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FIXED POINTS OF GENERALIZED WEAK CONTRACTIONS IN SYMMETRIC SPACES

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Abstract. In this paper we obtain some fixed point theorems for generalized weak contractions in symmetric spaces without using the completeness of the spaces and continuity conditions. Besides discussing special cases, we observe the usefulness of results on the setting of symmetric spaces.

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1. Introduction

The study of fixed points in symmetric spaces was initiated in the work of Hicks and Rhoades [10]. Recently, the existence of fixed points for single-valued and multi-valued operators in symmetric spaces has been studied extensively (see [1], [3], [5], [6], [11], [12], [14], [15], [19] and many others).

The purpose of this paper is to obtain some fixed and common fixed point theorems in symmetric spaces for a certain class of mappings on general settings. These mappings satisfy the property (E.A) introduced and studied by Aamri and Moutawakil [2] for the

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first time. Further, the completeness of the space and the continuity conditions on the mappings are omitted. We present some examples to show the usefulness of the property (E.A) and symmetric spaces.

2. Preliminaries

Throughout this paper $\mathbb{N}$ denotes the set of natural numbers. For the sake of completeness we recall the following definitions.

Definition 2.1. Let X be non-empty set and $d : X \times X \to [0, \infty)$ a functional. Then d is called a symmetric on X if

(S1): $d(x, y) \geq 0$;

(S2): $d(x, y) = 0$ iff $x = y$;

(S3): $d(x, y) = d(y, x)$.

The pair (X, d) is called a symmetric space of semi-symmetric space.

If we include the triangle inequality in the above definition then we get the usual definition of a metric space. However, a symmetric on X need not be a metric on X. Therefore the class of symmetric spaces is larger than metric spaces.

Example 2.2. The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) = \left\{ \{x_n\} \subset \mathbb{R} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty \right\}$ together with $d : l_p(\mathbb{R}) \times lp(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) := \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}, \text{ (where } x = \{x_n\},\ y = \{y_n\} \in l_p(\mathbb{R}))$$

is a symmetric space.

Let d be a symmetric on a (nonempty) set X and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $\tau(d)$ on X is given by $U \in \tau(d)$ if and only if, for each $x \in U$, $B(x, r) \in U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $\tau(d)$. Note that $\lim\limits_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$.

The following two axioms were given by Wilson [20].

Let (X,d) be a symmetric space.

(W.1): Given x, y and $\{x_n\}$ in X. $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ implies $x = y$.

(W.2): Given $\{x_n\}$, $\{y_n\}$ and x in X. $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ implies that $\lim_{n\to\infty} d(y_n, x) = 0$.

The following definition is essentially due to Aamri and Moutawakil [2] on a metric space.

Definition 2.3. [2] Let X be a symmetric space. Two mappings $S, T : X \to X$ satisfy the property (E.A) if there exits a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ for some } z \in X.$$

When S is an identity mapping on X. We obtain the corresponding definition for a (single) mapping satisfying the property (E.A) [16].

Example 2.4. Let $X = [0,1]$ and $d(x,y) = |x-y|^2$ for all $x, y \in X$. Let $S, T : X \to X$ defined by

$$Sx = x^2 \text{ and } Tx = \tfrac{x}{2} \text{ for all } x \in X.$$

Consider the sequence $x_n = \frac{1}{n}$ for $n \in N$ Then

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0.$$

Therefore S and T satisfy the property (E.A). Notice that d is a symmetric on X. Also it is easy to verify that d is not a metric on X.

There are mappings which do not satisfy the property (E.A).

Example 2.5. Let $X = [2,\infty)$ and $d(x,y) = |x-y|^2$ for all $x, y \in X$. Define $S, T : X \to X$ by $Sx = x+1$ and $Tx = 2x+1$ for all $x \in X$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Therefore $\lim_{n\to\infty} x_n = t-1$ and $\lim_{n\to\infty} x_n = \frac{t-1}{2}$. Then $t = 1$, which is a contradiction since $1 \notin X$. Hence S and T do not satisfy the property (E.A).

Remark 2.6. The class of mappings satisfying property (E.A) contain the class of the well known compatible mappings (see Jungck [13]) as well as the class of non-compatible mappings. The property (E.A) is very useful in the study of fixed points of non expansive mappings. In fact the property (E.A) ensure the existence of a coincidence point for a pair of non expansive type mappings in a metric space [16].

3. Fixed and common fixed point theorems

Throughout this section we shall use the following notations.

(1): X:= A symmetric space (X, d);

(2): Φ:= A class of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying:

(a): φ is continuous and monotone nondecreasing,

(b): $\varphi(t) = 0 \Leftrightarrow t = 0$.

(3): $M_T(x, y) := \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$

(4): $M_{S,T}(x, y) := \max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}$

Definition 3.1. (See [4], [9]). Let X be a metric space and $T : X \to X$. The mapping T is said to be (ψ, φ)-weak contraction if

$$\psi(d(Tx; Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{3.1}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

We extend the definition 3.1 as follows:

Definition 3.2. Let X be a metric space and $T : X \in X$. The mapping T will be called a quasi weak contraction if

$$\psi(d(Tx, Ty)) \leq \psi(M_T(x, y)) - \varphi(M_T(x, y) \tag{3.2}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

When $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ with $k \in (0, 1)$, in the Definition 3.2, we recover the well known quasi-contraction due to Ćirić [7].

We remark that Ćirić's quasi-contraction [7] is considered as the most general among contractions listed in Rhoades [18].

Now we present our first result.

Theorem 3.3. *Let X be a symmetric space and $T : X \in X$ a quasi weak contraction satisfying the property (E.A). Then T has a unique fixed point in X.*

Proof. Since T satisfies the property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \to X$. Since T is a quasi weak contraction on X, by (3.2), we get

$$\psi(d(Tz, Tx_n)) \leq \psi(M_T(z, x_n)) - \varphi(M_T(z, x_n)). \tag{3.3}$$

Notice that

$$\begin{aligned} \lim_{n\to\infty} M_T(z, x_n) &= \lim_{n\to\infty} \max\{d(z, x_n), d(z, Tz), d(x_n, Tx_n), d(z, Tx_n), d(x_n, Tz)\} \\ &= \max\{d(z, z), d(z, Tz), d(z, z), d(z, z), d(z, Tz)\} \\ &= \max\{0, d(z, Tz), 0, 0, d(z, Tz)\} \\ &= d(z, Tz). \end{aligned} \tag{3.4}$$

Since $\psi, \varphi \in \Phi$, (3.3) and (3.4) implies

$$\lim_{n\to\infty} \psi(d(Tz, Tx_n)) = \psi(d(Tz, z)) \leq \psi(d(z, Tz)) - \varphi(d(z, Tz)),$$

a contradiction, unless $d(Tz, z) = 0$. Thus $Tz = z$ and z is a fixed point of T.

To prove the uniqueness, we suppose that T has two distinct fixed points u and v in X.

Again notice that

$$\begin{aligned} M_T(u, v) &= \max\{d(u, v), d(u, Tu), d(v, Tv), d(u, Tv), d(v, Tu)\} \\ &= \max\{d(u, v), d(u, u), d(v, v), d(u, v), d(v, u)\} \\ &= d(u, v). \end{aligned} \tag{3.5}$$

Since $\psi, \varphi \in \Phi$, (3.2) and (3.5) implies

$$\begin{aligned} \psi(d(u, v)) &= \psi(d(Tu, Tv)) \leq \psi(M_T(u, v)) - \varphi(M_T(u, v)) \\ &= \psi(d(u, v)) - \varphi(d(u, v)), \end{aligned}$$

a contradiction, unless $d(u, v) = 0$.

When $\psi(t) = t$ in Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let X be a symmetric space and $T : X \to X$ a mapping satisfying the property (E.A) such that*

$$d(Tx, Ty) \leq M_T(x, y) - \varphi(MT(x, y)) \tag{3.6}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then T has a unique fixed point.

Corollary 3.5. *Let X be a symmetric space and $T : X \to X$ a (Ćirić) quasi-contraction satisfying the property (E.A). Then T has a unique fixed point.*

Proof. It comes from Theorem 3.3, when $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ with $k \in (0, 1)$.

Corollary 3.6. *Let X be a symmetric space and $T : X \to X$ a (ψ, φ)-weak contraction mapping satisfying the property (E.A). Then T has a unique fixed point.*

Proof. It comes from Theorem 3.3, when $M_T(x, y) = d(x, y)$.
When $M_T(x, y) = d(x, y), \psi(t) = t.$ and $\psi(t) = 0$, then we have the following non-expansive type result.

Corollary 3.7. *Let X be a symmetric space and $T : X \to X$ a mapping satisfying the property (E.A) such that*

$$d(Tx, Ty) \leq d(x, y) \tag{3.7}$$

for all $x, y \in X$. Then T has a unique fixed point.

In order to obtain results for common fixed points, we slightly modify the Definition 3.2 as follows:

Definition 3.8. Let X be a symmetric space. Two mappings $S, T : X \to X$ will (be called to) satisfy the modified property (E.A) if there exits a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n = z, \text{ for some } z \in X.$$

Theorem 3.9. *Let X be a symmetric space and $S, T : X \to X$ mappings satisfying the modified property (E.A) such that*

$$\psi(d(Sx, Tx)) \leq \psi(M_{S,T}(x, y)) - \varphi(M_{S,T}(x, y))$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$. Then S and T have a unique common fixed point in X.

Proof. Proof is similar to the proof of Theorem 3.3.

Corollary 3.10. (See also [8] and [17].) *Let X be a symmetric space and $S, T : X \to X$ mappings satisfying the modified property (E.A) such that*

$$d(Sx, Tx) \leq k(M_{S,T}(x, y))$$

for all $x, y \in X$, where $k \in (0, 1)$. Then S and T have a unique common fixed point in X.

Proof. It comes from Theorem 3.9, when $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ with $k \in (0, 1)$. Now we present an example to illustrate our results.

Example 3.11. Let $X = [0, 1)$ and $d(x, y) = |x - y|^2$. It can be easily verified that (X, d) is a symmetric space. Define $T : X \to X$ by $Tx = \frac{x}{2}$. Further, define $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = 2t$ and $\varphi(t) = \frac{3t}{2}$. Consider the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}, n \in N$. Clearly,

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} Tx_n$$

and T satisfies the property (E.A). For all $x, y \in X$.

$$\begin{aligned} \psi(d(Tx, Ty)) &= \frac{|x - y|^2}{2} \leq 2|x - y|^2 - \frac{3|x - y|^2}{2} \\ &= \psi(d(x, y)) - \varphi(d(x, y)). \end{aligned}$$

Therefore T satisfies all the hypotheses of Corollary 3.6 and $T0 = 0$. It is interesting to note that X is not complete and d is not a metric on X.

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