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INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR FORCED IMPULSIVE DIFFERENTIAL EQUATIONS WITH DAMPING TERM UNDER VARIABLE DELAY

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Abstract: In this paper, interval oscillation of second order forced impulsive differential equations with damping term under variable delay are studied. We follow the Kong and Philos type class functions and pair of functions (H_1, H_2) to obtain the oscillation criteria. The results obtained in this paper extend some of the existing results. An example is given to illustrate the main result.

Keywords: oscillation; second order; impulse; damping term; variable delay.

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1. INTRODUCTION

The theory of impulsive differential equations has applications in control theory, physics, population dynamics, industrial robotics etc. The oscillation of solutions of second order impulsive differential equations are systematically studied by several authors [5],[6],[7] and [8]. In [9], [10], [11] the authors studied the oscillation of solutions of second order differential equations with

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constant delay and in [12], [13], [14] authors studied the oscillation of solutions of second order differential equations with variable delay. Motivated by the work of [12], [14], we obtain the oscillation criteria for second order nonlinear forced impulsive differential equation with damping term under variable delay. The results obtained in this paper extend some of the existing results and are illustrated by an example.

Consider the second order following impulsive differential equation,

$$\begin{cases} (r(t)\Phi_\alpha(x'(t)))' + p(t)\Phi_\alpha(x'(t)) + q_0(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t)-\sigma(t)) = e(t),\ t \neq \tau_k \\ x(\tau_k^{+}) = a_k x(\tau_k),\quad x'(\tau_k^{+}) = b_k x'(\tau_k),\ t=\tau_k, k=1,2,..., \end{cases} \tag{1}$$

where $\Phi_*(s) = |s|^{*-1}$, $k \in N$, $t \geq t_0$, $\{\tau_k\}$ is the impulsive moments sequence with $0 \leq t_0 = \tau_0 < \tau_1 < ... < \tau_k < ... \lim_{k\to\infty} \tau_k = \infty$ and $x(\tau_k^{+}) = \lim_{t\to t_k^{+0}} x(t)$, $x'(\tau_k^{+}) = \lim_{h\to o^{+}} \frac{x(\tau_k+h)-x(\tau_k^{+})}{h}$, $x(\tau_k^{-}) = \lim_{t\to t_k^{-0}} x(t) = x(t)$, $x'(\tau_k^{-}) = \lim_{h\to o^{-}} \frac{x(\tau_k+h)-x(\tau_k)}{h} = x'(\tau_k)$. Let $J \subset R$ be an interval and define $PLC(J,R) = \{x : J \to R : x(t)$ is piecewise-left-continuous and has discontinuity of first kind at τ_k's$\}$. Define an interval delay function $D_k(t) = t - \tau_k - \sigma(t), t \in [\tau_k, \tau_{k+1}], k \in N.$ Throughout this paper, we always assume the following conditions hold:

(A1) $r \in C^1([t_0,\infty),(0,\infty)), p, q_i, e \in PLC([t_0,\infty),R), i = 0,1,2,\ldots,n$;

(A2) $\beta_1 > ... > \beta_m > \alpha > \beta_{m+1} > ... > \beta_n > 0$ are constants.

(A3) $b_k \geq a_k > 0, k \in N$ are constants.

(A4) $\sigma(t) \in C([t_0,\infty))$, there exists a nonnegative constant σ such that $0 \leq \sigma(t) \leq \sigma$ for all $t \geq 0$ and $\tau_{k+1} - \tau_k > \sigma$ for all $k \in N$;

(A5) there is one zero point $t_k \in (t_k, t_{k+1}]$ such that $D_k(t) < 0$ for $t \in (\tau_k, t_k)$, $D_k(t) > 0$ for $t \in (t_k, \tau_{k+1}]$ and $D_k(t_k) = 0.$

By a solution of (1) we mean a function $x \in PC([t_0,\infty),R)$ such that $x' \in PC([t_0,\infty),R)$ and

$x(t)$ satisfies (1) for $t \geq t_0$. A nontrivial solution is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called non-oscillatory. An equation is called oscillatory if all its solutions are oscillatory.

2. MAIN RESULTS

Use the notation, let $k(s) = \max\{i : t_0 < \tau_i < s\}$, and $r_j = \max\{r(t) : t \in [c_j, d_j]\}$. We define a function $\phi \in C([c,d], R)$ and an operator $\Omega : C([c,d], R) \to R$ by

$$\Omega_c^d[\phi] = \phi(\tau_{k(c)+1}) \frac{b_{k(c)+1}^{\alpha} - a_{k(c)+1}^{\alpha}}{a_{k(c)+1}^{\alpha}(\tau_{k(c)+1} - c)^{\alpha}} + \sum_{i=k(c)+2}^{k(d)} \phi(\tau_i) \frac{b_i^{\alpha} - a_i^{\alpha}}{a_i^{\alpha}(\tau_i - \tau_{i-1})^{\alpha}}.$$

For the discussion of the impulse moments of $x(t)$ and $x(t-\sigma(t))$, we need to consider the following four cases

$(C1)$ $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma < d_j$;

$(C2)$ $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma > d_j$;

$(C3)$ $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma < d_j$;

$(C4)$ $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma > d_j$;

and also assume that there exist points $\delta_j \in (c_j, d_j) \setminus \{\tau_k\}, j = 1, 2.$ which divides intervals $[c_j, d_j]$ in to two parts $[c_j, \delta_j]$ and $[\delta_j, d_j]$. In view of whether or not there are impulsive moments of $x(t)$ in $[c_j, \delta_j]$ and $[\delta_j, d_j]$, we should consider the following three cases

$(\bar{C}1)$ $k(c_j) < k(\delta_j) < k(d_j)$

$(\bar{C}2)$ $k(c_j) < k(\delta_j) = k(d_j)$

$(\bar{C}3)$ $k(c_j) = k(\delta_j) < k(d_j)$.

Also, we have the following relation between δ_j and $t_{k(\delta_j)}$

$(\overline{\overline{C}}1)\ t_{k(\delta_j)} < \delta_j$, $(\overline{\overline{C}}2)\ t_{k(\delta_j)} > \delta_j$, $(\overline{\overline{C}}3)\ t_{k(\delta_j)} = \delta_j$.

Throughout the paper we consider $(C1)$ with $(\overline{C}1)$ and $(\overline{\overline{C}}1)$ only. The discussions for other cases are similar and omitted. The following preparatory lemmas will be useful to prove our main theorem.

Lemma 1 Let $\{\beta_i\}$, $i = 1,2,...,n,$ be the n-tuple satisfying $\beta_1 > ... > \beta_m > \alpha > \beta_{m+1} > ... > \beta_n > 0.$ Then there exist an n-tuple $(\eta_1, \eta_2, ..., \eta_n)$ satisfying

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \tag{2}$$

which also satisfies

$$\sum_{i=1}^{n} \eta_i \le 1,\ 0 < \eta_i < 1. \tag{3}$$

Proof: The proof of Lemma 1 can be obtained easily from Lemma 1 of [4] by taking $\alpha_i = \beta_i / \alpha$.

The Lemma below can be found in [1].

Lemma 2 Let X and Y be non-negative real numbers. Then

$$\lambda XY^{\lambda-1} - X^{\lambda} \le (\lambda - 1)Y^{\lambda}, \lambda > 1 \tag{4}$$

where inequality holds if and only if $X = Y$.

Let $\alpha > 0, A > 0, B \ge 0$ and $y \ge 0.$ Put $X = B^{\frac{\alpha}{\alpha+1}}, Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} A^{\alpha} B^{\frac{-\alpha^2}{\alpha+1}}, \lambda = 1 + \frac{1}{\alpha}$ in Lemma 2 we have $Ay - By^{\frac{\alpha+1}{\alpha}} \le \left(\frac{A}{\alpha+1}\right)^{\alpha+1} \left(\frac{\alpha}{B}\right)^{\alpha}$.

Following Kong [3] and Philos [2], we define a class of functions: Let $F = \{(t,s) : t_0 \le s \le t\},$ $H_1, H_2 \in C^1(F, R).$ A pair of functions (H_1, H_2) is said to belong to a function class $\mathcal{H}$, if $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0, H_2(t,s) > 0$ for t > s and there exist $h_1, h_2 \in L_{loc}(F, \mathbb{R})$ such that $\frac{\partial H_1(t,s)}{\partial t} = h_1(t,s)H_1(t,s)$, $\frac{\partial H_2(t,s)}{\partial t} = -h_2(t,s)H_2(t,s).$

For convenience we use the following notation for the below expression over [c,d]

$$\Lambda_c^d[\psi]:\left[\int_c^{\tau_{k(c)+1}}\frac{(t-\tau_{k(c)}-\sigma(t))^\alpha}{(t-\tau_{k(c)})^\alpha}+\sum_{l=k(c)+1}^{k(d)-1}\left(\int_{\tau_l}^{t_l}\frac{(t-\tau_l)^\alpha}{b_l^\alpha(t+\sigma(t)-\tau_l)^\alpha}+\int_{t_l}^{\tau_{l+1}}\frac{(t-\tau_l-\sigma(t))^\alpha}{(t-\tau_l)^\alpha}\right)\right.$$
$$\left.+\int_{\tau_{k(d)}}^{t_{k(d)}}\frac{(t-\tau_{k(d)})^\alpha}{b_{k(d)}^\alpha(t+\sigma(t)-\tau_{k(d)})^\alpha}+\int_{t_{k(d)}}^{d}\frac{(t-\tau_{k(d)}-\sigma(t))^\alpha}{(t-\tau_{k(d)})^\alpha}\right]\psi(t)dt$$

Theorem 3 Suppose that for any $T\geq t_0$, there exist $c_j, d_j \notin \{\tau_k\}, j=1,2$ such that $T<c_1-\sigma$ $<c_1<\delta_1<d_1\leq c_2-\sigma<c_2<\delta_2<d_2$ and

$$p(t), q_i(t)\geq 0, (-1)^j e(t)\geq 0,\ t\in[c_j-\sigma(t), d_j]\setminus\{\tau_k\}, i=0,1,2,\ldots,n,\ j=1,2. \tag{5}$$

Let $\{\eta_i\}, i=1,2,\ldots,n,$ be an n-tuple satisfying (3) and (4). If there exist $(H_1,H_2)\in\mathcal{H}$ such that

$$\frac{1}{H_1(\delta_j,c_j)}\left(\Lambda_{c_j}^{\delta_j}\left[Q(t)H_1(t,c_j)\right]+\int_{c_j}^{\delta_j}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_j)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_1(t,c_j)\,dt\right)$$
$$+\frac{1}{H_2(d_j,\delta_j)}\left(\Lambda_{\delta_j}^{d_j}\left[Q(t)H_2(d_j,t)\right]+\int_{\delta_j}^{d_j}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_2(d_j,t)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_2(d_j,t)\,dt\right)$$
$$>\frac{r_j}{H_1(\delta_j,c_j)}\Omega_{c_j}^{\delta_j}\left[H_1(.,c_j)\right]+\frac{r_j}{H_2(d_j,\delta_j)}\Omega_{\delta_j}^{d_j}\left[H_2(d_j,.)\right] \tag{6}$$

where $Q(t)=\eta_0^{-\eta_0}\,|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i}, \eta_0=1-\sum_{i=1}^{n}\eta_i,$ then (1) is oscillatory.

Proof: Let us suppose that $x(t)$ is a non-oscillatory solution of (1). Without loss of generality, we may assume that $x(t)>0$ for $t\in[c_1,d_1]$. Define

$$w(t)=\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}. \tag{7}$$

Then for $t\in[c_1,d_1]$ and $t\neq\tau_k$, we have

$$w'(t)=\left[-\sum_{i=1}^{n}q_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma(t)))-\frac{|e(t)|}{\Phi_\alpha(x(t-\sigma(t)))}\right]\frac{\Phi_\alpha(x(t-\sigma(t)))}{\Phi_\alpha(x(t))}$$

$$-\frac{p(t)}{r(t)}w(t)-\frac{\alpha}{(r(t))^{1/\alpha}}|w(t)|^{\frac{\alpha+1}{\alpha}}-q_0(t). \tag{8}$$

By arithmetic-geometric mean inequality, $\sum_{i=1}^{n}\eta_i v_i \geq \prod_{i=0}^{n} v_i^{\eta_i}, v_i \geq 0.$

Take $v_0=\eta_0^{-1}\frac{|e(t)|}{\Phi_\alpha(x(t-\sigma(t)))}$ and $v_i=\eta_i^{-1}q_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma(t)))$, from (2) and (3) we get

$$-\sum_{i=1}^{n}q_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma(t)))-\frac{|e(t)|}{\Phi_\alpha(x(t-\sigma(t)))}\leq -\eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i}. \tag{9}$$

Now equation (8) becomes

$$w'(t)=-Q(t)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}-\frac{p(t)}{r(t)}w(t)-\frac{\alpha}{(r(t))^{1/\alpha}}|w(t)|^{\frac{\alpha+1}{\alpha}}-q_0(t) \tag{10}$$

where $Q(t)=\eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=0}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i}$.

If there are impulsive moments $\tau_{k(c_1)+1},\tau_{k(c_1)+2},...,\tau_{k(\delta_1)}$ in $[c_{1,}\delta_1]$, $\tau_{k(\delta_1)+1},\tau_{k(c_1)+2},...,\tau_{k(d_1)}$ in $[\delta_{1,}d_1]$ and zero point t_l of $D_l(t)$ in each (τ_l,τ_{l+1}) for $l=k(c_1)+1,k(c_1)+2,...,k(d_1)-1$. Multiplying both sides of (10) by $H_1(t,c_1)$ and integrating over $[c_{1,}\delta_1]$, also apply integration by parts on the left side we obtain

$$\sum_{l=k(c_1)+1}^{k(\delta_1)}H_1(t_l,c_1)[w(\tau_l)-w(\tau_l^{+})]+H_1(\delta_l,c_1)w(\delta_1)-\int_{c_1}^{\delta_1}H_1(t,c_1)h_1(t,c_1)w(t)dt$$

$$\leq -\left[\int_{c_1}^{\tau_{k(c_1)+1}}+\sum_{l=k(c_1)+1}^{k(\delta_1)-1}\left(\int_{\tau_l}^{t_l}+\int_{t_l}^{\tau_{l+1}}\right)+\int_{\tau_{k(\delta_1)}}^{t_{k(\delta_1)}}+\int_{t_{k(\delta_1)}}^{\delta_1}\right]Q(t)H_1(t,c_1)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}dt$$

$$+\int_{c_1}^{\delta_1}\left[-\frac{p(t)}{r(t)}w(t)-\frac{\alpha}{(r(t))^{1/\alpha}}|w(t)|^{\frac{\alpha+1}{\alpha}}-q_0(t)\right]H_1(t,c_1)\,dt$$

$$\leq -\left[\int_{c_1}^{\tau_{k(c_1)+1}}+\sum_{l=k(c_1)+1}^{k(\delta_1)-1}\left(\int_{\tau_l}^{t_l}+\int_{t_l}^{\tau_{l+1}}\right)+\int_{\tau_{k(\delta_1)}}^{t_{k(\delta_1)}}+\int_{t_{k(\delta_1)}}^{\delta_1}\right]Q(t)H_1(t,c_1)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}dt$$

$$+\int_{c_1}^{\delta_1}\left[\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right||w(t)|-\frac{\alpha}{(r(t))^{1/\alpha}}|w(t)|^{\frac{\alpha+1}{\alpha}}\right]H_1(t,c_1)\,dt-\int_{c_1}^{\delta_1}q_0(t)H_1(t,c_1)\,dt \tag{11}$$

Use Lemma 2 with $A=\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|, B=\frac{\alpha}{(r(t))^{1/\alpha}}, y=|w(t)|$, we have

$$\left[\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right||w(t)|-\frac{\alpha}{(r(t))^{1/\alpha}}|w(t)|^{\frac{\alpha+1}{\alpha}}\right]\leq\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\alpha+1} \tag{12}$$

Apply (12) in (11) we get,

$$\sum_{l=k(c_1)+1}^{k(\delta_1)}H_1(t_l,c_1)[w(\tau_l)-w(\tau_l^{+})]+H_1(\delta_l,c_1)w(\delta_1)$$

$$\leq-\left[\int_{c_1}^{\tau_{k(c_1)+1}}+\sum_{l=k(c_1)+1}^{k(\delta_1)-1}\left(\int_{\tau_l}^{t_l}+\int_{t_l}^{\tau_{l+1}}\right)+\int_{\tau_{k(\delta_1)}}^{t_{k(\delta_1)}}+\int_{t_{k(\delta_1)}}^{\delta_1}\right]Q(t)H_1(t,c_1)\frac{x^{\alpha}(t-\sigma(t))}{x^{\alpha}(t)}\,dt$$

$$+\int_{c_1}^{\delta_1}\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\alpha+1}H_1(t,c_1)\,dt-\int_{c_1}^{\delta_1}q_0(t)H_1(t,c_1)\,dt \tag{13}$$

For $t=\tau_k, k=1,2,\ldots,$ we have $w(\tau_k^{+})=\frac{b_k^{\alpha}}{a_k^{\alpha}}w(\tau_k)$. Therefore from (13) we get

$$\left[\int_{c_1}^{\tau_{k(c_1)+1}}+\sum_{l=k(c_1)+1}^{k(\delta_1)-1}\left(\int_{\tau_l}^{t_l}+\int_{t_l}^{\tau_{l+1}}\right)+\int_{\tau_{k(\delta_1)}}^{t_{k(\delta_1)}}+\int_{t_{k(\delta_1)}}^{\delta_1}\right]Q(t)H_1(t,c_1)\frac{x^{\alpha}(t-\sigma(t))}{x^{\alpha}(t)}\,dt$$

$$+\int_{c_1}^{\delta_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_1(t,c_1)\,dt$$

$$\leq\sum_{l=k(c_1)+1}^{k(\delta_1)}\left[\frac{b_l^{\alpha}-a_l^{\alpha}}{a_l^{\alpha}}\right]H_1(\tau_l,c_1)w(\tau_l)-H_1(\delta_l,c_1)w(\delta_1) \tag{14}$$

Now for $t\in[c_1,d_1]\setminus\{\tau_l\}$, from (1), it is clear that

$$(r(t)\Phi_{\alpha}(x'(t)))'+p(t)\Phi_{\alpha}(x'(t))=e(t)-q_0(t)\Phi_{\alpha}(x(t))-\sum_{i=1}^{n}q_i(t)\Phi_{\beta_i}(x(t))\leq 0.$$

Multiplying both side of above inequality by $\overline{p}(t)=\exp\left(\int^{t}\frac{p(s)}{r(s)}ds\right)$, we get

$(\overline{p}(t)\,r(t)\Phi_\alpha(x'(t)))' = \overline{p}(t)\left(e(t) - q_0(t)\,\Phi_\alpha(x(t)) - \sum_{i=1}^{n} q_i(t)\,\Phi_{\beta_i}(x(t)) \right) \le 0$ which implies that

$[\overline{p}(t)\,r(t)\Phi_\alpha(x'(t))]$ is non increasing on $[c_1, d_1]\setminus\{\tau_l\}$. Because there are different integrations in

(14), we will estimate $\dfrac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}$ in each interval of t.

Case (i) If $t_l < t \le \tau_{l+1}$, for $l = k(c_1)+1, k(c_1)+2, ..., k(d_1)-1, l \ne k(\delta_1)$ then $(t-\sigma(t), t) \subset (\tau_l, \tau_{l+1}]$.

Thus there is no impulsive moment in $(t-\sigma(t), t)$. Therefore for any $s \in (t-\sigma(t), t)$, there exists a

$\xi_l \in (\tau_l, s)$ such that $x(s) > x(s) - x(\tau_l^+) = x'(\xi_l)(s-\tau_l)$. Since $x(\tau_l^+) > 0$, the function $\Phi_\alpha(.)$ is

an increasing function and $[\overline{p}(t)\,r(t)\Phi_\alpha(x'(t)]$ is non-increasing on (τ_l, τ_{l+1}), we have

$$\Phi_\alpha(x(s)) > \Phi_\alpha(x'(\xi_l)(s-\tau_l)) = \frac{\overline{p}(\xi_l)r(\xi_l)}{\overline{p}(\xi_l)r(\xi_l)}\Phi_\alpha(x'(\xi_l))(s-\tau_l)^\alpha$$

$$\ge \frac{\overline{p}(s)r(s)}{\overline{p}(\xi_l)r(\xi_l)}\Phi_\alpha(x'(s))(s-\tau_l)^\alpha \tag{15}$$

Therefore, $\Phi_\alpha(x'(s)(s-\tau_l)) < \dfrac{\overline{p}(\xi_l)r(\xi_l)}{\overline{p}(s)r(s)}\Phi_\alpha(x(s)) \le \Phi_\alpha(x(s)),\ \xi_l \in (\tau_l, s)$. Thus $\dfrac{x'(s)}{x(s)} < \dfrac{1}{s-\tau_l}$.

Integrating both sides from $t-\sigma(t)$ to t, we obtain $\dfrac{x(t-\sigma(t))}{x(t)} > \dfrac{t-\tau_l-\sigma(t)}{t-\tau_l},\ t \in (t_l, \tau_{l+1}]$

$$\therefore \frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \frac{(t-\tau_l-\sigma(t))^\alpha}{(t-\tau_l)^\alpha},\ t \in (t_l, \tau_{l+1}] \tag{16}$$

Case (ii) If $\tau_l < t \le t_l$, for $l = k(c_1)+1, k(c_1)+2, ..., k(d_1)$, then $\tau_l - \sigma < t - \sigma(t) < \tau_l < t$. There is an impulsive moment τ_l in $(t-\sigma(t), t)$. For any $t \in (\tau_l, t_l)$, we have $x(t) > x(t) - x(\tau_l^+)$ $= x'(\zeta_l)(t-\tau_l),\ \zeta_l \in (\tau_l, t)$. Using the impulsive conditions and the monotone properties of $r(t)$, $\Phi_\alpha(.)$, $[\overline{p}(t)\,r(t)\Phi_\alpha(x'(t)]$, we get

$$\Phi_\alpha(x(t)-a_l x(\tau_l)) = \frac{\overline{p}(\zeta_l)r(\zeta_l)}{\overline{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha(x'(\zeta_l))(t-\tau_l)^\alpha$$
$$\le \frac{\overline{p}(\tau_l^+)r(\tau_l^+)}{\overline{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha(x'(\tau_l^+))(t-\tau_l)^\alpha = \frac{\overline{p}(\tau_l)r(\tau_l)}{\overline{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha(b_l x'(\tau_l))(t-\tau_l)).$$

Since $x(\tau_l) > 0$, we have

$$\Phi_\alpha\left(\frac{x(t)}{x(\tau_l)} - a_l\right) \le \frac{\overline{p}(\tau_l)r(\tau_l)}{\overline{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha\left(b_l\frac{x'(\tau_l)}{x(\tau_l)}(t-\tau_l)\right) \tag{17}$$

In addition, $x(\tau_l) > x(\tau_l) - x(\tau_l - \sigma(t)) = x'(\varsigma_l)\sigma(t),\ \varsigma_l \in (t-\sigma(t), \tau_l)$.

Similar to the analysis in case (1), we have

$$\frac{x'(\tau_l)}{x(\tau_l)} < \frac{1}{\sigma(t)} \tag{18}$$

From (17) and (18) and note that the monotone properties of $\Phi_\alpha(.), \overline{p}(t)$ and , we get

$$\frac{x(t)}{x(\tau_l)} < a_l + \frac{b_l}{\sigma(t)}(t-\tau_l)$$

In view of assumption(A3), we have

$$\frac{x(\tau_l)}{x(t)} > \frac{\sigma(t)}{\sigma(t)a_l + b_l(t-\tau_l)} \ge \frac{\sigma(t)}{b_l(t+\sigma(t)-\tau_l)} > 0. \tag{19}$$

On the other hand, using similar analysis of case(i), we get

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_l+\sigma(t)}, s \in (\tau_l - \sigma(t), \tau_l) \tag{20}$$

Integrating (20) from $t-\sigma(t)$ to τ_l, where $t \in (\tau_l, \tau_l + \sigma(t))$, we get

$$\frac{x(t-\sigma(t))}{x(\tau_l)} > \frac{t-\tau_l}{\sigma(t)} \ge 0. \tag{21}$$

From (19) to (21), we obtain $\dfrac{x(t-\sigma(t))}{x(t)} > \dfrac{t-\tau_l}{b_l(t+\sigma(t)-\tau_l)},\ t \in (\tau_l, t_l]$.

$$\therefore \frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \frac{(t-\tau_l)^\alpha}{b_l^\alpha(t+\sigma(t)-\tau_l)^\alpha},\ t \in (\tau_l, t_l]$$

Using same as the proof of case(i) and case(ii), we can prove the following cases

Case(iii) If $c_1 \le t \le \tau_{k(c_1)+1}$, then $\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \frac{(t-\tau_{k(c_1)}-\sigma(t))^\alpha}{(t-\tau_{k(c_1)})^\alpha}$.

Case(iv) If $\delta_1 < t \le \tau_{k(\delta_1)+1}$, then $\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \frac{(t-\tau_{k(\delta_1)}-\sigma(t))^\alpha}{(t-\tau_{k(\delta_1)})^\alpha}$.

Case(v) If $t_{k(d_1)} < t \le d_1,$, then $\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \frac{(t-\tau_{k(d_1)}-\sigma(t))^\alpha}{(t-\tau_{k(d_1)})^\alpha}$.

Case(vi) If $t_{k(\delta_1)} < t \le \delta_1,$, then $\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \frac{(t-\tau_{k(\delta_1)}-\sigma(t))^\alpha}{(t-\tau_{k(\delta_1)})^\alpha}$.

On the other hand, for $t \in (\tau_{l,}\tau_{l+1}) \subset [c_1, d_1], l = k(c_1)+1, ..., k(d_1)-1,$ there exists $\gamma_l \in (\tau_{l,} t)$ such that $x(t) - x(\tau_l^+) = x'(\gamma_l)(t-\tau_l)$. In view of $x(\tau_l^+) > 0$ and the monotone properties of $\Phi_\alpha(.)$, $[\overline{p}(t) r(t) \Phi_\alpha(x'(t)]$ we obtain $\Phi_\alpha(x(t)) > \Phi_\alpha(x'(\gamma_l))(t-\tau_l)^\alpha \ge \frac{\overline{p}(t)r(t)}{\overline{p}(\gamma_l)r(\gamma_l)} \Phi_\alpha(x'(t))(t-\tau_{l-1})^\alpha$

which implies $\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))} < \frac{\overline{p}(\gamma_l)r(\gamma_l)}{\overline{\overline{p}}(t)(t-\tau_l)^\alpha} \le \frac{r(\gamma_l)}{(t-\tau_l)^\alpha}$. This is $w(t) \le \frac{r(\gamma_l)}{(t-\tau_l)^\alpha}$.

Letting t tends to τ_{l+1}^-, we obtain

$$w(\tau_{l+1}) \le \frac{r_1}{(\tau_{l+1}-\tau_l)^\alpha}, \text{ for } \tau_{l+1} \in [c_1, d_1],\ l = k(c_1)+1, ..., k(d_1)-1, k \ne k(\delta_1). \tag{22}$$

Using similar analysis, we can get

$$w(\tau_l) \le \frac{r_1}{(\tau_l - c_1)^\alpha}, \quad \text{for } \tau_l \in [c_1, d_1],\ l = k(c_1)+1 \tag{23}$$

$$w(\tau_l) \le \frac{r_1}{(\tau_l - \delta_1)^\alpha}, \quad \text{for } \tau_l \in [c_1, d_1],\ l = k(\delta_1)+1 \tag{24}$$

Using case (i)-(vi), (14), (22)-(24) and (A3), we obtain

$$\left[\int_{c_1}^{\tau_{k(c_1)+1}} \frac{(t-\tau_{k(c_1)}-\sigma(t))^\alpha}{(t-\tau_{k(c_1)})^\alpha} + \sum_{l=k(c_1)+1}^{k(\delta_1)-1} \left(\int_{\tau_l}^{t_l} \frac{(t-\tau_l)^\alpha}{b_l^\alpha (t+\sigma(t)-\tau_l)^\alpha} + \int_{t_l}^{\tau_{l+1}} \frac{(t-\tau_l-\sigma(t))^\alpha}{(t-\tau_l)^\alpha} \right) \right.$$

$$+\int_{\tau_{k(\delta_1)}}^{t_{k(\delta_1)}} \frac{(t-\tau_{k(\delta_1)})^{\alpha}}{b_{k(\delta_1)}^{\alpha}(t+\sigma(t)-\tau_{k(\delta_1)})^{\alpha}} + \int_{t_{k(\delta_1)}}^{\delta_1} \frac{(t-\tau_{k(\delta_1)}-\sigma(t))^{\alpha}}{(t-\tau_{k(\delta_1)})^{\alpha}}\Bigg]Q(t)H_1(t,c_1)dt$$

$$+\int_{c_1}^{\delta_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_1(t,c_1)\,dt$$

$$\le \frac{b_{k(c_1)+1}^{\alpha}-a_{k(c_1)+1}^{\alpha}}{a_{k(c_1)+1}^{\alpha}} H_1(\tau_{k(c_1)+1},c_1)\frac{r_1}{(\tau_{k(c_1)+1}-c_1)^{\alpha}} + \sum_{l=k(c_1)+2}^{k(\delta_1)}\left[\frac{b_l^{\alpha}-a_l^{\alpha}}{a_l^{\alpha}}\right]H_1(\tau_l,c_1)\frac{r_1}{(\tau_{l+1}-\tau_l)^{\alpha}}$$

$$-H_1(\delta_l,c_1)w(\delta_1) \tag{25}$$

Next, multiplying both sides of (10) by $H_2(d_1,t)$ and then integrating over $[\delta_1,d_1]$ and using similar analysis to the above, we obtain

$$\Bigg[\int_{\delta_1}^{\tau_{k(\delta_1)+1}} \frac{(t-\tau_{k(\delta_1)}-\sigma(t))^{\alpha}}{(t-\tau_{k(\delta_1)})^{\alpha}} + \sum_{l=k(\delta_1)+1}^{k(d_1)-1}\left(\int_{\tau_l}^{t_l}\frac{(t-\tau_l)^{\alpha}}{b_l^{\alpha}(t+\sigma(t)-\tau_l)^{\alpha}} + \int_{t_l}^{\tau_{l+1}}\frac{(t-\tau_l-\sigma(t))^{\alpha}}{(t-\tau_l)^{\alpha}}\right)$$

$$+\int_{\tau_{k(d_1)}}^{t_{k(d_1)}} \frac{(t-\tau_{k(d_1)})^{\alpha}}{b_{k(d_1)}^{\alpha}(t+\sigma(t)-\tau_{k(d_1)})^{\alpha}} + \int_{t_{k(d_1)}}^{d_1} \frac{(t-\tau_{k(d_1)}-\sigma(t))^{\alpha}}{(t-\tau_{k(d_1)})^{\alpha}}\Bigg]Q(t)H_2(d_1,t)dt$$

$$+\int_{\delta_1}^{d_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_2(d_1,t)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_2(d_1,t)\,dt$$

$$\le \frac{b_{k(\delta_1)+1}^{\alpha}-a_{k(\delta_1)+1}^{\alpha}}{a_{k(\delta_1)+1}^{\alpha}} H_2(d_1,\tau_{k(\delta_1)+1})\frac{r_1}{(\tau_{k(\delta_1)+1}-\delta_1)^{\alpha}} + \sum_{l=k(\delta_1)+2}^{k(d_1)}\left[\frac{b_l^{\alpha}-a_l^{\alpha}}{a_l^{\alpha}}\right]H_2(d_1,\tau_l)\frac{r_1}{(\tau_{l+1}-\tau_l)^{\alpha}}$$

$$+H_2(d_1,\delta_1)w(\delta_1) \tag{26}$$

Dividing (25) and (26) by $H_1(\delta_l,c_1)$ and $H_2(d_1,\delta_1)$ respectively, then adding them, we get

$$\frac{1}{H_1(\delta_l,c_1)}\left(\Lambda_{c_1}^{\delta_1}\left[Q(t)H_1(t,c_1)\right]+\int_{c_1}^{\delta_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_1(t,c_1)\,dt\right)$$

$$+\frac{1}{H_2(d_1,\delta_1)}\left(\Lambda_{\delta_1}^{d_1}\left[Q(t)H_2(d_1,t)\right]+\int_{\delta_1}^{d_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_2(d_1,t)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_2(d_1,t)\,dt\right)$$

$$\le \frac{r_1}{H_1(\delta_l,c_1)}\Omega_{c_1}^{\delta_1}\left[H_1(.,c_1)\right]+\frac{r_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}\left[H_2(d_1,.)\right]$$

which contradicts (6) for j =1. This completes the proof when x(t) is positive. The proof when x(t) is eventually negative is analogous by repeating a similar argument on the interval [c_2, d_2].

Remark

When $p(t)=0$ and $\sigma(t)=\sigma$, Theorem 3 reduces to Theorem 2.8 of [9] and

when $p(t)=0$, Theorem 3 reduces to Theorem 2.1 of [14].

3. EXAMPLE

Consider the following impulsive differential equation

$$\begin{cases} x''(t)+p(t)x'(t)+v_1q_1(t)\Phi_{\frac{5}{2}}\left(x(t-\dfrac{\sin^2\pi t}{3})\right)+v_2q_2(t)\Phi_{\frac{1}{2}}\left(x(t-\dfrac{\sin^2\pi t}{3})\right)=e(t),\ t\neq\tau_k \\ x(\tau_k^{+})=a_kx(\tau_k),\quad x'(\tau_k^{+})=b_kx'(\tau_k),\ t=\tau_k, k=1,2,..., \end{cases} \tag{27}$$

where $p(t)=q_1(t)=q_2(t)=\begin{cases}(t-8n)^3, & t\in[8n,8n+3]\\ 27, & t\in[8n+3,8n+5]\\ (8n+8-t)^3, & t\in[8n+5,8n+8]\end{cases}$

$e(t)=\begin{cases}(t-8n)^3(t-8n-4)^3, & t\in[8n,8n+4]\\ (t-8n-4)^3(8n+8-t)^3, & t\in[8n+4,8n+8]\end{cases}$, v_2 and v_2 are positive constants,

$\tau_k:\tau_{n,1}=8n+\frac{3}{2},\ \tau_{n,2}=8n+\frac{5}{2},\ \tau_{n,3}=8n+\frac{11}{2},\ \tau_{n,4}=8n+\frac{13}{2},\ b_k\geq a_k>0,\ k\in\mathrm{N}$ are constants.

Here $r(t)=1,\ \alpha=1,\beta_1=\frac{5}{2},\beta_2=\frac{1}{2}$. Choose $\eta_1=\frac{1}{3},\eta_2=\frac{1}{3}$ then $\eta_0=\frac{1}{3}$. So the conditions of Lemma (1) are satisfied. For any $t_0>0$, we can choose n large enough such that $t_0<8n$ and $[c_1,d_1]=[8n+1,8n+3],\ [c_2,d_2]=[8n+5,8n+7],\ \delta_1=8n+2,\ \delta_2=8n+6$. It is easily verify that zero points of interval delay function $D_k(t)=t-\tau_k-(\sin^2\pi t)/3$ are $t_1\approx1.709\in(c_1,\delta_1]$, $t_2\approx2.709\in(\delta_1,d_1],\ t_3\approx5.709\in(c_2,\delta_2]$, $t_4\approx6.709\in(\delta_2,d_2]$. The variable delay function $\sigma(t)=\frac{1}{3}\sin^2\pi t$ satisfies $0\leq\sigma(t)\leq\sigma=\frac{1}{3}$. Also conditions $(C1),(\overline{C}1),(\overline{\overline{C}}1)$ and (A5) are satisfied. For $t\in[c_1,d_1]$, let $H_1(t,s)=H_2(t,s)=(t-s)^2.\ \therefore\ H_1(t,c_1)=(t-8n-1)^2, H_2(d_1,t)=(8n+3-t)^2$,

$$H_1(\delta_1,c_1)=H_2(d_1,\delta_1)=1. \quad h_1(t,s)=-h_2(t,s)=\frac{2}{(t-s)}. \quad h_1(t,c_1)=\frac{2}{(t-8n-1)}, \quad h_2(d_1,t)=\frac{2}{(8n+3-t)}.$$

$$Q(t)H_1(t,c_1)=\left(\frac{1}{3}\right)^{-1/3}\left|(t-8n)^3(t-8n-4)^3\right|^{1/3}\left[\left(\frac{1}{3}\right)^{-2/3}(v_1)^{1/3}(v_2)^{1/3}(t-8n)^2\right](t-8n-1)^2$$

$$=3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(t-8n-1)^2$$

$$\Lambda_{c_1}^{\delta_1}\left[Q(t)H_1(t,c_1)\right]=\int_{8n+1}^{8n+\frac{3}{2}}\frac{t-8n+\frac{3}{2}-\frac{\sin^2\pi t}{3}}{t-8n+\frac{3}{2}}3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(t-8n-1)^2\,dt$$

$$+\int_{8n+\frac{3}{2}}^{8n+1.709}\frac{t-8n-\frac{3}{2}}{b_{n,1}(t-8n-\frac{3}{2}+\frac{\sin^2\pi t}{3})}3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(t-8n-1)^2\,dt$$

$$+\int_{8n+1.709}^{8n+2}\frac{t-8n-\frac{3}{2}-\frac{\sin^2\pi t}{3}}{t-8n-\frac{3}{2}}3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(t-8n-1)^2\,dt$$

$$=3(v_1v_2)^{1/3}\left[\int_1^{\frac{3}{2}}\frac{u+\frac{3}{2}-\frac{\sin^2\pi u}{3}}{u+\frac{3}{2}}(u)^3(4-u)(u-1)^2\,du+\int_{\frac{3}{2}}^{1.709}\frac{(u-\frac{3}{2})(u)^3(4-u)(u-1)^2}{b_{n,1}(u-\frac{3}{2}+\frac{\sin^2\pi u}{3})}du\right.$$
$$\left.+\int_{1.709}^{2}\frac{u-\frac{3}{2}-\frac{\sin^2\pi u}{3}}{u-\frac{3}{2}}(u)^3(4-u)(u-1)^2\,du\right]$$

$$=3(v_1v_2)^{1/3}\left[0.2588+\frac{0.2304}{b_{n,1}}+2.3630\right]=3(v_1v_2)^{1/3}\left[2.6218+\frac{0.2304}{b_{n,1}}\right]$$

$$\int_{c_1}^{\delta_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_1(t,c_1)\,dt$$

$$=\int_{8n+1}^{8n+2}\left[0-\frac{1}{4}\left[\frac{2}{(t-8n-1)}-(t-8n)^3\right]^2\right](t-8n-1)^2\,dt$$

$$=-\frac{1}{4}\int_1^2\left[\left[\frac{2}{(u-1)}-u^3\right]^2\right](u-1)^2\,du=-1.3427.$$

$$\therefore\frac{1}{H_1(\delta_l,c_1)}\left(\Lambda_{c_1}^{\delta_1}\left[Q(t)H_1(t,c_1)\right]+\int_{c_1}^{\delta_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_1(t,c_1)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_1(t,c_1)\,dt\right)$$

$$=3(v_1v_2)^{1/3}\left[2.6218+\frac{0.2304}{b_{n,1}}\right]-1.3427.$$

$$Q(t)H_2(d_1,t)==3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(8n+3-t)^2$$

$$\Lambda_{\delta_1}^{d_1}\left[Q(t)H_2(d_1,t)\right]=\int_{8n+2}^{8n+\frac{5}{2}}\frac{t-8n-\frac{3}{2}-\frac{\sin^2\pi t}{3}}{t-8n-\frac{3}{2}}3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(3+8n-t)^2\,dt$$

$$+\int_{8n+\frac{5}{2}}^{8n+2.709}\frac{t-8n-\frac{5}{2}}{b_{n,2}(t-8n-\frac{5}{2}+\frac{\sin^2\pi t}{3})}3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(3+8n-t)^2\,dt$$

$$+\int_{8n+2.709}^{8n+3}\frac{t-8n-\frac{5}{2}-\frac{\sin^2\pi t}{3}}{t-8n-\frac{5}{2}}3(v_1v_2)^{1/3}(t-8n)^3(4+8n-t)(3+8n-t)^2\,dt$$

$$=3(v_1v_2)^{1/3}\left[\int_{2}^{\frac{5}{2}}\frac{u-\frac{3}{2}-\frac{\sin^2\pi u}{3}}{u-\frac{3}{2}}(u)^3(4-u)(3-u)^2\,du+\int_{\frac{5}{2}}^{2.709}\frac{(u-\frac{5}{2})(u)^3(4-u)(3-u)^2}{b_{n,2}(u-\frac{5}{2}+\frac{\sin^2\pi u}{3})}du\right.$$
$$\left.+\int_{2.709}^{3}\frac{u-\frac{5}{2}-\frac{\sin^2\pi u}{3}}{u-\frac{5}{2}}(u)^3(4-u)(3-u)^2\,du\right]$$

$$=3(v_1v_2)^{1/3}\left[4.6324+\frac{0.1785}{b_{n,2}}+0.1002\right]=3(v_1v_2)^{1/3}\left[4.7326+\frac{0.1785}{b_{n,2}}\right]$$

$$\int_{\delta_1}^{d_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_2(d_1,t)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_2(d_1,t)\,dt$$
$$=\int_{8n+2}^{8n+3}\left[0-\frac{1}{4}\left[\frac{2}{(8n+3-t)}-(t-8n)^3\right]^2\right](8n+3-t)^2\,dt.$$

$$=-\frac{1}{4}\int_{2}^{3}\left[\left[\frac{2}{3-u}-u^3\right]^2\right](3-u)^2\,du=-6.6118.$$

$$\therefore\frac{1}{H_2(d_1,\delta_1)}\left(\Lambda_{\delta_1}^{d_1}\left[Q(t)H_2(d_1,t)\right]+\int_{\delta_1}^{d_1}\left[q_0(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_2(d_1,t)-\frac{p(t)}{r(t)}\right|^{\alpha+1}\right]H_2(d_1,t)\,dt\right)$$

$$= \quad 3(v_1 v_2)^{1/3}\left[4.7326+\frac{0.1785}{b_{n,2}}\right]-6.6118 .$$

Since $r_1 = 1$ then the right side of the inequality (6) with j =1 is

$$\frac{r_1}{H_1(\delta_l, c_1)}\Omega_{c_1}^{\delta_1}\left[H_1(.,c_1)\right]+\frac{r_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}\left[H_2(d_1,.)\right]=\left[\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}\right]+\left[\frac{b_{n,2}-a_{n,2}}{4a_{n,2}}\right].$$

Thus (6) satisfied with j =1 if

$$3(v_1 v_2)^{1/3}\left[7.3544+\frac{0.2304}{b_{n,1}}+\frac{0.1785}{b_{n,2}}\right]>7.9545+\left[\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}\right]+\left[\frac{b_{n,2}-a_{n,2}}{4a_{n,2}}\right].$$

In a similar way, the inequality (6) satisfied with j =2 if

$$3(v_1 v_2)^{1/3}\left[7.1185+\frac{0.4714}{b_{n,3}}+\frac{0.0493}{b_{n,4}}\right]>12.8608+\left[\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}\right]+\left[\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}\right].$$

So, if we choose the constants v_1, v_2 large enough such that

$$\begin{cases} 3(v_1 v_2)^{1/3}\left[7.3544+\frac{0.2304}{b_{n,1}}+\frac{0.1785}{b_{n,2}}\right]>7.9545+\left[\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}\right]+\left[\frac{b_{n,2}-a_{n,2}}{4a_{n,2}}\right] \\ 3(v_1 v_2)^{1/3}\left[7.1185+\frac{0.4714}{b_{n,3}}+\frac{0.0493}{b_{n,4}}\right]>12.8608+\left[\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}\right]+\left[\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}\right] \end{cases}$$

Then by Theorem 3 equation (6) is oscillatory.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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