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FAULHABER POLYNOMIALS

R. SIVARAMAN*

Department of Mathematics, D. G. Vaishnav College, Chennai, India

Abstract: The process of summing of powers of natural numbers has been an interesting quest right from ancient time to modern time. German Mathematician Johann Faulhaber introduced wonderful ways to determine the sum of powers of natural numbers through explicit polynomials which were called today as "Faulhaber's Polynomials". In this paper, we try to obtain those polynomials by proving an important theorem and deducing from it.

Keywords: sum of powers of natural numbers; triangular numbers; Nicomachus theorem; Faulhaber polynomials.

1. INTRODUCTION

Mathematicians like Johann Faulhaber, Pierre-de-Fermat, Pascal, Johann Bernoulli, Euler, Jacobi made great strides in finding sums of powers of natural numbers and proving most important theorems in number theory that we know today. Their works form the fundamental of modern number theory. Though the works of giants like Fermat, Bernoulli, Euler and Jacobi were quite well known, the basic idea for summing powers of natural numbers was first provided rigorously by the German mathematician Johann Faulhaber during early part of seventeenth century. In this paper, we try to prove the basic ideas that Faulhaber thought in his days, which are now called "Faulhaber Polynomials" in his honour.

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*Corresponding author

E-mail address: rsivaraman1729@yahoo.co.in

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2. DEFINITION

The sum of kth powers of first n natural numbers denoted by $\sigma_k(n)$ is defined as

$$\sigma_k(n) = 1^k + 2^k + \cdots + n^k \quad (2.1).$$

We may also denote $\sigma_k(n)$ simply by σ_k.

2.1 The sum of first n natural numbers is given by

$$\sigma_1(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad (2.2)$$

The numbers in the expression $\sigma_1(n) = \sigma_1 = \dfrac{n(n+1)}{2}$ are called "Triangular Numbers".

2.2 Knowing the value $\sigma_1(n)$ as in equation (2.2) we get the following equations representing the sums of powers of natural numbers (see [4],[5]).

$$\sigma_1(n) = 1^1 + 2^1 + \cdots + n^1 = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sigma_2(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\sigma_3(n) = 1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$\sigma_4(n) = 1^4 + 2^4 + \cdots + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$\sigma_5(n) = 1^5 + 2^5 + \cdots + n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$\sigma_6(n) = 1^6 + 2^6 + \cdots + n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$\sigma_7(n) = 1^7 + 2^7 + \cdots + n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$\sigma_8(n) = 1^8 + 2^8 + \cdots + n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

$$\sigma_9(n) = 1^9 + 2^9 + \cdots + n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2$$

$$\sigma_{10}(n) = 1^{10} + 2^{10} + \cdots + n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

The expressions mentioned above provide the sums up to first ten powers of first n natural numbers. We now try to establish relationship between various powers of sums of natural numbers.

3. BASIC RELATIONSHIP

Ancient Greek mathematician Nicomachus proved that the sum of the first n cubes is the square of the nth triangular number. This is now called "Nicomachus Theorem" in his honour. In view of definitions and formulas mentioned in section 2, by Nicomachus theorem, we have $\sigma_3 = \sigma_1^{\,2} \quad (3.1)$. Thus for any positive integer k, this gives $\sigma_3^{\,k} = \sigma_1^{\,2k} \quad (3.2)$.

We now prove an important theorem to establish further relationships.

4. MAIN RESULTS

Theorem 1. The kth power of triangular numbers $\sigma_1^{\,k}$ is expressed as the linear combination of σ_k up to σ_{2k}.

Proof: We first note that $\sigma_k = \sigma_k(n)$ for all natural numbers n and k. Now computing the kth power of triangular numbers (the kth power of sum of first n natural numbers) we have

$$\sigma_1^{\,k} = \sigma_1^{\,k}(n) = \left[\frac{n(n+1)}{2}\right]^k = \sum_{p=1}^{n}\left[\left(\frac{p(p+1)}{2}\right)^k - \left(\frac{p(p-1)}{2}\right)^k\right]$$

$$= \sum_{p=1}^{n}\sum_{r=0}^{k}\left(\frac{p}{2}\right)^k \binom{k}{r} p^r \left[1-(-1)^{k-r}\right]$$

$$= \frac{1}{2^k}\sum_{p=1}^{n}\sum_{r=0}^{k} p^{k+r}\binom{k}{r}\left[1-(-1)^{k-r}\right] = \frac{1}{2^k}\sum_{r=0}^{k}\sum_{p=1}^{n} p^{k+r}\binom{k}{r}\left[1-(-1)^{k-r}\right]$$

$$\sigma_1^{\,k} = \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}\left[1-(-1)^{k-r}\right]\sigma_{k+r} \quad (4.1)$$

From equation (4.1) we see that $\sigma_1^{\,k}$ is indeed expressed as linear combination of σ_k up to σ_{2k}.

This completes the proof.

Some special identities leading to Faulhaber's polynomials can be obtained from equation (4.1) of Theorem 1 by considering specific values of *k*.

Theorem 2. The cubes, fourth, fifth and sixth powers of triangular numbers are given by the expressions

$$\sigma_1^{\,3} = \frac{1}{4}\sigma_3 + \frac{3}{4}\sigma_5 \qquad (4.2)$$

$$\sigma_1^{\,4} = \frac{1}{2}\sigma_5 + \frac{1}{2}\sigma_7 \qquad (4.3)$$

$$\sigma_1^{\,5} = \frac{1}{16}\sigma_5 + \frac{5}{8}\sigma_7 + \frac{5}{16}\sigma_9 \qquad (4.4)$$

$$\sigma_1^{\,6} = \frac{3}{16}\sigma_7 + \frac{5}{8}\sigma_9 + \frac{3}{16}\sigma_{11} \qquad (4.5)$$

Proof: We will first consider $k = 3$ in equation (4.1) of Theorem 1.

$$\sigma_1^{\,3} = \frac{1}{2^3}\sum_{r=0}^{3}\binom{3}{r}\left[1-(-1)^{3-r}\right]\sigma_{3+r} = \frac{1}{8}\left[2\sigma_3 + 6\sigma_5\right] = \frac{1}{4}\sigma_3 + \frac{3}{4}\sigma_5 \text{ giving equation (4.2)}$$

Now taking $k = 4$ in equation (4.1) of Theorem 1, we get

$$\sigma_1^{\,4} = \frac{1}{2^4}\sum_{r=0}^{4}\binom{4}{r}\left[1-(-1)^{4-r}\right]\sigma_{4+r} = \frac{1}{16}\left[8\sigma_5 + 8\sigma_7\right] = \frac{1}{2}\sigma_5 + \frac{1}{2}\sigma_7$$

This proves equation (4.3)

Now, considering $k = 5$ in equation (4.1) of Theorem 1, we have

$$\sigma_1^{\,5} = \frac{1}{2^5}\sum_{r=0}^{5}\binom{5}{r}\left[1-(-1)^{5-r}\right]\sigma_{5+r} = \frac{1}{32}\left[2\sigma_5 + 20\sigma_7 + 10\sigma_9\right] = \frac{1}{16}\sigma_5 + \frac{5}{8}\sigma_7 + \frac{5}{16}\sigma_9$$

This proves equation (4.4)

Similarly, considering $k = 6$ in equation (4.1), we have

$$\sigma_1^{\,6} = \frac{1}{2^6}\sum_{r=0}^{6}\binom{6}{r}\left[1-(-1)^{6-r}\right]\sigma_{6+r} = \frac{1}{64}\left[12\sigma_7 + 40\sigma_9 + 12\sigma_{11}\right] = \frac{3}{16}\sigma_7 + \frac{5}{8}\sigma_9 + \frac{3}{16}\sigma_{11}$$

This proves equation (4.5). This completes the proof.

Theorem 3. The fifth, seventh, ninth and eleventh powers of sums of first *n* natural numbers are given by the expressions

$$\sigma_5 = \frac{\sigma_1^2(4\sigma_1 - 1)}{3} \qquad (4.6)$$

$$\sigma_7 = \frac{\sigma_1^2(6\sigma_1^2 - 4\sigma_1 + 1)}{3} \qquad (4.7)$$

$$\sigma_9 = \frac{\sigma_1^2(16\sigma_1^3 - 20\sigma_1^2 + 12\sigma_1 - 3)}{5} \qquad (4.8)$$

$$\sigma_{11} = \frac{\sigma_1^2(16\sigma_1^4 - 32\sigma_1^3 + 34\sigma_1^2 - 20\sigma_1 + 5)}{3} \qquad (4.9)$$

Proof: From equations (3.1) and (4.2), we have $\sigma_1^3 = \frac{1}{4}\sigma_1^2 + \frac{3}{4}\sigma_5$. Simplifying this, we get

$$\sigma_5 = \frac{4\sigma_1^3 - \sigma_1^2}{3} = \frac{\sigma_1^2(4\sigma_1 - 1)}{3}$$ which is equation (4.6)

Using equations (4.3) and (4.6), we get $\sigma_1^4 = \frac{1}{2}\sigma_5 + \frac{1}{2}\sigma_7 = \frac{\sigma_1^2(4\sigma_1 - 1)}{6} + \frac{1}{2}\sigma_7$

Simplifying this expression we get $\sigma_7 = \frac{\sigma_1^2(6\sigma_1^2 - 4\sigma_1 + 1)}{3}$ which is equation (4.7)

Now using equations (4.4), (4.6) and (4.7), we get

$$\sigma_1^5 = \frac{1}{16}\sigma_5 + \frac{5}{8}\sigma_7 + \frac{5}{16}\sigma_9 = \frac{\sigma_1^2(4\sigma_1 - 1)}{48} + \frac{5\sigma_1^2(6\sigma_1^2 - 4\sigma_1 + 1)}{24} + \frac{5}{16}\sigma_9$$

Simplifying this, we get $\sigma_9 = \frac{\sigma_1^2(16\sigma_1^3 - 20\sigma_1^2 + 12\sigma_1 - 3)}{5}$. This proves equation (4.8)

Similarly, using equations (4.5), (4.7), (4.8), we get

$$\sigma_1^6 = \frac{3}{16}\sigma_7 + \frac{5}{8}\sigma_9 + \frac{3}{16}\sigma_{11} = \frac{\sigma_1^2(6\sigma_1^2 - 4\sigma_1 + 1)}{16} + \frac{\sigma_1^2(16\sigma_1^3 - 20\sigma_1^2 + 12\sigma_1 - 3)}{8} + \frac{3}{16}\sigma_{11}$$

Simplifying this, we get $\sigma_{11} = \frac{\sigma_1^2(16\sigma_1^4 - 32\sigma_1^3 + 34\sigma_1^2 - 20\sigma_1 + 5)}{3}$ which is equation (4.9) This completes the proof.

5. FAULHABER'S POLYNOMIALS

Considering $\sigma_1 = x$, and using the notation $F_k(x) = \sigma_k$ to denote k th Faulhaber polynomial, and using the equations using (3.1), (4.6), (4.7), (4.8) and (4.9), we can write the polynomial equations as shown below.

$$F_3(x) = x^2 \tag{5.1}$$

$$F_5(x) = \frac{x^2(4x-1)}{3} \tag{5.2}$$

$$F_7(x) = \frac{x^2(6x^2-4x+1)}{3} \tag{5.3}$$

$$F_9(x) = \frac{x^2(16x^3-20x^2+12x-3)}{5} \tag{5.4}$$

$$F_{11}(x) = \frac{x^2(16x^4-32x^3+34x^2-20x+5)}{3} \tag{5.5}$$

6. CONCLUSION

Through Theorem 1, we have proved an important fact that $\sigma_1^{\,k}$ can be expressed as linear combination of σ_k up to σ_{2k}. This fact helps us to derive the relationship between successive powers of sums of natural numbers as derived in Theorem 2. Through equations (4.6) to (4.9) of Theorem 3, we have obtained Faulhaber's Polynomials listed as equations (5.1) to (5.5) respectively. We observe that for odd values of k, $F_k(x)$ is a polynomial in x of degree $\frac{k+1}{2}$. Here x represent the nth triangular number. Hence the Faulhaber's polynomials of odd orders can be computed in terms of triangular numbers and we further notice that x^2 is a factor for each of them. With these observations, we conclude that for odd values of k, the Faulhaber polynomial is a polynomial of degree $\frac{k+1}{2}$ with a double zero at the origin such that $\sigma_k = F_k(\sigma_1)$. Because of this, we see that the Faulhaber polynomials will immediately provide the answers for sums of odd powers of natural numbers in terms of triangular numbers which are readily known to us.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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