Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 6, 2390-2407
https://doi.org/10.28919/jmcs/4888
ISSN: 1927-5307

EXPLORATION OF SOME DISTRIBUTIONAL PROPERTIES, PARAMETER ESTIMATES AND APPLICATIONS OF THE WRAPPED QUASI LINDLEY DISTRIBUTION

SAHANA BHATTACHARJEE*

Department of Statistics, Gauhati University, Guwahati-781014, Assam, India

Abstract. A circular distribution called Wrapped Quasi Lindley distribution with two parameters has been recently proposed, but apart from the expressions for pdf, cdf, their circular representations, characteristic function and the maximum likelihood equations for the proposed distribution, no other properties of the distribution as well as the characteristics of the parameter estimates were explored by the authors. A slight error has also been observed in the expression for pdf of the distribution. Also, the application of the distribution in modeling real life data was not exhibited. Further, the form of the characteristic function in the paper is not compact and there is no closed form of the expression of the trigonometric moments. This paper thus aims to rectify the expression for pdf and explore a few descriptive measures and distributional properties of the Wrapped Quasi Lindley distribution and derive closed form expressions for the characteristic function and hence the trigonometric moments using an identity. it is found that the operations of wrapping and convoluting linear distributions around unit circle are commutative. The maximum likelihood estimates of the parameters of the distribution are shown to be consistent through a simulation study. The utility of the Wrapped Quasi Lindley model to a real-life data set on orientations is shown and the goodness-of-fit of the distribution is assessed and compared to that of the Wrapped Exponential and Wrapped Lindley distribution with the help of the log-likelihood, AIC and BIC measures. Further, the probabilities of the orientations to lie in a certain interval are estimated on the basis of the fitted Wrapped Quasi Lindley distribution.

*Corresponding author
E-mail address: sahana.bhattacharjee@hotmail.com

Received July 29, 2020

The distribution is found to be more appropriate in modeling the situations where the directions having lower magnitude have higher likelihood of occurrence.

Keywords: wrapped quasi Lindley distribution; circular data; trigonometric moments; convolution; simulation study.

2010 AMS Subject Classification: 60E05, 60E10, 62F10.

1. Introduction

Circular statistics is the specialized branch of statistics which deals with circular data, i.e. data which arise in terms of angles. Circular data is two-dimensional in nature and is represented either as a point on a circle of unit radius, centered at the origin or as a unit vector in the plane, connecting the origin to the corresponding point[17]. Circular data arise in various fields of science such as Geology (orientations of cross-beds in rivers, measured in degrees), Meteorology (wind direction), Biology (vanishing angles of birds soon after their release) [19], Medicine (times of arrival of patients in a casualty ward of a hospital), etc.

A sub class of circular probability distributions can be generated from the distributions on the real line through the Wrapping approach. In this approach, a linear random variable (r.v.) is transformed into a circular r.v. by reducing its modulo 2π. In other words, the circular r.v. corresponding to the linear r.v. X is $\theta = X (\bmod\ 2\pi)$. [12] introduced Wrapped distributions and obtained wrapped variables from the corresponding symmetric as well as non-symmetric distributions on the real line. Many authors, since then, have carried out comprehensive work on wrapped distributions. For instance, [9] studied the Wrapped Exponential and Wrapped Laplace distributions, discussed their properties and statistical inference. [1] and [18] derived the Wrapped Chi-square distribution and Wrapped Exponential distribution respectively and studied their properties. [7] introduced the Wrapped Geometric distribution and also obtained some generalizations of this distribution. [13] proposed the Lindley distribution and [5] extensively studied its several properties. The same authors also showed it to be better in modeling certain data sets in comparison to the classical Exponential distribution. [11] projected the Wrapped Lindley distribution and explored its various properties. [20] introduced the Quasi Lindley distribution, a generalization of the Lindley distribution and showed its flexibility over the Lindley and exponential distribution. [2] proposed the Wrapped Quasi Lindley distribution wherein they derived expressions for pdf, cdf, their circular graphical representations, characteristic function and the maximum likelihood equations. The pdf of the Wrapped Quasi Lindley distribution (WQLD) with parameters β and α has been found to be

$$g(\theta) = \frac{\theta \exp(-\beta\theta)}{\alpha+1}\left[\frac{(\alpha+\theta\beta)}{1-\exp(-2\pi\beta)} + \frac{\{2\pi\beta\exp(2\pi\beta)\}}{\{-1+\exp(2\pi\beta)\}^2}\right]$$

However, the second term within the parentheses is incorrect; and consequently, the correct expression for the pdf of the Wrapped Quasi Lindley distribution with parameters θ and α has been derived and is shown below:

$$\begin{aligned}
f_w(\beta) &= \sum_{k=0}^{\infty} f(\beta+2k\pi) \\
&= \sum_{k=0}^{\infty} \frac{\theta\{\alpha+\theta(\beta+2k\pi)\}}{\alpha+1}\exp\{-\theta(\beta+2k\pi)\} \\
&= \frac{\theta\exp(-\beta\theta)}{\alpha+1}\left[\alpha\sum_{k=0}^{\infty}\exp(-2k\pi\theta)+\theta\beta\sum_{k=0}^{\infty}\exp(-2k\pi\theta)+\right. \\
&\qquad \left. 2\pi\theta\sum_{k=0}^{\infty}k\exp(-2k\pi\theta)\right] \\
&= \frac{\theta\exp(-\beta\theta)}{\alpha+1}\left[\frac{(\alpha+\theta\beta)}{1-\exp(-2\pi\theta)}+\frac{\{2\pi\theta\exp(-2\pi\theta)\}}{\{1-\exp(-2\pi\theta)\}^2}\right], \qquad (1) \\
&\qquad \theta>0, \alpha>0, \beta\in(0,2\pi]
\end{aligned}$$

where the r.v. β follows the Wrapped Quasi Lindley distribution with parameters θ and α, denoted by $WQLD(\theta,\alpha)$.

The closed form of the expressions for the characteristic function and hence, the trigonometric moments and other related measures of this distribution are derived in Section 2. In the same section, the commutativity of the operations of wrapping and convoluting linear distributions on the unit circle is established. Simulation study to show the consistency of the maximum likelihood estimators of the parameters of the distribution is displayed in Section 3. In Section 4, the proposed model is applied to the data on orientation of 50 starhead topminnows and its goodness-of-fit to the data set considered is assessed. Also, the comparison of the fit of the proposed model to the data with that of the wrapped Exponential and Wrapped Lindley distribution is carried out. The estimation of the probabilities of the orientations to lie in a certain interval, on the basis of the fitted Wrapped Quasi Lindley distribution is also presented in this section. Finally, Section 5 summarizes the findings of the paper.

2. PROPERTIES OF $WQLD(\theta,\alpha)$

In this section, some distributional properties of $WQLD(\theta,\alpha)$ are explored and the expressions for the characteristic function, trigonometric moments, coefficient of skewness and kurtosis and the median direction of the $WQL(\theta,\alpha)$ are derived. Further, a necessary and sufficient condition for a circular r.v. to follow WQLD is also established.

2.1. Derivation of WQLD density from mixture of Wrapped Exponential and Wrapped Gamma densities. We know that the operations of wrapping and mixing commute [10]. Also, the Quasi Lindley distribution defined on the real line arises as a mixture of the Exponential and the Gamma distribution [20]. Consequently, the $WQL(\theta,\alpha)$ arises as a mixture of Wrapped Exponential (θ) and Wrapped Gamma $(2,\theta)$ distribution as shown below:
The mixture of the Wrapped exponential (θ) and Wrapped Gamma $(2,\theta)$ distribution with mixing parameter $\frac{\alpha}{\alpha+1}$ has the density function given by

$$\begin{aligned} f^{w}(\beta) &= \frac{\alpha}{\alpha+1}\left[\frac{\theta\exp(-\beta\theta)}{1-\exp(-2\pi\theta)}\right]+\frac{1}{\alpha+1}\left[\sum_{k=0}^{\infty}\theta^{2}\exp\{-\theta(\beta+2\pi k)\}(\beta+2\pi k)\right] \\ &= \frac{\theta\exp(-\beta\theta)}{\alpha+1}\left[\frac{(\alpha+\theta\beta)}{1-\exp(-2\pi\theta)}+\frac{\{2\pi\theta\exp(-2\pi\theta)\}}{\{1-\exp(-2\pi\theta)\}^{2}}\right] \end{aligned}$$

which is the p.d.f. of $WQL(\theta,\alpha)$. Consequently, the Wrapped Quasi Lindley $WQL(\theta,\alpha)$ r.v. Θ admits the representation

$$\Theta \stackrel{d}{=} I\Theta_1+(1-I)\Theta_2$$

where Θ_1 and Θ_2 are independent Wrapped exponential (θ) and Wrapped Gamma $(2,\theta)$ r.v.'s respectively; I is an indicator r.v. which takes on values 1 and 0 with probabilities $\frac{\alpha}{1+\alpha}$ and $\frac{1}{1+\alpha}$ respectively, independently of Θ_1 and Θ_2. Here, $\stackrel{d}{=}$ denotes distributional equivalence.

2.2. Characteristic Function. In this paper, the characteristic function and hence, the trigonometric moments of the WQLD have been obtained in terms of simpler closed form expressions as shown below:

The characteristic function of a wrapped circular variable, say φ_p at an integer value p can be obtained from the characteristic function of the corresponding unwrapped linear r.v, say $\phi_X(t)$ via the following relation [17]:

$$\varphi_p = \phi_X(p) \tag{2}$$

The characteristic function of the Quasi Lindley (θ, α) distribution is given by

$$\phi_X(t) = \frac{\theta\alpha(\theta - it) + \theta^2}{(\alpha+1)(\theta - it)^2}; \quad i = \sqrt{-1} \tag{3}$$

Therefore, the characteristic function of $WQL(\theta, \alpha)$ is

$$\varphi_p = \frac{\left[\theta^2(\alpha+1) - ip\theta\alpha\right](\theta - ip)^{-2}}{(\alpha+1)} \quad p = 0, \pm 1, \pm 2, \ldots \tag{4}$$

Using the result of [18] which gives $\forall a,b,r \in R^+$, $(a-ib)^{-r} = \left(a^2+b^2\right)^{\frac{-r}{2}} e^{ir\arctan\left(\frac{b}{a}\right)}$ or $\left(a^2+b^2\right)^{\frac{-r}{2}} e^{ir\tan^{-1}\left(\frac{b}{a}\right)}$, the following expressions are obtained

$$(\theta - ip)^{-2} = \left(\theta^2 + p^2\right)^{-1} \exp\left\{2i\tan^{-1}\left(\frac{p}{\theta}\right)\right\}$$

$$\left[\theta^2(\alpha+1) - ip\theta\alpha\right]^{-1} = \left[\theta^4(\alpha+1)^2 + p^2\theta^2\alpha^2\right]^{\frac{-1}{2}} \exp\left\{i\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right\}$$

So, the characteristic function of $WQL(\theta, \alpha)$ is finally obtained as,

$$\varphi_p = \frac{\left[\theta^4(\alpha+1)^2 + p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)} \exp\left[2i\tan^{-1}\left(\frac{p}{\theta}\right) - i\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right] \tag{5}$$

This provides a closed form and simpler expression of the characteristic function of WQLD. Again, an alternative expression for φ_p is

$$\varphi_p = \rho_p \exp(i\mu_p).$$

Comparing the above two equations, ρ_p and μ_p are obtained as

$$\rho_p = \frac{\left[\theta^4(\alpha+1)^2 + p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)} \tag{6}$$

$$\mu_p = 2\tan^{-1}\left(\frac{p}{\theta}\right) - \tan^{-1}\left(\frac{p\alpha}{\theta\left(1+\alpha\right)}\right) \tag{7}$$

A necessary and sufficient condition for a circular r.v. to follow $WQLD(\theta,\alpha)$ is presented in the following remark:

Remark 1: $\Theta \sim WLD(\theta,\alpha)$ *if and only if* $2\pi - \Theta \sim WLD(-\theta,\alpha)$, *where* $\theta,\alpha > 0$.

Proof: We have, the c.f. of $\beta \sim WQL(\theta,\alpha)$ as

$$\begin{aligned}
\varphi_p\left(\beta\right) &= E\left(e^{ip\beta}\right) \\
&= \frac{\left[\theta^4\left(\alpha+1\right)^2 + p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{\left(\alpha+1\right)\left(\theta^2+p^2\right)} \exp\left[2i\arctan\left(\frac{p}{\theta}\right) - i\arctan\left(\frac{p\alpha}{\theta\left(1+\alpha\right)}\right)\right]
\end{aligned}$$

Therefore, the c.f. of $(2\pi - \beta)$ is

$$\begin{aligned}
\varphi_p\left(2\pi-\beta\right) &= E\left(e^{ip(2\pi-\beta)}\right) \\
&= e^{ip2\pi}E\left(e^{-ip\beta}\right) \\
&= \left(\cos 2\pi + i\sin 2\pi\right)\frac{\left[\theta^4\left(\alpha+1\right)^2 + \left(-p\right)^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{\left(\alpha+1\right)\left(\theta^2+\left(-p\right)^2\right)} \\
&\quad \exp\left[2i\arctan\left(\frac{-p}{\theta}\right) - i\arctan\left(\frac{-p\alpha}{\theta\left(1+\alpha\right)}\right)\right] \\
&= \frac{\left[\theta^4\left(\alpha+1\right)^2 + p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{\left(\alpha+1\right)\left(\theta^2+p^2\right)} \exp\left[2i\arctan\left(\frac{-p}{\theta}\right) - i\arctan\left(\frac{-p\alpha}{\theta\left(1+\alpha\right)}\right)\right]
\end{aligned}$$

which is the c.f. of $WQL(-\theta,\alpha)$.

Conversely, suppose $2\pi - \beta = \beta' \sim WQL(-\theta,\alpha)$. Then the c.f. of β' is

$$\begin{aligned}
\varphi_p\left(\beta'\right) &= E\left(e^{ip\beta'}\right) \\
&= \frac{\left[\theta^4\left(\alpha+1\right)^2 + p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{\left(\alpha+1\right)\left(\theta^2+p^2\right)} \exp\left[2i\arctan\left(\frac{-p}{\theta}\right) - i\arctan\left(\frac{-p\alpha}{\theta\left(1+\alpha\right)}\right)\right]
\end{aligned}$$

So, the c.f. of β is

$$\begin{aligned}\varphi_p(\beta) &= E\left(e^{ip\beta}\right)\\ &= E\left(e^{ip(2\pi-\beta')}\right)\\ &= e^{ip2\pi}E\left(e^{-ip\beta'}\right)\\ &= \frac{\left[\theta^4(\alpha+1)^2+p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)}\\ &\quad \exp\left[2i\arctan\left(\frac{p}{\theta}\right)-i\arctan\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right]\end{aligned}$$

which is the c.f. of $WQL(\theta,\alpha)$.

2.3. Trigonometric moments and related descriptive measures. Let $\beta \sim WQL(\theta,\alpha)$. The p^{th} non-central trigonometric moment of β is given by [4]

$$\varphi_p = \alpha_p + i\beta_p$$

where $\alpha_p = \rho_p\cos\mu_p$ and $\beta_p = \rho_p\sin\mu_p$. Therefore, equation (6) and (7) give

$$\alpha_p = \frac{\left[\theta^4(\alpha+1)^2+p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)}\cos\left[2\tan^{-1}\left(\frac{p}{\theta}\right)-\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right] \tag{8}$$

$$\beta_p = \frac{\left[\theta^4(\alpha+1)^2+p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)}\sin\left[2\tan^{-1}\left(\frac{p}{\theta}\right)-\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right] \tag{9}$$

We see that the expressions of the trigonometric moments are also simpler in comparison to those obtained in the paper by [2]. We know that the non-central trigonometric moments of β are the Fourier coefficients in the Fourier series expansion of the p.d.f $f(\beta)$. Consequently, we have the following Fourier representation of the $WQL(\theta,\alpha)$ density:

$$
\begin{aligned}
f(\beta) &= \sum_{p=-\infty}^{\infty} \varphi_p e^{-ip\beta} \\
&= \frac{1}{2\pi}\left[1+2\sum_{p=1}^{\infty}\left(\alpha_p \cos p\beta + \beta_p \sin p\beta\right)\right] \\
&= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{p=1}^{\infty}\left\{\frac{\left[\theta^4(\alpha+1)^2+p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)}\cos\left[2\tan^{-1}\left(\frac{p}{\theta}\right)-\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right]\right. \\
&\left.\cos p\beta+\frac{\left[\theta^4(\alpha+1)^2+p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)}\sin\left[2\tan^{-1}\left(\frac{p}{\theta}\right)-\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right]\sin p\beta\right\}
\end{aligned}
$$

In particular, the mean resultant length of β is

$$
\rho_1=\rho=\frac{\left[\theta^4(\alpha+1)^2+\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+1)} \tag{10}
$$

and the mean direction of β is

$$
\mu_1=\mu=2\tan^{-1}\left(\frac{1}{\theta}\right)-\tan^{-1}\left(\frac{\alpha}{\theta(1+\alpha)}\right) \tag{11}
$$

ρ indicates the extent of concentration of β towards the mean direction μ and it lies between 0 and 1. The closer it is to 1, the higher is the concentration towards μ. The c.f. and the trigonometric moments of $WQL(\theta,\alpha)$ can also be obtained through the c.f. and moments of the mixing distributions of $WQL(\theta,\alpha)$: $WE(\theta)$ and $WG(2,\theta)$.

The circular variance of β is defined by

$$
V=1-\rho=1-\frac{\left[\theta^4(\alpha+1)^2+\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+1)} \tag{12}
$$

The interpretation of V is contrary to that of ρ.

The p^{th} central trigonometric moments of β are given by

$$
\begin{aligned}
\bar{\alpha_p} &= \rho_p\cos(\mu_p-p\mu) \\
\bar{\beta_p} &= \rho_p\sin(\mu_p-p\mu)
\end{aligned}
$$

Therefore, the p^{th} central trigonometric moments of the proposed distribution are

$$\bar{\alpha_p} = \frac{\left[\theta^4(\alpha+1)^2+p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)}\cos\left[\left\{2\tan^{-1}\left(\frac{p}{\theta}\right)-\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right\}- p\left\{2\tan^{-1}\left(\frac{1}{\theta}\right)-\tan^{-1}\left(\frac{\alpha}{\theta(1+\alpha)}\right)\right\}\right] \tag{13}$$

$$\bar{\beta_p} = \frac{\left[\theta^4(\alpha+1)^2+p^2\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)(\theta^2+p^2)}\sin\left[\left\{2\tan^{-1}\left(\frac{p}{\theta}\right)-\tan^{-1}\left(\frac{p\alpha}{\theta(1+\alpha)}\right)\right\}- p\left\{2\tan^{-1}\left(\frac{1}{\theta}\right)-\tan^{-1}\left(\frac{\alpha}{\theta(1+\alpha)}\right)\right\}\right] \tag{14}$$

The measures of skewness and kurtosis, denoted by ζ_1^0 and ζ_2^0 respectively are defined as

$$\zeta_1^0 = \frac{\bar{\beta}_2}{V^{\frac{3}{2}}}, \qquad \zeta_2^0 = \frac{\bar{\alpha}_2-\rho^4}{V^2}$$

Thus, ζ_1^0 and ζ_2^0 for $WQL(\theta,\alpha)$ is given by

$$\zeta_1^0 = \frac{\bar{\beta}_2}{\left\{1-\frac{\left[\theta^4(\alpha+1)^2+\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)\left(\theta^2+1\right)}\right\}^{\frac{3}{2}}} \tag{15}$$

$$\zeta_2^0 = \frac{\bar{\alpha}_2-\rho^4}{\left\{1-\frac{\left[\theta^4(\alpha+1)^2+\theta^2\alpha^2\right]^{\frac{1}{2}}}{(\alpha+1)\left(\theta^2+1\right)}\right\}^{2}} \tag{16}$$

ζ_1^0 is nearly zero for unimodal symmetric data sets and ζ_2^0 is close to zero for the data sets which are single peaked and for which the Wrapped normal distribution provides a good fit ([16]).

The median direction of a circular distribution having density $f(.)$, denoted by ξ_0 is the solution of the following equation in the interval $[0,2\pi)$ [9]:

$$\int_{\xi_0}^{\xi_0+\pi} f(\beta)\,d\beta=\int_{\xi_0+\pi}^{\xi_0+2\pi} f(\beta)\,d\beta = \frac{1}{2}$$

where f is such that $f(\xi_0) > f(\xi_0+\pi)$.

Thus, we have

$$\int_{\xi_0}^{\xi_0+\pi}\frac{\theta\exp(-\beta\theta)}{\alpha+1}\left[\frac{(\alpha+\theta\beta)}{1-\exp(-2\pi\theta)}+\frac{\{2\pi\theta\exp(-2\pi\theta)\}}{\{1-\exp(-2\pi\theta)\}^2}\right]d\beta = \frac{1}{2} \tag{17}$$

and

$$\int_{\xi_0+\pi}^{\xi_0+2\pi} \frac{\theta \exp(-\beta\theta)}{\alpha+1}\left[\frac{(\alpha+\theta\beta)}{1-\exp(-2\pi\theta)}+\frac{\{2\pi\theta\exp(-2\pi\theta)\}}{\{1-\exp(-2\pi\theta)\}^2}\right]d\beta = \frac{1}{2} \tag{18}$$

Simplifying and then adding (17) and (18), we get

$$\begin{aligned}
1 &= \frac{\alpha}{(\alpha+1)\{1-\exp(-2\pi\theta)\}}\left[\exp(-\theta\xi_0)-\exp(-\theta(\xi_0+2\pi))\right]+\\
&\quad \frac{\theta^2}{(\alpha+1)\{1-\exp(-2\pi\theta)\}}\left[\xi_0\exp(-\theta\xi_0)-(\xi_0+2\pi))\exp(-\theta(\xi_0+2\pi))\right]+\\
&\quad \frac{1}{(\alpha+1)\{1-\exp(-2\pi\theta)\}}\left[\exp(-\theta\xi_0)-\exp(-\theta(\xi_0+2\pi))\right]+\\
&\quad \frac{2\pi\theta\exp(-2\pi\theta)}{(\alpha+1)\{1-\exp(-2\pi\theta)\}^2}\left[\exp(-\theta\xi_0)-\exp(-\theta(\xi_0+2\pi))\right]\\
\Rightarrow 1 &= \exp(-\theta\xi_0)\left[1+\frac{2\pi\theta\exp(-2\pi\theta)}{(\alpha+1)\{1-\exp(-2\pi\theta)\}}+\theta^2\left\{\frac{2\pi\exp(-2\pi\theta)}{(\alpha+1)\{1-\exp(-2\pi\theta)\}}+\right.\right.\\
&\quad \left.\left.\frac{\xi_0}{(\alpha+1)}\right\}\right] \qquad (19)
\end{aligned}$$

ξ_0 is obtained by solving equation (19). The values of the above measures for different values of θ and α are listed in Table 1. Table 1 shows that when α and θ is kept fixed, concentration towards the mean direction and kurtosis increase with an increase in θ and α respectively whereas the distribution becomes more negatively skewed. The median diection of the distribution approaches the zero direction with an increase in the values of both θ and α.

TABLE 1. Values of the trigonometric moments and related measures of $WQL(\theta,\alpha)$ for the different values of θ and α

	$\alpha=0.6$				$\alpha=1.5$			
Measure	$\theta=0.2$	$\theta=0.7$	$\theta=1.4$	$\theta=2.5$	$\theta=0.2$	$\theta=0.7$	$\theta=1.4$	$\theta=2.5$
μ	1.665	1.428	0.978	0.612	1.497	1.211	0.835	0.525
ρ	0.081	0.373	0.685	0.871	0.121	0.433	0.720	0.886
V	0.919	0.627	0.315	0.129	0.878	0.566	0.279	0.113
ζ_1^0	-0.043	-0.301	-1.067	-2.292	-0.072	-0.426	-1.299	-2.627
ζ_2^0	-0.006	0.095	1.023	3.063	0.008	0.256	1.519	3.971
ξ_0	1.748	0.3339	0.136	0.0107	1.149	0.0713	0.0020	0.0010
	$\theta=0.5$				$\theta=1.7$			
Measure	$\alpha=0.3$	$\alpha=0.8$	$\alpha=1.5$	$\alpha=2.3$	$\alpha=0.3$	$\alpha=0.8$	$\alpha=1.5$	$\alpha=2.3$
μ	1.781	1.487	1.338	1.265	0.928	0.807	0.724	0.674
ρ	0.220	0.267	0.312	0.343	0.749	0.767	0.787	0.802
V	0.780	0.733	0.688	0.657	0.251	0.233	0.213	0.198
ζ_1^0	-0.115	-0.188	-0.250	-0.292	-1.318	-1.509	-1.704	-1.849
ζ_2^0	-0.015	0.032	0.096	0.148	1.374	1.754	2.230	2.628
ξ_0	0.445	0.296	0.200	0.146	0.303	0.004	0.0004	0.0002

We already know that the operations of wrapping and mixing linear distributions around a unit circle commute. The operations of wrapping and convoluting linear distributions around a unit circle also commute and the result is expressed in Remark 2 which is as stated below:

Remark 2: The circular distribution obtained by wrapping the convolution of two linear distributions around a unit circle coincide with the convolution of their corresponding wrapped distributions.

Proof: Suppose X and Y be two independently distributed r.v.'s with the density functions $f_X(.)$ and $f_Y(.)$ respectively. Further, let $X \in S \subseteq \mathbb{R}$. Then the density function of $Z = X + Y$ is obtained as

$$f_Z(z) = \int_S f_X(t)\, f_Y(z-t)\, dt$$

The wrapped distribution corresponding to $f_Z(.)$ is given by

$$\begin{aligned}
f_Z^w(z) &= \sum_{k=-\infty}^{\infty} f_z(z+2\pi k) \\
&= \sum_{k=-\infty}^{\infty} \int_S f_X(t)\, f_Y(z+2\pi k-t)\, dt \\
&= \sum_{k=-\infty}^{\infty} \int_S f_Y(z-t+2\pi k)\, f_X(t)\, dt \\
&= \int_S \sum_{k=-\infty}^{\infty} f_Y(z-t+2\pi k)\, f_X(t)\, dx \\
&= \sum_{-\infty}^{\infty} \left\{ \int_0^{2\pi} f_Y^w(z-(t+2\pi j))\, f_X(t+2\pi j) \right\} dt \\
&= \sum_{-\infty}^{\infty} \left\{ \int_0^{2\pi} f_Y^w(z-t-2\pi j)\, f_X(t+2\pi j) \right\} dt \\
&= \sum_{-\infty}^{\infty} \int_0^{2\pi} f_Y^w(z-t)\, f_X(t+2\pi j)\, dt \qquad [f(\theta) = f(\theta+2\pi)] \\
&= \int_0^{2\pi} \sum_{-\infty}^{\infty} f_X(t+2\pi j)\ f_Y^w(z-t)\, dt \\
&= \int_0^{2\pi} f_X^w(t)\ f_Y^w(z-t)\, dt
\end{aligned}$$

which coincides with the convolution of the wrapped densities corresponding to X and Y. By virtue of the Fubini's theorem, since the integral is bounded and the integrands are non-negative, the interchanging of the order of integration and summation is valid.

The maximum likelihood estimates of the parameters θ and α of $WQL(\theta,\alpha)$ maximize the log-likelihood function given by

$$\log L = n\log\theta - \theta\sum_{i=1}^{n}\beta_i - n\log(\alpha+1) + \sum_{i=1}^{n}\log\left[\left\{\frac{\alpha+\theta\beta_i}{1-\exp(-2\pi\theta)}\right\} + \left\{\frac{2\pi\theta\exp(-2\pi\theta)}{(1-\exp(-2\pi\theta))^2}\right\}\right] \tag{20}$$

w.r.t variations in θ and α, where $\beta_1,\beta_2,\ldots\beta_n$ is a random sample of size n from $WQL(\theta,\alpha)$. Since the maximum likelihood equations are non-linear in nature and difficult to be solved analytically, a suitable numerical technique is employed to obtain solutions for θ and α [2].

3. SIMULATION STUDY

A simulation study to generate random variables from $WQL(\theta,\alpha)$ is carried out and m.l.e of the parameters θ and α are obtained. For different values of θ and α, samples of size 100, 250, 500 and 800 are generated. The program is replicated $N = 1000$ times to get the m.l.e. of θ and α. Steps of the simulation algorithm to obtain the m.l.e. of the parameters are as given below:

- *Step 1:* A random variable is generated from the $U(0,1)$ distribution, say u.
- *Step 2:* The expression of c.d.f. of $WQL(\theta,\alpha)$ is equated with u and is solved for β, which is a r.v. from $WQL(\theta,\alpha)$. Steps 1 and 2 are repeated to get a sample of the desired size.
- *Step 3:* The m.l.e of θ and α is obtained by maximizing the log-likelihood function in (20) for values of β generated in Step 2 w.r.t variations in θ and α respectively.

To calculate the average bias and MSE of the m.l.e., the following formulae are used:

Let the true value of the parameter θ be θ^* and the m.l.e be $\hat{\theta}$. Then the Bias and MSE of $\hat{\theta}$ in estimating θ is given by

$$\begin{aligned} Bias\left(\hat{\theta}\right) &= \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta^*\right) \\ MSE\left(\hat{\theta}\right) &= \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta^*\right)^2 \end{aligned}$$

where N is the number of replications and $\hat{\theta}_i$ is the m.l.e. of θ obtained in the i^{th} replicate. Similarly, the Bias and MSE of the m.l.e. of α are calculated. The m.l.e. is consistent if the Bias and MSE decreases (approaches to zero) with an increase in the sample size. Table 2 shows the average values of the Bias and MSE of the m.l.e. of θ and α for the different sample sizes and for different set of values of θ and α. The results in Table 2 show that the Bias and MSE of the m.l.e of both θ and α approaches towards zero with an increase in the sample size. This shows that the estimates of the parameters are accurate, precise, and hence, consistent.

TABLE 2. Average values of Bias and MSE of the m.l.e of α and θ for different sample sizes and for different values of α and θ

	$\alpha = 0.5, \theta = 0.7$				$\alpha = 0.8, \theta = 1.5$			
n	$Bias(\alpha)$	$MSE(\alpha)$	$Bias(\theta)$	$MSE(\theta)$	$Bias(\alpha)$	$MSE(\alpha)$	$Bias(\theta)$	$MSE(\theta)$
100	-0.0063	0.0052	-0.0098	0.0087	-0.0084	0.0077	-0.0165	0.0313
250	-0.0013	0.0004	-0.0021	0.0014	-0.0047	0.0031	-0.0068	0.0121
500	-0.0007	0.0002	-0.0015	0.0008	-0.0016	0.0020	-0.0021	0.0067
800	-0.0004	0.0001	-0.0011	0.0003	-0.0009	0.0013	-0.0007	0.0033

The calculation of the trigonometric moments, other related measures is carried out using the **R** software, version 3.4.0, through the user-contributed packages viz. *CircStats* [14] and *circular* [15] with the help of self-programmed codes. The *maxLik* package [22] is used to obtain the maximum likelihood estimates of the parameters and *rootSolve* package [21] is used to generate random variables from $WQL(\theta, \alpha)$ and to solve non-linear equations.

4. Analysis and Results

This section comprises of the application of WQLD to a real-life data set. Further, the fit of this distribution to the data is compared with that of the Wrapped Lindley distribution [11] and Wrapped Exponential distribution [8] with the help of the statistics - log likelihood, AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion). The data set considered here is the sun compass orientations of 50 starhead topminnows, measured under heavily overcast conditions, which is procured from [6] and published in [4], Appendix B4.

The Wrapped Quasi Lindley distribution $WQL(\theta,\alpha)$, Wrapped Lindley distribution $WL(\theta)$ and Wrapped Exponential distribution $WE(\theta)$ are applied to the data set and the parameters are estimated. Table 3 summarizes the estimated parameters.

TABLE 3. Values of the statistics and other measures for the $WQL(\theta)$, $WL(\theta)$ and $WE(\theta)$ fitted to the data on orientations of starhead topminnows

Distribution	m.l.e of the parameters	Log-likelihood	AIC	BIC
$WQL(\theta,\alpha)$	$\theta = 0.1209, \alpha = 19.0512$	-90.8215	185.643	189.4671
$WL(\theta)$	$\theta = 0.5138$	-94.7979	191.5958	193.5079
$WE(\theta)$	$\theta = 0.1149$	-95.8932	193.5157	195.1783

The goodness-of-fit of $WQL(\theta,\alpha)$ to the data is checked using Watson's U^2 one sample test [3]. The p-value of the test is > 0.05, which shows that the Wrapped Quasi Lindley distribution is a good fit to the given data. Figure 1 exhibits the distribution function plot of $WQL(0.1209, 19.0512)$ fitted to the data. Smaller AIC and BIC values corresponding to $WQL(\theta,\alpha)$ in comparison to $WL(\theta)$ and $WE(\theta)$ is a consequence of the proposed distribution being a two-parameter model, whereas the other two are one-parameter model. However, smaller value of the log-likelihood corresponding to the $WQL(\theta,\alpha)$ and $WE(\theta)$ clearly shows that the Wrapped Quasi Lindley distribution fits the data better than the Wrapped Lindley and Wrapped Exponential distribution.

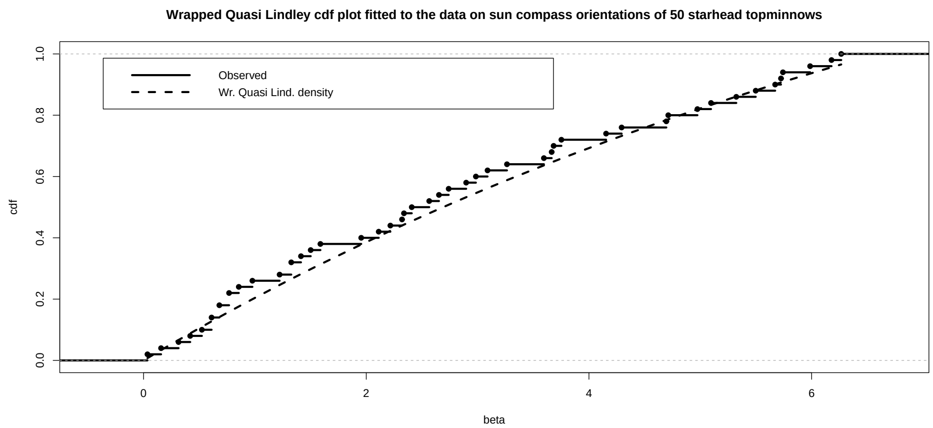

FIGURE 1. Distribution function plot of $WQL(0.1209, 19.0512)$ fitted to the data on sun compass orientations of 50 starhead topminnows

Table 4 shows the estimated probabilities of the orientation of the 50 starhead topminnows to lie in a certain interval, based on the best fitting Wrapped Quasi Lindley distribution with parameters $\theta = 0.1209, \alpha = 19.0512$.

TABLE 4. Estimated probabilities of the orientation of the 50 starhead topminnows to lie in a certain interval, based on the best fitting $WQL(0.1209, 19.0512)$

Range (in degrees)	Estimated probability
$[0, 50)$	0.1758
$[50, 100)$	0.1591
$[100, 150)$	0.1440
$[150, 200)$	0.1303
$[200, 250)$	0.1179
$[250, 300)$	0.1067
$[300, 360)$	0.1151

It is clear from Table 4 that the starhead topminnows are most likely to move in the direction $[0^o, 50^o)$ as measured by the sun compass. This also supports the intuitive claim made in section (2) that the Wrapped Quasi Lindley model is more appropriate for data for which the directions of lower magnitude have a higher probability of occurrence.

5. CONCLUDING REMARKS

In this paper, a few descriptive measures and distributional properties of the Wrapped Quasi Lindley distribution are explored and closed form expressions for the characteristic function and hence the trigonometric moments using an identity are derived. The behaviour of the various descriptive measures of the distribution are studied and a necesaary and sufficient condition for a circular r.v. to follow the WQLD is established. The operations of wrapping and convoluting linear distributions around unit circle are found to be commutative. A simulation study is performed to check the consistency of the maximum likelihood estimates of the parameters of the distribution. Lastly, to show application of the proposed model, the real data set on orientations of 50 starhead topminnows is modeled using this distribution. The goodness-of-fit test applied to the data showed that the Wrapped Quasi Lindley distribution is a good fit. Also, the multiple goodness-of-fit statistics showed the Wrapped Quasi Lindley distribution to give a better fit than Wrapped Exponential and Wrapped Lindley distribution. From the density plots for the Wrapped Quasi Lindley distribution and with the help of estimated probabilities for the data set considered, it is found that this distribution is more appropriate in modeling the situations where the directions having lower magnitude have higher likelihood of occurrence. This exhibits the usefulness of the distribution.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] M. Adnan, S. Roy, Wrapped chi-square distribution. J. Appl. Stat. Sci. 18 (2011), 307-317.

[2] A.M.H. Al-khazaleh, S. Alkhazaleh, On Wrapping of Quasi Lindley Distribution. Mathematics. 7 (2019), 930.

[3] S. Bhattacharjee, K. K. Das, Comparison of estimation methods of the joint density of a circular and linear variable. J. Data Sci. 15 (2017), 129-154.

[4] N. I. Fisher, Statistical Analysis of Circular Data. Cambridge University Press, Cambridge, 1993.

[5] M. Ghitany, B. Atieh, S. Nadarajah, Lindley distribution and its applications. Math. Comput. Simul. 78(4)(2008), 493-506.

[6] C. P. Goodyear, Terrestrial and Aquatic Orientation in the Starhead Topminnow, Fundulus Notti, Science. 168 (1970), 603–605.

[7] S. Jacob, K. Jayakumar, Wrapped geometric distribution: A new probability model for circular data. J. Stat. Theory Appl. 12 (4) (2013), 348-355.

[8] S. R. Jammalamadaka, T. Kozubowski, A wrapped exponential circular model. Proc. AP Acad. Sci. 5 (2001), 43–56.

[9] S. R. Jammalamadaka, T. Kozubowski, New families of wrapped distributions for modeling skew circular data. Commun. Stat., Theory Meth. 33 (2004), 2059-2074.

[10] S. R. Jammalamadaka, T.J. Kozubowski, A General Approach for Obtaining Wrapped Circular Distributions via Mixtures, Sankhya A. 79 (2017), 133–157.

[11] S. Joshi, K. Jose, Wrapped Lindley distribution, Commun. Stat., Theory Meth. 47 (5) (2017), 1013-1021.

[12] P. L. Lévy, Addition des variables aléatoires définies sur une circonférence. Bull. Soc. Math. Fr. 67 (1939), 1-41.

[13] D. Lindley, Fiducial distributions and Baye's theorem, J. R. Stat. Soc. Ser. B. 20 (1) (1958), 102-107.

[14] U. Lund, C. Agostinelli, CircStats: Circular Statistics, from "Topics in Circular Statistics" (2001), R package version 0.2-4 (2018). URL: https://cran.r-project.org/web/packages/CircStats/CircStats.pdf.

[15] Lund, U. and Agostinelli, C. circular: Circular Statistics, R package version 0.4-7 (2017). URL: https://cran.r-project.org/web/packages/circular/circular.pdf.

[16] K. V. Mardia, P. E. Jupp, Directional Statistics. Wiley, Chichester, 2000.

[17] J. S. Rao, A. Sengupta, Topics in Circular Statistics. World Scientific Publishing, Singapore, 2001.

[18] S. Roy, M. Adnan, Wrapped weighted exponential distribution, Stat. Probab. Lett. 87 (2012), 77-83.

[19] K. Schmidt-Koenig, On the role of loft, the distance and site of release in pigeon homing (the "cross-loft" experiment), Biometrika. 125 (1963), 154-164.

[20] R. Shanker, A. Mishra, A quasi lindley distribution, Afr. J. Math. Computer Sci. Res. 6 (4) (2013), 64-71.

[21] K. Soetaert. rootSolve: Nonlinear Root Finding, Equilibrium and Steady- state Analysis of Ordinary Differential Equations. R package version 1.7 (2016). URL:https://cran.r-project.org /web/packages/rootSolve/rootSolve.pdf, .

[22] O. Toomet, A. Henningsen, Maximum Likelihood Estimation and Related Tools, R package version 1.3-4 (2019). URL: https://cran.r-project.org/web/packages/maxLik/maxLik.pdf.