 0 \tag{4}$$

$$r_3 z + \beta yz - Eq_3 z = 0 \tag{5}$$

From equation (3) and (4) we have $x^* = k_1 e^{-\frac{\sigma_1}{r_1}}$, $y^* = \frac{Eq_3 - r_3}{\beta}$, with $Eq_3 > r_3$ and using this in (5) we get $z^* = \frac{\left(r_2 \ln\frac{k_2}{y^*} - Eq_2\right)}{\alpha} + \frac{\sigma_1}{\alpha}\frac{x^*}{y^*}$.

4. Local Stability Analysis

The variational matrix can be written as:

$$V(x, y, z) = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \tag{6}$$

Where,

$V_{11} = \left(r_1 \ln\frac{k_1}{x} - \sigma_1\right) - r_1$, $V_{12} = 0$, $V_{13} = 0$, $V_{21} = \sigma_1$, $V_{22} = \left(r_2 \ln\frac{k_2}{y} - \alpha z - Eq_2\right) - r_2$, $V_{23} = -\alpha y$, $V_{31} = 0$, $V_{32} = \beta z$ and $V_{33} = r_3 + \beta y - Eq_3$.

Therefor for $P_7(x^*, y^*, z^*)$, variational matrix is

$$V(x^*, y^*, z^*) = \begin{pmatrix} -r_1 & 0 & 0 \\ \sigma_1 & -r_2 - \sigma_1 \frac{x^*}{y^*} & -\alpha y^* \\ 0 & \beta z^* & 0 \end{pmatrix} \tag{7}$$

The characteristic equation for (7) can be written as $\lambda^3 + A\lambda^2 + B\lambda + C = 0$

Where, $A = r_1 + \left(r_2 + \sigma_1 \frac{x^*}{y^*}\right)$, $B = r_1\left(r_2 + \sigma_1 \frac{x^*}{y^*}\right) + \alpha\beta y^* z^*$ and $C = r_1 \alpha\beta y^* z^*$.

So, by Routh-Hurwitz condition, P_7 will be stable if $\begin{vmatrix} A & C \\ 1 & B \end{vmatrix} = AB - C > 0$.

Now, $AB - C = \left\{r_1 + \left(r_2 + \sigma_1 \frac{x^*}{y^*}\right)\right\}\left\{r_1\left(r_2 + \sigma_1 \frac{x^*}{y^*}\right) + \alpha\beta y^* z^*\right\} - r_1 \alpha\beta y^* z^*$

$$= r_1^2\left(r_2+\sigma_1\frac{x^*}{y^*}\right)+r_1\left(r_2+\sigma_1\frac{x^*}{y^*}\right)^2+\alpha\beta y^* z^*\left(r_2+\sigma_1\frac{x^*}{y^*}\right)$$

Since, all the parameters and x^*, y^*, z^* are positive, so from the above expression it is clear that

$$AB-C = r_1^2\left(r_2+\sigma_1\frac{x^*}{y^*}\right)+r_1\left(r_2+\sigma_1\frac{x^*}{y^*}\right)^2+\alpha\beta y^* z^*\left(r_2+\sigma_1\frac{x^*}{y^*}\right)>0. \quad (8)$$

Therefore, the interior equilibrium point $P_7(x^*, y^*, z^*)$ is always asymptotically stable.

5. BIONOMIC EQUILIBRIUM

The biological equilibrium of the system (2) is given by the solution of $\frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=0$. Now for bionomic equilibrium (in which the net revenue obtained by selling the harvested species equals to the total cost of harvesting) we have to solve the given system (2) together with the equation in which economic rent is zero for a combine steady state. So if c be the constant fishing cost per unit effort with and , which are the constant prices per unit biomass of the landed prey and predator respectively from the unreserved area, then the economic rent i.e. the revenue at any time is given by

$$\pi(x,y,z,E)=(p_2q_2y+p_3q_3z-c)E \quad (9)$$

6. OPTIMAL HARVESTING POLICY

In this present discussion of optimal harvesting policy, let the present value J of continuous time-stream of revenues is given by

$$J=\int_0^\infty e^{-\delta t}\pi(x,y,z,E,t)dt \quad (10)$$

Where $\pi(x,y,z,E,t)=(p_2q_2y+p_3q_3z-c)E$ and δ denotes the annual discount rate. Now we need to maximize J subject to the system of differential equations (2) with the help of Pontryagin's Maximal Principle [41]. Here the harvesting effort $E(t)$ is the control variable and is subjected to the constraints $0\le E(t)\le E_{max}$, so that $V_t=[0,E_{max}]$ is the control set with E_{max} is the feasible upper limit for the harvesting effort.

The Hamiltonian for this model can be written as

$$H=e^{-\delta t}(p_2q_2y+p_3q_3z-c)E+\mu_1(t)\left[r_1x\,ln\frac{k_1}{x}-\sigma_1x\right]+\mu_2(t)\left[r_2y\,ln\frac{k_2}{y}+\sigma_1x-\alpha yz-Eq_2y\right]+\mu_3(t)[r_3z+\beta yz-Eq_3z] \quad (11)$$

The corresponding adjoint equations are

$$\frac{d\mu_1}{dt} = -\frac{\partial H}{\partial x}, \frac{d\mu_2}{dt} = -\frac{\partial H}{\partial y}, \frac{d\mu_3}{dt} = -\frac{\partial H}{\partial z} \tag{12}$$

Therefore using (11) and (12) and with the help of the biological equilibrium P_7 by omitting super script we have

$$\begin{aligned} \frac{d\mu_1}{dt} &= r_1\mu_1 - \sigma_1\mu_2 \\ \frac{d\mu_2}{dt} &= \sigma_1\mu_2 x - \mu_3\beta z - p_2q_2Ee^{-\delta t} \\ \frac{d\mu_3}{dt} &= \alpha y\mu_2 - p_3q_3Ee^{-\delta t} \end{aligned} \tag{13}$$

The solution of the above system of linear differential equation can be written as

$$\mu_1 = A_1e^{m_1t} + A_2e^{m_2t} + A_3e^{m_3t} + \frac{M_1}{N}e^{-\delta t} \tag{14}$$

In which m_1, m_2 and m_3 are the roots of the cubic equation

$$a_0m^3 + a_1m^2 + a_2m + a_3 = 0 \tag{15}$$

Where,

$a_0 = 1$, $a_1 = -(r_1 + \sigma_1 x)$, $a_2 = r_1\sigma_1 x + \alpha\beta yz$, $a_3 = -r_1\alpha\beta yz$

μ_1 is bounded $m_i < 0$, $i = 1, 2, 3$ or $A_i'^s = 0$.

The Huriwtz matrix for the above cubic equation (15) is

$$\begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{pmatrix} \text{ and } \Delta_1 = a_1\ (< 0),\ \Delta_2 = a_1a_2 - a_3,\ \Delta_3 = a_3(a_1a_2 - a_3) \tag{16}$$

Therefore, the roots of the cubic equation are all real negative or complex conjugate having negative real part if and only if Δ_1, Δ_2 and Δ_3 are positive. But since $\Delta_1 < 0$, so it is quite difficult to check whether $m_i < 0$, so we take $A_i = 0$ $(i = 1, 2, 3)$.

Hence from (14)

$$\mu_1(t) = \frac{M_1}{N}e^{-\delta t} \tag{17}$$

So, by similar process we have

$$\mu_2(t) = \frac{M_2}{N}e^{-\delta t} \tag{18}$$

And

$$\mu_3(t) = \frac{M_3}{N}e^{-\delta t} \tag{19}$$

Where,

$$\frac{M_1}{N} = \frac{\sigma_1[\delta p_2q_2 + \beta z p_3q_3]E}{(r_1+1)(\delta^2 + \delta\sigma_1 x + \alpha\beta yz)} \tag{20}$$

$$\frac{M_2}{N} = \frac{[\delta p_2q_2 + \beta z p_3q_3]E}{(\delta^2 + \delta\sigma_1 x + \alpha\beta yz)} \tag{21}$$

$$\frac{M_3}{N} = \frac{\{p_3q_3(\delta+\sigma_1x)-\alpha p_2q_2y\}E}{(\delta^2+\delta\sigma_1x+\alpha\beta yz)} \tag{22}$$

For positive biological equilibrium P_7, $\delta^2 + \delta\sigma_1x + \alpha\beta yz \neq 0$. Also the shadow price $\mu_i(t)e^{\delta t}, i = 2, 3$. for two species of unreserved area remain bounded as $t \to \infty$ and hence they satisfy the transversality condition at ∞.

So, the Hamiltonian must be maximized for $E \in [0, E_{max}]$. Assuming that the control constraint $0 \leq E(t) \leq E_{max}$ are not binding that means the optimal equilibrium does not occur at $E = 0$ *or* $E = E_{max}$. Therefor we consider singular control.

So,

$$\frac{\partial H}{\partial E} = e^{-\delta t}(p_2q_2y + p_3q_3z - c) - \mu_2q_2y - \mu_3q_3z = 0 \tag{23}$$

Or,

$$e^{-\delta t}\frac{d\pi}{dE} = \mu_2q_2y + \mu_3q_3z \tag{24}$$

Also we have from (9) that

$$\frac{d\pi}{dE} = (p_2q_2y + p_3q_3z - c) \tag{25}$$

This equation (25) explains that the total user cost of harvest per unit effort must be equal to the discounted value of the future profit at the steady-state effort level [41].

Now using (24) and (25) we have

$$e^{-\delta t}(p_2q_2y + p_3q_3z - c) = \mu_2q_2y + \mu_3q_3z \tag{26}$$

Therefore substituting the values of μ_2 and μ_3 from equation (18) and (19) respectively to (26) we get

$$\left(p_2 - \frac{M_2}{N}\right)q_2y + \left(p_3 - \frac{M_3}{N}\right)q_3z = c \tag{27}$$

Above equation (27) together with the system (2) gives the optimal equilibrium population densities at $x = x_\delta, y = y_\delta$ and $z = z_\delta$. So, when $\delta \to \infty$, above equation (27) leads to the result

$$p_2q_2y_\infty + p_3q_3z_\infty = c \tag{28}$$

Which gives $\pi(x_\infty, y_\infty, z_\infty, E) = 0$

Therefore using (27) we have,

$$\pi = (p_2q_2y + p_3q_3z - c) = \frac{(M_2q_2y+M_3q_3z)E}{N} \tag{29}$$

Since $\frac{M_2}{N}$ and $\frac{M_3}{N}$ are of $o(\delta^{-1})$, so π is a decreasing function of $\delta\ (\geq 0)$. We then conclude that $\delta = 0$, leads to the maximization of π.

7. STOCHASTIC MODEL

Now to incorporate stochasticity in our existing system, we are going to perturbed the variables around their respective values corresponding to the positive equilibrium point $P_7(x^*, y^*, z^*)$ in $\mathbb{R}^3_+$, assuming the feasibility and local asymptotic stability of P_7. In our work local asymptotic stability of the system for the non-trivial equilibrium is obvious by the condition of existence of , which we have already checked in previous section. So, using white noise type stochastic perturbation of the variables x, y, z around their equilibrium values x^*, y^*, z^*, which is proportional to the distance of x, y, z from the values x^*, y^*, z^* respectively and with this the system (2) can be represented as

$$\begin{aligned} dx &= \left(r_1 x\, ln\frac{k_1}{x} - \sigma_1 x\right)dt + \nu_1(x - x^*)d\xi_t^1 \\ dy &= \left(r_2 y\, ln\frac{k_2}{y} + \sigma_1 x - \alpha yz - Eq_2 y\right)dt + \nu_2(y - y^*)d\xi_t^2 \\ dz &= (r_3 z + \beta yz - Eq_3 z)dt + \nu_3(z - z^*)d\xi_t^3 \end{aligned} \tag{30}$$

Where $d\xi_t^i = \xi_i(t), i = 1, 2, 3$ are standard wiener process [42- 44] independent to each other, along with the real constants $\nu_i,\ i = 1, 2, 3$. In the next section we investigate the asymptotic stability of the equilibrium point (x^*, y^*, z^*) under stochasticity for the Ito stochastic differential system (30).

8. STOCHASTIC STABILITY OF THE POSITIVE EQUILIBRIUM

Since $P_7(x^*, y^*, z^*)$ is the positive co-existential equilibrium point in $\mathbb{R}^3_+$, then the Ito stochastic differential system (30) can be centred at this positive equilibrium (x^*, y^*, z^*), considering the change of variables as

$$u_1 = x - x^*, u_2 = y - y^*, u_3 = z - z^* \tag{31}$$

So, after linearization, the linearized stochastic differential equations around (x^*, y^*, z^*) is of the form

$$du(t) = f\big(u(t)\big)dt + g\big(u(t)\big)d\xi(t) \tag{32}$$

Where, $u(t) = col\big(u_1(t), u_2(t), u_3(t)\big)$ and

$$f\big(u(t)\big) = \begin{pmatrix} -r_1 & 0 & 0 \\ \sigma_1 & -r_2 - \sigma_1\frac{x^*}{y^*} & -\alpha y^* \\ 0 & \beta z^* & 0 \end{pmatrix} u(t) \tag{33}$$

And

$$g(u) = \begin{pmatrix} \nu_1 u_1 & 0 & 0 \\ 0 & \nu_2 u_2 & 0 \\ 0 & 0 & \nu_3 u_3 \end{pmatrix} \tag{34}$$

In equation (32) the positive equilibrium (x^*, y^*, z^*) corresponds to a trivial solution $u(t) = 0$. Let U be the set $U = (t \geq t_0) \times \mathbb{R}^3, t_0 \in \mathbb{R}^+$. Hence $V \in C_2^0(U)$ is a twice continuously differentiable function with respect to t (*cf.* Afanas'ev [45]).

So, in connection with the above equation (32),

$$\boldsymbol{L}\, V(t,u) = \frac{\partial V(t,u)}{\partial t} + f^T(u)\frac{\partial V(t,u)}{\partial u} + \frac{1}{2} Tr\left[g^T(u)\frac{\partial^2 V(t,u)}{\partial u^2} g(u)\right] \tag{35}$$

where $\frac{\partial V}{\partial u} = col\left(\frac{\partial V}{\partial u_1}, \frac{\partial V}{\partial u_2}, \frac{\partial V}{\partial u_3}\right)$ and $\frac{\partial^2 V}{\partial u^2} = \left(\frac{\partial^2 V}{\partial u_i \partial u_j}\right)$; $i, j = 1,2,3$ and T denotes transposition.

Theorem-1: *Suppose there exists a function $V(t,u) \in C_2^0(U)$ satisfying the inequalities*

$$K_1|u|^p \leq V(t,u) \leq K_2|u|^p,\ \boldsymbol{L}\, V(t,u) \leq -K_3|u|^p,\ K_i > 0, p > 0. \tag{36}$$

Then the trivial solution of (32) is exponentially $p-stable$ for $t \geq 0$.

If in (36), $p = 2$, then the trivial solutions of (32) is globally asymptotically stable in probability (*cf.* Afanas'ev [45]).

Theorem-2: *If $r_1 > \frac{1}{2}\nu_1^2$ and $r_2 > \frac{1}{2}\nu_2^2$, then the zero solution of (32) is asymptotically mean square stable.*

Proof: Let us consider the Lyapunov function $L(u) = \frac{1}{2}\,[w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2]$ (37)

Where w_i are real positive constants. Then the first inequality of (36) is true for $p = 2$.

Now,

$$\boldsymbol{L}\, L(u) = w_1(-r_1 u_1)u_1 + w_2(\sigma_1 u_1 - r_2 u_2 - \alpha y^* u_3)u_2 + w_3(\beta z^* u_2)u_3$$
$$+ \frac{1}{2} Tr\left[g^T(u)\frac{\partial^2 L}{\partial u^2} g(u)\right] \tag{38}$$

In which, $\frac{\partial^2 L}{\partial u^2} = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix}$, so $g^T(u)\frac{\partial^2 L}{\partial u^2} g(u) = \begin{pmatrix} w_1\nu_1^2 u_1^2 & 0 & 0 \\ 0 & w_2\nu_2^2 u_2^2 & 0 \\ 0 & 0 & w_3\nu_3^2 u_3^2 \end{pmatrix}$.

Therefore $\frac{1}{2} Tr\left[g^T(u)\frac{\partial^2 L}{\partial u^2} g(u)\right] = \frac{1}{2}[w_1\nu_1^2 u_1^2 + w_2\nu_2^2 u_2^2 + w_3\nu_3^2 u_3^2]$ (39)

Now choosing $w_2\alpha y^* u_2 u_3 = w_3\beta z^* u_2 u_3 + w_2\sigma_1 u_1 u_2 + \frac{1}{2} w_3\nu_3^2 u_3^2$ in (38) and using (39) in (38) we have

$$\boldsymbol{L}\, L(u) = -\left(r_1 - \frac{1}{2}\nu_1^2\right) w_1 u_1^2 - \left(r_2 - \frac{1}{2}\nu_2^2\right) w_2 u_2^2 \tag{40}$$

This completes the proof of the theorem.

9. NUMERICAL EXPERIMENTS

Let, $r_1 = 6.09$, $r_2 = 4.07$, $r_3 = 2$, $k_1 = 300$, $k_2 = 500$, $\alpha = 0.05$, $\beta = 0.005$, $q_2 = 0.001$, $q_3 = 0.3$, $\sigma_1 = 0.5$, $p_2 = 50$, $p_3 = 100$, $c = 20$, $\delta = 0.5$ and $E = 10$.

With these set of data, the biological equilibrium point for this problem is $(276.40, 200.00, 88.20)$, corresponding bionomic and optimal equilibrium points are respectively (276.4, 8.92, 12.33) and (276.4, 10.19, 28.46).

Also the stability diagram and Phase diagram of system (2) are respectively depicted in Figure-1 & 2.

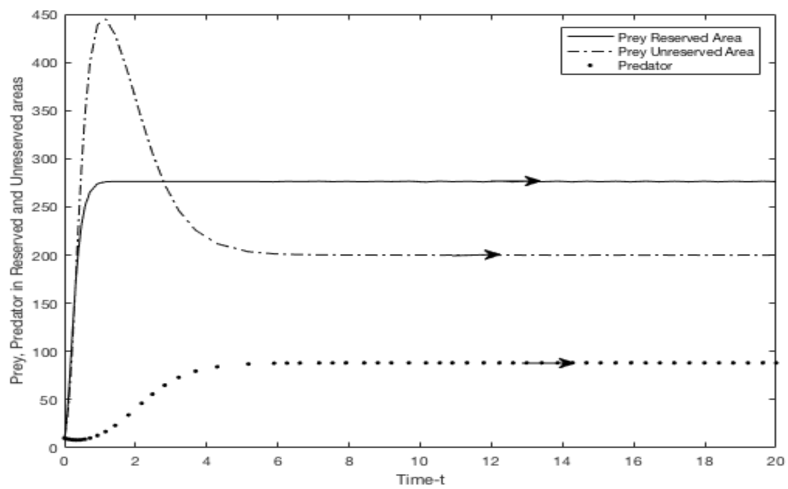

Figure-1: Stability diagram of the system with initial value $x = 10, y = 10, z = 10$.

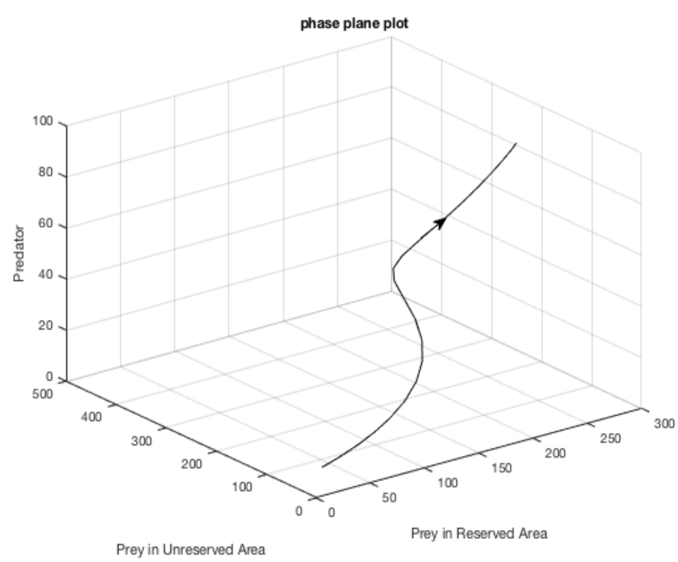

Figure-2: Phase-Space trajectory.

CONCLUSION

In this study a prey-predator model is formulated in marine aquatic ecosystem considering a reserved lake area adjacent to the sea, in which the prey fish species can spend some time in their juvenile stage. As in this age-stage, there is no harvesting and predation in reserved area so the species can grow normally and attained a high level of bio-mass and after that the prey species migrated to open access fisheries zones, where they have to face predation, harvesting and other natural hazards due to environmental fluctuation.

In this work we consider Gompertzian law of growth for prey species, both for reserved and un-reserved area, this growth law is much suitable specially for fish species and Holling type-I response function for prey-predator interaction. Then we check local stability for the system about the interior equilibrium point and find that for any feasible parametric value the system is always stable, which is a very good justification for this kind of fisheries. Also we discuss about the bionomic equilibrium, to justify the system under harvesting phenomena. Then to investigate the effect of combined harvesting in un-reserved area, we study optimal harvesting policy with the help of Pontryagin's Maximal Principle and find the optimal equilibrium point numerically. Also we discuss our model through a numerical example, stability and phase diagram (Figure-1 & 2) under deterministic environment.

Finally using Gaussian white noise we perturb the system to study its stability behavior under environmental fluctuation and with the use of suitable Lyapunov function we find that the system is asymptotically mean square stable under some conditions, which is also justified in real life scenario, because any system which is stable under deterministic environment may not be so under fluctuation and which is one of the main cause for marine species extinction due to Global Warming.

However, to make the model much more closer to the real life situation one can introduce extra complexity to this system by incorporating time delay, fuzziness or randomness etc. in some or all parameters in our model.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] J. Lotka, Elements of Physical Biology, Williams and Wilkins, Baltimore. 1925.

[2] V. Volterra, Variazioni e fluttuazioni del numero di individui in species animali conviventi, Mem. Accd. Lincei. 2 (1926), 31-113.

[3] A. J. Nicholson, V. A. Bailey, J. G. Skellam, M. V. Brain, J. R. Proctor, The simultaneous growth of interacting system, Acta Biotheor. 13 (1960), 131-144.

[4] M. E. Gurtin, R. C. Maccamy, Non-linear age-dependent population dynamics, Arch. Rat. Mech. Anal. 54 (1974), 281-300.

[5] D. L. De Angelis, Global asymptotic stability criteria for models of density-dependent population growth, J. Theor. Biol. 50 (1975), 35-43.

[6] H. A. Dekker, A simple mathematical model of rodent population cycles, J. Math. Biol. 2 (1975), 57-67.

[7] H. D. Landel, B. D. Hansen, A three stage population model with cannibalism, Bull. Math. Biol. 37 (1975), 11-17.

[8] J. N. Kapur, At stability analysis of continuous and discrete population models, Indian J. Pure Appl. Math. 9 (1976), 702-708.

[9] Maynard, J. Smith, Models in Ecology, Cambridge: Cambridge University Press. (1971).

[10] Maynard, J. Smith, Models in Ecology, Cambridge: Cambridge University Press. (1974).

[11] K. S. Chaudhuri, A bioeconomic model of harvesting a multi species fishery. Ecol. Model. 32 (1986), 267–279.

[12] K. S. Chaudhuri, K. S.: Dynamic optimization of combined harvesting of a two species fishery. Ecol. Model. 41 (1987), 21–25.

[13] T. K. Kar, K. S. Chaudhuri, On non-selective harvesting of a multi-species fishery. Int. J. Math. Educ. Sci. Technol. 33(4) (2002), 543–556.

[14] T. K. Kar, K. S. Chaudhuri, Regulation of a prey-predator fishery by taxation. Int. J. Math. Educ. Sci. Technol. 34 (3) (2003), 543–556.

[15] D. Purohit, K. S. Chaudhuri, A bioeconomic model of non-selective harvesting of two competing fish species. Int. J. Math. Educ. Sci. Technol. 33 (4) (2002), 543–556.

[16] K. S. Chaudhuri, S. Saha Ray, On the combined harvesting of a prey-predator system. J. Biol. Syst. 4 (1996), 376–389.

[17] T. K. Kar, S. Mishra, Influence of prey reserve in a prey-predator fishery. Nonlinear Anal., Theory Meth. Appl. 65 (2006), 1725–1735.

[18] T. K. Kar, H. Matsuda, Controllability of a harvested prey-predator system with time delay. J. Biol. Syst. 14 (2) (2006), 243–254.

[19] T. K. Kar, U. K. Pahari, Modelling and analysis of a prey-predator system with age-structure and harvesting. Nonlinear Anal., Real World Appl. 8 (2007), 601–609.

[20] T. K. Kar, Selective harvesting in a prey-predator fishery with time delay. Math. Comput. Model. 38 (2003), 449–458.

[21] T. K. Kar, K. S. Chaudhuri, On selective harvesting of two competing fish species in the presence of environmental fluctuation. Nat. Res. Model. 17 (3) (2004), 251–273.

[22] W. Clark, Profit maximization and the extinction of annual species. Politic. Econ. 18 (1973), 950–961.

[23] C.W. Clark, Possible effects of schooling on the dynamics of exploited fish populations, ICES J. Marine Sci. 36 (1974), 7–14.

[24] W. Clark, Mathematical bio-economics: The optimal management of renewable resources. Wiley, New York, 1976.

[25] C. W. Clark, A delayed-recruitment model of population dynamics, with an application to baleen whale populations, J. Math. Biol. 31 (1976), 381-391.

[26] C. W. Clark, Mathematical bio-economics: The optimal management of renewable resources, John Wiley and Son, New York. (1979).

[27] C. W. Clark, Bioeconomic Modelling and Fisheries Management, Wiley, New York. 1985.

[28] C. W. Clark, The effect of fisherman's quotas on expected catch rate, Marine Resource Econ. 1 (1985), 419-427.

[29] C. W. Clark, Mathematical Bio-economics: The Optimal Management of Renewable Resources, 2nd ed., Wiley, New York. 1990.

[30] C. W. Clark, Renewable resource: Fisheries. In J. C. J. M. van den Bergh, ed., Handbook of Environmental and Resource Economics, Chelttenham, Edward Elgar, England. 1999.

[31] M. Mesterton-Gibbons, On the optimal policy for combined harvesting of independent species. Nat. Res. Model. 2 (1987), 109–134.

[32] M. Mesterton-Gibbons, On the optimal policy for combined harvesting of predator-prey. Nat. Res. Model. 3 (1988), 63–90.

[33] S. Ganguly, K. S. Chaudhuri, Regulations of a single species fishery by taxation, Ecol. Model. 82 (1995), 51-60.

[34] B. Dubay, P. Chandra, P. Sinha, A model for fishery resource with reserve area. Nonlinear Anal., Real World Appl. 4 (2003), 625–637.

[35] T. K. Kar, S. Misra, Influence of prey reserve in a prey-predator fishery. Nonlinear Anal., Theory Meth. Appl. 65 (2006), 1725–1735.

[36] B. Dubay, A prey-predator model with reserved area. Nonlinear Anal., Model. Control 12(4) (2007), 479–494.

[37] Y. Louartassi, N. Elalami, E. Mazoudi, Stability analysis and static output feedback design for a model the fishery resource with reserve area. Appl. Math. Sci. 6 (66) (2012), 3289–3299.

[38] K. Chakraborty, T. K., Kar, Economic perspective of marine reserve in fisheries: a bionomic model. Math. Biosci. 240 (2012), 212-222.

[39] D. Sadhukhan, B. Mondal, M. Maiti, Bio-economic Prey–Predator Fishery Model with Intratrophic Predation, Time Delay in Reserved and Unreserved Area, in: S. Kar, U. Maulik, X. Li (Eds.), Operations Research and Optimization, Springer Singapore, Singapore, 2018: pp. 227–245.

[40] B. Gompertz, On the nature of the function expressive of the law of hu- man mortality, and on a new mode of determining the value of life contingencies, Phil. Trans. R. Soc. Lond. 115 (1825), 513-583.

[41] L. S. Pontryagin, V. S. Boltyanskii, R. V. Gamkrelizre, E. F. Mishchenko, The Mathematical Theory of Optimal Process. Pergamon Press, London, 1962.

[42] I. Gikhman, A. V. Skorokhod, The Theory of Stochastic Processes, Springer, Berlin, vol. I, 1974.

[43] I. Gikhman, A. V. Skorokhod, The Theory of Stochastic Processes, Springer, Berlin, vol. II, 1975.

[44] I. Gikhman, A. V. Skorokhod, The Theory of Stochastic Processes, Springer, Berlin, vol. III, 1979.

[45] V. N. Afanas’ev, V. B. Kolmanowski, V.R. Nosov, Mathematical Theory of Global Systems Design. Kluwer Academic, Dordrecht, 1996.