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ON COMMON FIXED POINTS OF SUBCOMPATIBLE MAPPINGS IN S-METRIC SPACES

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Abstract. In this paper, we prove two common fixed point theorems for two pairs of subcompatible mappings which are also subsequentially continuous under different generalized contractions in S-metric spaces. We also give examples to support our results.

Keywords: S-metric space; subcompatible mappings; reciprocally continuous mappings; subsequentially continuous mappings.

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1. Introduction

In 2006, Mustafa and Sims [3] introduced G-metric spaces as a generalization of metric spaces and proved the existence of fixed points under different contractions. In 2012, Sedghi, Shobe and Aliouche [1] introduced a new concept called an S-metric space and studied its some properties. They also stated that an S-metric space is a generalization of a G-metric space. But, in 2014 Dung, Hieu and Radojevic [4] showed by an example that an S-metric space is not a generalization of a G-metric space and conversely. Thus the class of S-metric spaces and the

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class of G-metric spaces are distinct. On the other hand, in 2011, H. Bouhadjera et al. [6] introduced new concepts in metric spaces called subcompatibility and subsequential continuity by generalizing occasionally weakly compatibility and reciprocal continuity respectively.

In this paper, we define subcompatibility and subsequential continuity in S-metric spaces and establish two common fixed point theorems.

In the following, we present some definitions which are frequently used in this paper.

2. PRELIMINARIES

Definition 2.1. [1] Let X be a non empty set. Then we say that a function
S: $X^3 \to [0,\infty)$ is an S-metric on X iff it satisfies the following for all α, β, γ and $\theta \in X$
P1) S(α, β, γ)=0 iff $\alpha = \beta = \gamma$.
P2) S(α, β, γ). $\leq$S(α, α, θ)+S(β, β, θ)+S(γ, γ, θ).
Here (X, S) is called an S-metric space.

Example 2.2. (X, S) is an S-metric space ,
where $X = [0,1]$ and $S(\alpha,\beta,\gamma)= \begin{cases} 0, \text{for } \alpha = \beta = \gamma \\ \max\{\alpha,\beta,\gamma\}, \text{ otherwise} \end{cases}$ for $\alpha,\beta,\gamma \in X$.

Example 2.3. [2] (X, S) is an S-metric space ,
where $X = \mathbb{R}$ and $S(\alpha,\beta,\gamma)=|\alpha-\gamma|+|\beta-\gamma|$ for α, β, $\gamma \in X$.

Example 2.4. (X, S) is an S-metric space ,
where $X = [0,4]$ and $S(\alpha,\beta,\gamma)=\max\{|\alpha-\gamma|,|\beta-\gamma|\}$ for α, β, $\gamma \in X$.

Definition 2.5. [1] We say that a sequence (α_n) in an S-metric space (X, S) converges to some $\alpha \in X$ iff $S(\alpha_n,\alpha_n,\alpha) \to 0$ as $n \to \infty$.

Lemma 2.6. [1] In an S-metric space (X, S), we have S(α, α, γ)=S(γ, γ, α) for all α, $\gamma \in X$.

Lemma 2.7. [1] In an S-metric space (X, S), if there exist sequences (α_n) and (β_n) in X such that $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} \beta_n = \beta$, then $\lim_{n\to\infty} S(\alpha_n,\alpha_n,\beta_n)=S(\alpha,\alpha,\beta)$.

Definition 2.8. We say that two self maps f and R of an S-metric space (X, S) are subcompatible iff there exists a sequence (α_n) in X such that $\lim_{n\to\infty} R(\alpha_n)=\lim_{n\to\infty} f(\alpha_n) = \gamma$ for some $\gamma \in X$ and $\lim_{n\to\infty} S(fR\alpha_n, fR\alpha_n, Rf\beta_n)=0$.

Example 2.9. Consider an S-metric space (X, S),
where $X = [0,\infty)$ and $S(\alpha,\beta,\gamma)=|\alpha-\gamma|+|\beta-\gamma|$ for α, β, $\gamma \in X$.

Define two self maps f, R on X by f$\alpha = \alpha$ and R$\alpha = 1$ for $\alpha \in X$. Now consider $\alpha_n = 1 + \frac{1}{n}$ for $n \in \mathbb{N}$. Then f$\alpha_n = 1 + \frac{1}{n}$ and R$\alpha = 1$ for all $n \in \mathbb{N}$. This will imply that

S(fα_n, fα_n, 1)=S$(1+\frac{1}{n}, 1+\frac{1}{n}, 1)$=$\frac{2}{n}$ and S(Rα_n, Rα_n, $1) = 0$ for every $n \in \mathbb{N}$. It follows that f$\alpha_n \to 1$ and R$\alpha_n \to 1$, as $n \to \infty$. Note that (fR)$\alpha = 1$ and (Rf)$\alpha = 1$ for $\alpha \in X$. This implies that S(fRα_n, fRα_n, Rfα_n)=S(1, 1, 1)=0$\to$ 0, as $n \to \infty$. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty}$R(α_n)=$\lim_{n\to\infty}$f$(\alpha_n) = 1 \in X$ and $\lim_{n\to\infty}$S(fRα_n, fRα_n, Rfα_n)=0. Therefore f and R are subcompatible.

Definition 2.10. [5] We say that a mapping f of an S-metric sapce (X, S) into another S-metric space (Y, S$'$) is continuous at a point $\alpha \in X$ iff (f(α_n)) converges to f(α) in Y, whenever any sequence (α_n) converges to α in X.

Definition 2.11. We say that two self maps f and R of an S-metric space (X, S) are reciprocal continuous iff any sequence (α_n) in X such that $\lim_{n\to\infty}$R(α_n)=$\lim_{n\to\infty}$f$(\alpha_n) = \gamma$ for some $\gamma \in X$ implies $\lim_{n\to\infty}$fR(α_n)=fγ and $\lim_{n\to\infty}$Rf(α_n)=Rγ.

Clearly if f and g are continuous, then they are reciprocal continuous. Its converse in general need not be true.

Definition 2.12. We say that two self maps f and R of an S-metric space (X, S) are subsequentially continuous iff there exists a sequence (α_n) in X such that $\lim_{n\to\infty}$R(α_n)=$\lim_{n\to\infty}$f$(\alpha_n) = \gamma$ for some $\gamma \in X$ satisfying $\lim_{n\to\infty}$fR(α_n)=fγ and $\lim_{n\to\infty}$Rf(α_n)=Rγ.

Clearly if f and g are continuous or reciprocal continuous, then they are subsequentially continuous. In general, its converse need not be true.

Example 2.13 Consider an S-metric space (X, S),

where $X = [0, \infty)$ and S(α, β, γ)=$|\alpha - \gamma| + |\beta - \gamma|$ for $\alpha, \beta, \gamma \in X$.

Now we define f, R:$X \to X$ by f(α)= $\begin{cases} \frac{\alpha}{4}, \text{for } \alpha \in [0,1] \\ 4\alpha - 3, \text{ for } \alpha \in (1, \infty) \end{cases}$ and

R(α)= $\begin{cases} \frac{\alpha}{3}, \text{for } \alpha \in [0,1] \\ 3\alpha - 2, \text{ for } \alpha \in (1, \infty) \end{cases}$ for $\alpha \in X$.

Case(i): We first show that f and R are not continuous.

For this, consider $\alpha_n = 1 + \frac{1}{n}$ for $n \in \mathbb{N}$. Then we have f$\alpha_n = 1 + \frac{4}{n}$ and R$\alpha_n = 1 + \frac{3}{n}$ for $n \in \mathbb{N}$. This will imply that S$(\alpha_n, \alpha_n, 1) = 2|\alpha_n - 1| = 2\left|1 + \frac{1}{n} - 1\right| \to 0$, as $n \to \infty$. This shows that $\alpha_n \to 1$. Note that S(fα_n, fα_n, 1)=S$(1+\frac{4}{n}, 1+\frac{4}{n}, 1)$=$\frac{8}{n}$ and S(Rα_n, Rα_n, 1)=S$(1+\frac{3}{n}, 1+\frac{3}{n}, 1)$=$\frac{6}{n}$ for all $n \in \mathbb{N}$. This imply that f$(\alpha_n) \to 1 \neq \frac{1}{4}$=f(1) and R$(\alpha_n) \to 1 \neq \frac{1}{3}$=R(1). Thus there exists a sequence (α_n) in X such that α_n converges to 1, but f(α_n) does not converge to f(1) and also R(α_n) does not converge to R(1). This shows that f and R are not continuous functions on X.

Case(ii): Now let us show that f and R are subsequentially continuous. For this, consider $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Then we have f$\alpha_n = \frac{1}{4n}$ and R$\alpha_n = \frac{1}{3n}$ for all $n \in \mathbb{N}$. Now look at S(fα_n, fα_n, $0) = \frac{2}{4n}$ and S(Rα_n, Rα_n, $0) = \frac{2}{3n}$ for every $n \in \mathbb{N}$. This will imply that f$\alpha_n \to 0$ and R$\alpha_n \to 0$, as $n \to \infty$. Also note that fR$\alpha_n = \frac{1}{12n}$ and Rf$\alpha_n = \frac{1}{12n}$ and S(fRα_n, fRα_n, $0) = \frac{1}{6n}$ and S(Rfα_n, Rfα_n, $0) = \frac{1}{6n}$ for every $n \in \mathbb{N}$. This will imply that fR$\alpha_n \to$ 0=f0 and Rf$\alpha_n \to$0=R0, as $n \to \infty$. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty}$R(α_n)=$\lim_{n\to\infty}$f$(\alpha_n) = 0 \in X$ for implies $\lim_{n\to\infty}$fR(α_n)=f(0) and $\lim_{n\to\infty}$Rf(α_n)=R(0). Therefore f and R are subsequentially continuous.

Case(iii): Finally, we show that f and R are not reciprocal continuous.

For this, let $\alpha_n = 1 + \frac{1}{n}$ for $n \in \mathbb{N}$. By case(i), we have f$(\alpha_n) \to 1$ and R$(\alpha_n) \to 1$. Now look at fR $\alpha_n = 1 + \frac{1}{12n}$ and Rf$\alpha_n = 1 + \frac{1}{9n}$. This will imply that S(fRα_n, fRα_n, $1) = \frac{24}{n}$ and S(Rfα_n, Rfα_n, $1) = \frac{18}{n}$ for every $n \in \mathbb{N}$. This imply that fR$\alpha_n \to 1$ $\neq$f(1)=$\frac{1}{4}$ and Rfα_n $\to$1$\neq$R1=$\frac{1}{3}$, as $n \to \infty$. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty}$R(α_n)=$\lim_{n\to\infty}$f$(\alpha_n) = 1 \in X$ and $\lim_{n\to\infty}$fR(α_n) $\neq$f(1) and $\lim_{n\to\infty}$Rf$(\alpha_n) \neq R(1)$. Therefore f and R are not reciprocal continuous.

3. Main Results

Theorem 3.1. Suppose in an S-metric space X, there are four self maps f, g, R and T on X satisfying i) S(fα, fα, gβ)$\leq \phi$(S(Rα, Rα, Tβ)) for all $\alpha, \beta \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $0 < \phi(k) < k$ for every $k > 0$

ii) (f, R) and (g, T) are subcompatible

iii) (f, R) and (g, T) are subsequentially continuous.

Then f, g, R and T have a unique common fixed point.

Proof : Since (f, R) is subcompatible, we can find a sequence (α_n) in X such that

$\lim_{n\to\infty} R\alpha_n = \lim_{n\to\infty} f\alpha_n = \gamma$ for some $\gamma \in X$ and $\lim_{n\to\infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n)=0$. Now (g, T) is subcompatible implies that there exists a sequence (β_n) in X such that $\lim_{n\to\infty} g\beta_n = \lim_{n\to\infty} T\beta_n = \delta$ for some $\delta \in X$ and $\lim_{n\to\infty} S(gT\alpha_n, gT\alpha_n, Tg\alpha_n)=0$. Since (f, R) is subsequentially continuous, $(f\alpha_n)$ and $(R\alpha_n)$ converge to γ, we have $\lim_{n\to\infty} fR\alpha_n = f\gamma$ and $\lim_{n\to\infty} Rf\alpha_n = R\gamma$.Similarly, since (g, T) is subsequentially continuous, $(g\alpha_n)$ and $(T\alpha_n)$ converge to δ, we have $\lim_{n\to\infty} gT\alpha_n = g\delta$ and $\lim_{n\to\infty} Tg\alpha_n = T\delta$.

Now $\lim_{n\to\infty} fR\alpha_n = f\gamma$, $\lim_{n\to\infty} Rf\alpha_n = R\gamma$ and $\lim_{n\to\infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n)=0$ imply that $S(f\gamma, f\gamma, R\gamma)=0$ and hence $f\gamma=R\gamma$. Since $\lim_{n\to\infty} gT\alpha_n = g\delta$, $\lim_{n\to\infty} Tg\alpha_n = T\delta$ and $\lim_{n\to\infty} S(gT\alpha_n, gT\alpha_n, Tg\alpha_n)=0$, $S(g\delta, g\delta, T\delta)=0$ and hence $g\delta=T\delta$. For each $n \in \mathbb{N}$, we consider

$$S(f\alpha_n, f\alpha_n, g\beta_n) \leq \phi(S(R\alpha_n, R\alpha_n, T\beta_n)).$$

Letting $n \to \infty$, we have

$S(\gamma, \gamma, \delta) \leq \phi(S(\gamma, \gamma, \delta)) = \phi(0) = 0$, since ϕ is continuous. This implies $S(\gamma, \gamma, \delta) = 0$ and hence $\gamma=\delta$. Now let us show that $f\gamma=\gamma$. For each $n \in \mathbb{N}$, we consider

$$S(f\gamma, f\gamma, g\beta_n) \leq \phi(S(R\gamma, R\gamma, T\beta_n)).$$

Now letting $n \to \infty$, we have

$$\begin{aligned} S(f\gamma, f\gamma, \delta) &\leq \phi(S(R\gamma, R\gamma, \delta)) \\ &= \phi(S(f\gamma, f\gamma, \delta)). \end{aligned}$$

Therefore $S(f\gamma, f\gamma, \delta) \leq \phi(S(f\gamma, f\gamma, \delta))$. This will imply that $S(f\gamma, f\gamma, \gamma) \leq \phi(S(f\gamma, f\gamma, \gamma))$, since

$\gamma = \delta$. If $S(f\gamma, f\gamma, \gamma) \neq 0$, then by definition of ϕ, $S(f\gamma, f\gamma, \gamma) < S(f\gamma, f\gamma, \gamma)$-contradiction. Therefore, we must have $S(f\gamma, f\gamma, \gamma)=0$ and hence $f\gamma = \gamma$.

Now we show that $g\gamma = \gamma$. Note that $S(f\gamma, f\gamma, g\gamma) \leq \phi(S(R\gamma, R\gamma, T\gamma))$. Since $f\gamma=R\gamma$, we have $S(f\gamma, f\gamma, g\gamma) \leq \phi(S(f\gamma, f\gamma, g\gamma))$. This will imply that $S(\gamma, \gamma, g\gamma) \leq \phi(S(\gamma, \gamma, g\gamma))$, since $f\gamma=\gamma$.

If $S(\gamma, \gamma, g\gamma) \neq 0$, then $S(\gamma, \gamma, g\gamma) > 0$. By definition of ϕ, we have $\phi(S(\gamma, \gamma, g\gamma)) < S(\gamma, \gamma, g\gamma)$. This will imply that $S(\gamma, \gamma, g\gamma) < S(\gamma, \gamma, g\gamma)$ -contradiction. Therefore $S(\gamma, \gamma, g\gamma) = 0$ and hence $g\gamma = \gamma$. Since $f\gamma=R\gamma$ and $f\gamma=\gamma$, then $f\gamma=R\gamma = \gamma$ and hence γ is a common fixed point of f and R. Similarly, $g\gamma=\gamma$ and $g\gamma=T\gamma$ imply that $g\gamma=T\gamma=\gamma$ and hence γ is a common fixed point of g and T. Therefore γ is a common fixed point of f, R, g and T.

Let us now show the uniqueness of common fixed point of f, g, R and T. For this, let θ be another common fixed point of f, g, R and T.Then fθ=gθ=Rθ=Tθ=θ and fγ=gγ=Rγ=Tγ=γ. Now we consider S(θ, θ, γ)=S(fθ, fθ, gγ) $\leq$ ϕ(S(Rθ, Rθ, Tγ))=ϕ(S(θ, θ, γ)). If S(θ, θ, γ) $\neq 0$, then we must have S(θ, θ, γ) $<$ S(θ, θ, γ)-contradiction. Therefore $\theta = \gamma$ and the result is proved.

Corollary 3.2. Suppose in an S-metric space X, there are two self maps f and R on X satisfying

i) S(fα, fα, fβ)$\leq$ ϕ(S(Rα, Rα, Rβ)) for all $\alpha,\beta \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $0 < \phi(k) < k$ for every $k > 0$

ii) (f, R) is subcompatible

iii) (f, R) is subsequentially continuous.

Then f and R have a unique common fixed point.

Proof : Follows from the Theorem 3.1 by taking g=f and T=R on X.

Corollary 3.3. Suppose in an S-metric space X, there are four self maps f, g, R and T on X satisfying i) S(fα, fα, gβ)$\leq$ q(S(Rα, Rα, Tβ)) for all $\alpha,\beta \in X$ and for some q$\in [0,1)$

ii) (f, R) and (g, T) are subcompatible

iii) (f, R) and (g, T) are subsequentially continuous.

Then f, g, R and T have a unique common fixed point.

Proof : Let $\phi : [0,\infty) \to [0,\infty)$ be a function defined by $\phi(k)$ =qk for $k \in [0,\infty)$. Clearly it is continuous function on $[0,\infty)$ such that $\phi(0) = 0$ and $0 < \phi(k) < k$ for all $k > 0$. Therefore all the conditions of Theorem 3.1 are satisfied and hence the result proved.

Corollary 3.4. Suppose in an S-metric space X, there is a self map f on X satisfying

i) S(fα, fα, fβ)$\leq$ q(S(α, α, β)) for all $\alpha,\beta \in X$ and for some q$\in [0,1)$

ii) (f, I) is subcompatible

iii) (f, I) is subsequentially continuous, where I is the identity self map on X.

Then f has a unique fixed point.

Proof : Let $\phi : [0,\infty) \to [0,\infty)$ be a function defined by $\phi(k)$ =qk for $k \in [0,\infty)$.Clearly it is continuous function on $[0,\infty)$ such that $\phi(0) = 0$ and $0 < \phi(k) < k$ for all $k > 0$. Now we set g=f and R=T=I on X in the Theorem 3.1 and hence the result proved.

Theorem 3.5. Suppose in an S-metric space X, there are four self maps f, g, R and T on X satisfying

i) (f, R) and (g, T) are subcompatible

ii) (f, R) and (g, T) are subsequentially continuous

iii) $\Psi(S(f\alpha, f\alpha, g\beta)) \leq \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta))$ for all $\alpha,\beta \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $0 < \phi(k) < k$ for every $k > 0$ and $\Psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\Psi(0) = 0$ and

$\chi(\alpha,\beta)=\max\{S(R\alpha,R\alpha,T\beta),S(R\alpha,R\alpha,f\alpha),S(T\beta,T\beta,g\beta),S(R\alpha,R\alpha,g\beta),S(T\beta,T\beta,f\alpha)\}$ for $\alpha,\beta \in X$.

Then f, g, R and T have a unique common fixed point.

Proof : Suppose that (f, R) is subcompatible. Then we can find a sequence (α_n) in X such that $\lim_{n\to\infty} R\alpha_n = \lim_{n\to\infty} f\alpha_n = \gamma$ for some $\gamma \in X$ and $\lim_{n\to\infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n)=0$. Now (g, T) is subcompatible implies that there exists a sequence (β_n) in X such that $\lim_{n\to\infty} g\beta_n = \lim_{n\to\infty} T\beta_n = \delta$ for some $\delta \in X$ and $\lim_{n\to\infty} S(gT\alpha_n, gT\alpha_n, Tg\alpha_n)=0$. Since (f, R) is subsequentially continuous, $(f\alpha_n)$ and $(R\alpha_n)$ converge to γ, we have $\lim_{n\to\infty} fR\alpha_n = f\gamma$ and $\lim_{n\to\infty} Rf\alpha_n = R\gamma$. Similarly, since (g, T) is subsequentially continuous, $(g\beta_n)$ and $(T\beta_n)$ converge to δ, we have $\lim_{n\to\infty} gT\beta_n = g\delta$ and $\lim_{n\to\infty} Tg\beta_n = T\delta$.

Now $\lim_{n\to\infty} fR\alpha_n = f\gamma$, $\lim_{n\to\infty} Rf\alpha_n = R\gamma$ and $\lim_{n\to\infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n)=0$ imply that $S(f\gamma, f\gamma, R\gamma)=0$ and hence $f\gamma=R\gamma$.Since $\lim_{n\to\infty} gT\beta_n = g\delta$, $\lim_{n\to\infty} Tg\beta_n = T\delta$ and $\lim_{n\to\infty} S(gT\beta_n, gT\beta_n, Tg\beta_n)=0$, we have $S(g\delta, g\delta, T\delta)=0$ and hence $g\delta=T\delta$. Now we show that $\gamma = \delta$. For each $n \in \mathbb{N}$, we have

$$\Psi(S(f\alpha_n, f\alpha_n, g\beta_n)) \leq \Psi(\chi(\alpha_n,\beta_n)) - \phi(\chi(\alpha_n,\beta_n)), \text{ where}$$

$$\chi(\alpha_n,\beta_n)=\max\{S(R\alpha_n,R\alpha_n,T\beta_n),S(R\alpha_n,R\alpha_n,f\alpha_n),$$
$$S(T\beta_n,T\beta_n,g\beta_n),S(R\alpha_n,R\alpha_n,g\beta_n),S(T\beta_n,T\beta_n,f\alpha_n)\}$$

for every $n \in \mathbb{N}$. Now letting $n \to \infty$, we have

$\lim_{n\to\infty} \chi(\alpha_n,\beta_n))=\max\{S(\gamma,\gamma,\delta),S(\gamma,\gamma,\gamma),S(\delta,\delta,\delta),S(\gamma,\gamma,\delta),S(\delta,\delta,\gamma)\}=S(\gamma,\gamma,\delta)$. Then we have

$\Psi(S(\gamma, \gamma, \delta)) \leq \Psi(S(\gamma, \gamma, \delta))-\phi(S(\gamma, \gamma, \delta))$. This will imply that $\phi(S(\gamma, \gamma, \delta)) \leq 0$ and hence $\phi(S(\gamma, \gamma, \delta))=0$.

By definition of ϕ, we must have $S(\gamma, \gamma, \delta)=0$ and hence $\gamma = \delta$. Now let us show that $f\gamma = \gamma$. For each $n \in \mathbb{N}$, we have

$$\Psi(S(f\gamma, f\gamma, g\beta_n)) \leq \Psi(\chi(\gamma,\beta_n)) - \phi(\chi(\gamma,\beta_n)), \text{ where}$$

$\chi(\gamma,\beta_n)$=max$\{$S(Rγ,Rγ,Tβ_n), S(Rγ,Rγ,fγ),

S(Tβ_n,Tβ_n,gβ_n), S(Rγ,Rγ,gβ_n), S(Tβ_n,Tβ_n,fγ)$\}$

for every $n \in \mathbb{N}$.Now letting $n \to \infty$, we have

$\lim_{n\to\infty} \chi(\gamma,\beta_n)$=max$\{$S(R$\gamma$,R$\gamma$,$\gamma$), S(R$\gamma$,R$\gamma$,f$\gamma$), S($\gamma$,$\gamma$,$\gamma$), S(R$\gamma$,R$\gamma$,$\gamma$), S($\gamma$,$\gamma$,f$\gamma$)$\}$

=max$\{$S(fγ,fγ,γ), S(fγ,fγ,fγ), S(γ,γ,γ), S(fγ,fγ,γ), S(γ,γ,fγ)$\}$=S(fγ,fγ,δ).

Then we have Ψ(S(fγ, fγ, γ))$\leq$ Ψ(S(fγ, fγ, γ))-ϕ(S(fγ, fγ, γ)). This will imply that ϕ(S(fγ, fγ, γ)) ≤ 0 and hence ϕ(S(fγ, fγ, γ))=0.By definition of ϕ, we must have S(fγ, fγ, γ)=0 and hence f$\gamma = \gamma =$ Rγ. Now we show that g$\gamma = \gamma$. For this, we have

Ψ(S(fγ, fγ, gγ))$\leq$ $\Psi(\chi(\gamma,\gamma)) - \phi(\chi(\gamma,\gamma))$, where

$\chi(\gamma,\gamma)$=max$\{$S(Rγ,Rγ,Tγ), S(Rγ,Rγ,fγ), S(Tγ,Tγ,gγ), S(Rγ,Rγ,gγ), S(Tγ,Tγ,fγ)$\}$

=max$\{$S(γ,γ,gγ), S(γ,γ,γ), S(gγ,gγ,gγ), S(γ,γ,gγ), S(gγ,gγ,γ)$\}$ =S(gγ,gγ,γ).

Therefore Ψ(S(fγ, fγ, gγ))$\leq$ Ψ(S(gγ, gγ, γ))-ϕ(S(gγ, gγ, γ)). This will imply that Ψ(S(γ, γ, gγ))$\leq$ Ψ(S(gγ, gγ, γ))-ϕ(S(gγ, gγ, γ)). It follows that ϕ(S(gγ, gγ, γ))$\leq$ 0. This implies that ϕ(S(gγ, gγ, γ))=0 and hence g$\gamma = \gamma$=Tγ.

Now let us show the uniqueness of common fixed point of f, g, R and T. For this, let $\rho \in X$ be another common fixed point of f, g, R and T.Then fρ=gρ=Rρ=Tρ=ρ and fγ=gγ=Rγ=Tγ=γ. Note that Ψ(S(γ, γ, ρ))$\leq$ $\Psi(\chi(\gamma,\rho)) - \phi(\chi(\gamma,\rho))$, where

$\chi(\gamma,\rho)$=max$\{$S(Rγ,Rγ,Tρ), S(Rγ,Rγ,fγ), S(Tρ,Tρ,gρ), S(Rγ,Rγ,gρ), S(Tρ,Tρ,fγ)$\}$

=max$\{$S(γ,γ,ρ), S(γ,γ,γ), S(ρ,ρ,ρ), S(γ,γ,ρ), S(ρ,ρ,γ)$\}$=S(γ,γ,ρ).

Therefore Ψ(S(γ, γ, ρ))$\leq$ Ψ(S(γ, γ, ρ))-ϕ(S(γ, γ, ρ)). This will imply that ϕ(S(γ, γ, ρ))$\leq$ 0 and hence ϕ(S(γ, γ, ρ))=0. Then by definition of ϕ, S(γ, γ, ρ)=0 and therefore $\gamma = \rho$. Hence the result is proved.

Corollary 3.6. Suppose in an S-metric space X, there are three self maps f, g and R on X satisfying

i) (f, R) and (g, R) are subcompatible

ii) (f, R) and (g, R) are subsequentially continuous

iii) Ψ(S(fα, fα, gβ))$\leq$ $\Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta))$ for all $\alpha,\beta \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $0 < \phi(k) < k$ for every $k > 0$ and $\Psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\Psi(0) = 0$ and

$\chi(\alpha,\beta)$=max$\{S(R\alpha,R\alpha,R\beta),S(R\alpha,R\alpha,f\alpha),S(R\beta,R\beta,g\beta),S(R\alpha,R\alpha,g\beta),S(R\beta,R\beta,f\alpha)\}$ for $\alpha,\beta \in X$.

Then f, g and R have a unique common fixed point.

Proof : Follows from the Theorem 3.5 by taking T=R.

Now we give examples in support of main results.

Example 3.7. Consider an S-metric space (X, S),

where $X=[0,1]$ and $S(\alpha,\beta,\gamma)= \begin{cases} 0, \text{for } \alpha=\beta=\gamma \\ \max\{\alpha,\beta,\gamma\}, \text{ otherwise} \end{cases}$ for all $\alpha,\beta,\gamma \in X$.

Define four self maps f, g, , R and T on X as follows:

For $\alpha \in X$, $f\alpha=\frac{\alpha}{6}$, $g\alpha=\frac{\alpha}{6}$, $T\alpha=\alpha$ and $R\alpha=\frac{\alpha}{2}$. We also define $\phi:[0,\infty)\to[0,\infty)$ by $\phi(\alpha)=\frac{\alpha}{2}$ for $\alpha\in[0,\infty)$. Clearly ϕ is continuous on $[0,\infty)$ satisfying $\phi(0)=0$ and $0<\phi(\alpha)<\alpha$ for all $\alpha>0$. Let $\alpha,\beta\in X$. Now consider the following cases.

Case(i): Let $\alpha<\beta$. Then we have

$$S(f\alpha, f\alpha, g\beta)=\max\{\tfrac{\alpha}{6},\tfrac{\alpha}{6},\tfrac{\beta}{6}\}=\tfrac{1}{6}\max\{\alpha,\alpha,\beta,\}=\tfrac{\beta}{6}$$

and $\phi(S(R\alpha, R\alpha, T\beta))=\frac{1}{2}S(R\alpha, R\alpha, T\beta)=\frac{1}{2}\max\{\frac{\alpha}{2},\frac{\alpha}{2},\beta\}=\frac{\beta}{2}$, since $\frac{\alpha}{2}<\frac{\beta}{2}<\beta$.

Therefore $S(f\alpha, f\alpha, g\beta)\leq \phi(S(R\alpha, R\alpha, T\beta))$.

Now consider the case $\alpha>\beta$. This will imply that

$$S(f\alpha, f\alpha, g\beta)=\max\{\tfrac{\alpha}{6},\tfrac{\alpha}{6},\tfrac{\beta}{6}\}=\tfrac{1}{6}\max\{\alpha,\alpha,\beta\}=\tfrac{\alpha}{6}$$

and $\phi(S(R\alpha, R\alpha, T\beta))=\frac{1}{2}S(R\alpha, R\alpha, T\beta)=\frac{1}{2}\max\{\frac{\alpha}{2},\frac{\alpha}{2},\beta\}$.

subcase(i) : Let $\frac{\alpha}{2}>\beta$. Then we must have

$\phi(S(R\alpha, R\alpha, T\beta))=\frac{1}{2}(\frac{\alpha}{2})=\frac{\alpha}{4}\geq S(R\alpha, R\alpha, T\beta)$.

subcase(ii) : Let $\frac{\alpha}{2}<\beta$. Then we have

$\phi(S(R\alpha, R\alpha, T\beta))=\frac{\beta}{2}>\frac{\alpha}{6}= S(R\alpha, R\alpha, T\beta)$. From both cases, we conclude that $S(f\alpha, f\alpha, g\beta)\leq \phi(S(R\alpha, R\alpha, T\beta))$ for all $\alpha,\beta\in X$.

Case(ii):Consider $\alpha_n=\frac{1}{n}$ for $n\in\mathbb{N}$. Then $f\alpha_n=\frac{1}{6n}$ and $R\alpha_n=\frac{1}{2n}$ for $n\in\mathbb{N}$. This will imply that $S(f\alpha_n, f\alpha_n,0)=\max\{\frac{1}{6n},\frac{1}{6n},0\}=\frac{1}{6n}$ and $S(R\alpha_n, R\alpha_n,0)=\max\{\frac{1}{2n},\frac{1}{2n},0\}=\frac{1}{2n}$ for $n\in\mathbb{N}$. This shows $f\alpha_n\to 0$ and $R\alpha_n\to 0$, as $n\to\infty$. Now look at $fR\alpha_n=\frac{1}{12n}$ and $Rf\alpha_n=\frac{1}{12n}$ for $n\in\mathbb{N}$. It follows that $S(fR\alpha_n, fR\alpha_n, Rf\alpha_n)=0$ for all $n\in\mathbb{N}$. This will imply that $S(fR\alpha_n, fR\alpha_n, Rf\alpha_n)$ $\to 0$, as $n\to\infty$. Thus there exists a sequence (α_n) in X such that $\lim\limits_{n\to\infty}R(\alpha_n)=\lim\limits_{n\to\infty}f(\alpha_n)=0\in X$

and $\lim_{n\to\infty}$ S(fRα_n, fRα_n, Rfα_n)=0. Therefore (f, R) is subcompatible.

Also note that fR$\alpha_n \to$f(0)=0 and Rf$\alpha_n \to$R(0)=0. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty}$R(α_n)=$\lim_{n\to\infty}$f$(\alpha_n) = 0 \in X$ and also $\lim_{n\to\infty}$fR(α_n)=f(0) and $\lim_{n\to\infty}$Rf(α_n)=R(0). Therefore (f, R) is subsequentially continuous.

Case(iii):Now we show that the pair (g, T) is both subcompatible and subsequentially continuous. For this, we consider $\alpha_n = \frac{1}{1+n}$ for $n \in \mathbb{N}$. Then g$\alpha_n = \frac{1}{6(n+1)}$ and T$\alpha_n = \frac{1}{n+1}$ for $n \in \mathbb{N}$. This will imply that S(gα_n, gα_n,0)=max$\{\frac{1}{6(n+1)}, \frac{1}{6(n+1)}, 0\} = \frac{1}{6(n+1)}$ and

S(Tα_n, Tα_n,0)=max$\{\frac{1}{1+n}, \frac{1}{1+n}, 0\} = \frac{1}{1+n}$ for $n \in \mathbb{N}$.T his shows g$\alpha_n \to 0$ and T$\alpha_n \to 0$. Now look at gT$\alpha_n = \frac{1}{6(n+1)}$ and Tg$\alpha_n = \frac{1}{6(n+1)}$ for $n \in \mathbb{N}$. It follows that S(gTα_n, gTα_n, Tgα_n)=0 for all $n \in \mathbb{N}$.This will imply that S(gTα_n, gTα_n, Tgα_n) $\to 0$, as $n \to \infty$. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty}$g(α_n)=$\lim_{n\to\infty}$T$(\alpha_n) = 0 \in X$ and $\lim_{n\to\infty}$S(gTα_n, gTα_n, Tgα_n)=0. This shows that (g T) is subcompatible. Also note that gT$\alpha_n \to$g(0)=0 and Tg$\alpha_n \to$T(0)=0. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty}$T(α_n)=$\lim_{n\to\infty}$g$(\alpha_n) = 0 \in X$ and also $\lim_{n\to\infty}$gT(α_n)=g(0) and $\lim_{n\to\infty}$Tg(α_n)=T(0). Hence (g, T) is subsequentially continuous.Therefore the hypothesis of the Theorem 3.1 is satisfied and f, g, R and T have a unique common fixed point, namely zero.

Example 3.8. Consider an S-metric space (X, S),

where $X = [2,13)$ and S(α,β,γ)=$|\alpha-\gamma|+|\beta-\gamma|$ for $\alpha,\beta,\gamma \in X$.

Now we define f, g, R and T:$X \to X$ by f(α)= $\begin{cases} 2, \text{for } \alpha \in \{2\}\cup(3,13) \\ 8, \text{ for } \alpha \in (2,3] \end{cases}$,

g(α)= $\begin{cases} 2, \text{for } \alpha \in \{2\}\cup(3,13) \\ 3, \text{ for } \alpha \in (2,3] \end{cases}$, R(α)= $\begin{cases} 2, \text{for } \alpha = 2 \\ 9, \text{ for } \alpha \in (2,3] \\ \frac{\alpha+3}{2}, \text{ for } \alpha \in (3,13) \end{cases}$ and

T(α)= $\begin{cases} 2, \text{for } \alpha = 2 \\ 7, \text{ for } \alpha \in (2,3] \\ \alpha-1, \text{ for } \alpha \in (3,13) \end{cases}$ for $\alpha \in X$.

Also we define $\phi, \Psi : [0,\infty) \to [0,\infty)$ by $\phi(k) = 2k$ and $\Psi(k) = \frac{2k}{7}$ for $k \in [0,\infty)$. Note that $\chi(\alpha,\beta)$=max$\{$S(Rα,Rα,Tβ),S(Rα,Rα,fα),S(Tβ,Tβ,gβ),S(Rα,Rα,gβ),S(Tβ,Tβ,fα)$\}$

for $\alpha,\beta \in X$. Clearly ϕ and Ψ are continuous satisfying $\phi(0) = 0 = \Psi(0)$ and $0 < \phi(k) < k$ for every $k > 0$. Now consider the following cases.

Case(I):Consider the first sub case for $\alpha = \beta = 2$. Then we have

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = \Psi(\mathrm{S}(2,2,2)) = 0 \leq \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta)) = 2\chi(\alpha,\beta) - \frac{2}{7}\chi(\alpha,\beta)$.

Subcase(ii) : Let $\alpha = 2$ and $\beta \in (2,3]$. Then $\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 4$ and $\chi(\alpha,\beta)$=max$\{\mathrm{S}(2,2,7),\mathrm{S}(2,2,2),\mathrm{S}(7,7,3),\mathrm{S}(2,2,7),\mathrm{S}(7,7,2)\}$=$2|2-7| = 10$. This will imply that $\Psi(\chi(\alpha,\beta)) = 2(10) = 20$ and $\phi(\chi(\alpha,\beta)) = \frac{2}{7}(10)$. Therefore

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 4 < 20 - \frac{20}{7} = \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta))$.

Subcase(iii) : Let $\alpha = 2$ and $\beta \in (3,13)$. Then we have

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = \Psi(\mathrm{S}(2,2,2)) = 0 \leq \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta)) = 2\chi(\alpha,\beta) - \frac{2}{7}\chi(\alpha,\beta)$.

Subcase(iv) : Let $\alpha \in (2,3]$ and $\beta = 2$. Then

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(8,8,2) = 24$ and

$\chi(\alpha,\beta)$=max$\{\mathrm{S}(9,9,2),\mathrm{S}(9,9,8),\mathrm{S}(2,2,2),\mathrm{S}(9,9,2),\mathrm{S}(2,2,8)\}$=$2|9-2| = 14$. This will imply that $\Psi(\chi(\alpha,\beta)) = 2(14) = 28$ and $\phi(\chi(\alpha,\beta)) = \frac{2}{7}(14) = 4$.

Therefore $\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 24 \leq 28 - 4 = \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta))$.

Subcase(v) : Let $\alpha,\beta \in (2,3]$. Then $\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(8,8,3) = 20$ and $\chi(\alpha,\beta)$=max$\{\mathrm{S}(9,9,7),\mathrm{S}(9,9,8),\mathrm{S}(7,7,3),\mathrm{S}(9,9,3),\mathrm{S}(7,7,8)\}$=$2|9-3| = 12$. This will imply that $\Psi(\chi(\alpha,\beta)) = 2(12) = 24$ and $\phi(\chi(\alpha,\beta)) = \frac{2}{7}(12) = \frac{24}{7}$.

Therefore $\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 20 < 24 - \frac{24}{7} = \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta))$.

Subcase(vi) : Let $\alpha \in (2,3]$ and $\beta \in (3,13)$. Then

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(8,8,2) = 24$ and

$\chi(\alpha,\beta)$=max$\{\mathrm{S}(9,9,\beta-1),\mathrm{S}(9,9,2),\mathrm{S}(\beta-1,\beta-1,2),\mathrm{S}(9,9,2),\mathrm{S}(\beta-1,\beta-1,8)\}$=20. This will imply that $\Psi(\chi(\alpha,\beta)) = 2(20) = 40$ and $\phi(\chi(\alpha,\beta)) = \frac{2}{7}(40) = \frac{80}{7}$. Therefore

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 24 < 40 - \frac{80}{7} = \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta))$.

Subcase(vii) : Let $\alpha \in (3,13)$ and $\beta = 2$. Then we have

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = \Psi(\mathrm{S}(2,2,2)) = 0 \leq \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta)) = 2\chi(\alpha,\beta) - \frac{2}{7}\chi(\alpha,\beta)$.

Subcase(viii) : Let $\alpha \in (3,13)$ and $\beta \in (2,3]$. Then we have

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 2\mathrm{S}(2,2,3) = 4$ and

$\chi(\alpha,\beta)$=max$\{\mathrm{S}(\frac{\alpha+3}{2},\frac{\alpha+3}{2},7),\mathrm{S}(\frac{\alpha+3}{2},\frac{\alpha+3}{2},2),\mathrm{S}(7,7,3),\mathrm{S}(\frac{\alpha+3}{2},\frac{\alpha+3}{2},3),\mathrm{S}(7,7,2)\}$=12. This will imply that $\Psi(\chi(\alpha,\beta)) = 2(12) = 24$ and $\phi(\chi(\alpha,\beta)) = \frac{2}{7}(12) = \frac{24}{7}$. It follows that $\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = 4 < 24 - \frac{24}{7} = \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta))$.

Subcase(ix) : Let $\alpha,\beta \in (3,13)$.Then we have $\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) = \Psi(\mathrm{S}(2,2,2)) = 0 \leq \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta)) = 2\chi(\alpha,\beta) - \frac{2}{7}\chi(\alpha,\beta)$.

From all sub cases, we conclude that

$\Psi(\mathrm{S}(\mathrm{f}\alpha,\mathrm{f}\alpha,\mathrm{g}\beta)) \leq \Psi(\chi(\alpha,\beta)) - \phi(\chi(\alpha,\beta)))$ for all $\alpha,\beta \in X$.

Case(II):Now let us show that the pairs (f, R) and (g, T) are subcompatible. For this, we choose $\alpha_n = 2$ for all $n \in \mathbb{N}$. Then we have $\mathrm{f}\alpha_n = 2$ and $\mathrm{R}\alpha_n = 2$ for all $n \in \mathbb{N}$. Also we have $\mathrm{fR}\alpha_n = 2$ and $\mathrm{Rf}\alpha_n = 2$ for $n \in \mathbb{N}$. This will imply that $\mathrm{f}\alpha_n \to 2$ and $\mathrm{R}\alpha_n \to 2$ and also S(fRα_n, fRα_n, Rfα_n)$\to 0$, as $n \to \infty$. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty} \mathrm{R}(\alpha_n)$=$\lim_{n\to\infty} \mathrm{f}(\alpha_n) = 2 \in X$ and $\lim_{n\to\infty}$S(fRα_n, fRα_n, Rfα_n)=0. Therefore (f, R) is subcompatible. Similarly, we can easily show that the pair (g, T) is also subcompatible.

Case(III):Now we show that the pairs (f, R) and (g, T) are both subsequentially continuous. Clearly fRα_n $\to$f(2)=2 and Rfα_n $\to$R(2)=2 and also $\mathrm{f}\alpha_n \to 2$ and $\mathrm{R}\alpha_n \to 2$, as $n \to \infty$. Thus there exists a sequence (α_n) in X such that $\lim_{n\to\infty} \mathrm{R}(\alpha_n)$=$\lim_{n\to\infty} \mathrm{f}(\alpha_n) = 2 \in X$ and $\lim_{n\to\infty} \mathrm{fR}(\alpha_n)$=f(2) and $\lim_{n\to\infty} \mathrm{Rf}(\alpha_n)$=R(2). Therefore (f, R) is subsequentially continuous. Similarly, we can show that the pair (g, T) is subsequentially continuous. Therefore the hypothesis of the Theorem 3.5 is satisfied and f, R, g and T have a unique common fixed point, namely 2.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorem in S-metric spaces. Mat. Vesn. 64 (2012), 258–266.

[2] S. Sedghi, N.V. Dung, Fixed point theorems on S-metric spaces. Mat. Vesn. 66 (2014), 113–124.

[3] Z. Mustafa, B. Sims, A New Approach To Generalized Metric Spaces, J. Nonlinea Convex Anal. 7 (2006), 289–297.

[4] N.V. Dung, N.T. Hieu, S. Radojevic, Fixed point theorems for g-monotone maps on partially ordered S-metric spaces, Filomat. 28 (2014), 1885-1898.

[5] G.V.R. Babu, L.B. Kumssa, Fixed Points of (α, ψ, ϕ)- Generalized Weakly Contractive Maps and Property (P) in S-metric spaces, Filomat. 31 (2017), 4469-4481.

[6] H. Bouhadjera, C. Godet-Thobie, Common fixed point theorems for pairs of subcompatible maps, ArXiv:0906.3159 [Math]. (2011).