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HIGH-ORDER ITERATIVE METHODS FOR SOLVING NONLINEAR SYSTEMS

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Abstract: In this paper, we combining the Halley method with Householder method and using predictor–corrector technique to solve systems of nonlinear equations. One of the obtained methods have order eighteen of convergence. In addition, the proposed methods has been tested on a series of examples published in the literature and show good results when compared it with the previous literature.

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1. Introduction

Recently, several iterative methods have been made on the development for solving nonlinear equations and system of nonlinear equations. These methods have been improved using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques, see [1–9] and references therein. He [10] suggested an iterative method for solving the nonlinear equations by rewriting the given nonlinear equation as a system of coupled equations. This technique has been used by Chun [11] and Noor [12,13] to suggest some higher order convergent iterative methods for solving nonlinear equations. Newton method is the well-known

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iterative method for finding the solution of the nonlinear equations. There exist several classical multipoint methods with fourth-order and sixth-order convergence for solving nonlinear equations. It is well known [9] that the two-step Newton method has fourth-order convergence, which has been suggested by using the technique of updating the solution. Hafiz and Bahgat [14-15] modified Householder and Halley iterative methods for solving system of nonlinear equations. He also show that this new methods includes famous two step Newton method as a special case. In this paper we combining the Halley method with Householder method and using predictor–corrector technique to solve systems of nonlinear equations to construct new high-order iterative methods using predictor–corrector technique and used it for solving systems of nonlinear equations. Some illustrative examples have been presented, to demonstrate our methods and the results are compared with those derived from the previous methods. All test problems reveals the accuracy and fast convergence of the new methods. In the next sections, we use the same notation for the *n*-dimensional case as for the one-dimensional case, interpreting the symbols appropriately.

2. Iterative methods

Suppose we have system of nonlinear equations of the following form

$$\begin{gathered} f_1(x_1, x_2, \cdots, x_n) = 0, \\ f_2(x_1, x_2, \cdots, x_n) = 0, \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0, \end{gathered}$$

where $f_i : \mathrm{R}^n \to \mathrm{R}$ and the functions f_i is differentiable up to any desired order [16], can be thought of as mapping a vector $\mathrm{X} = (x_1, x_2, \ldots, x_n)^T$ of the *n*-dimensional space R^n, into the real line R. The system can alternatively be represented by defining a functional $F : \mathrm{R}^n \to \mathrm{R}^n$, by

$$F(x_1, x_2, \ldots x_n) = [f_1(x_1, x_2, \ldots x_n), \ldots, f_n(x_1, x_2, \ldots x_n)]^T .$$

Using vector notation to represent the variables $x_1, x_2, \ldots x_n$, the previous system then assumes the form:

$$F(\mathrm{x}) = 0 \tag{1}$$

For simplicity, we assume that X* is a simple root of Eq. (1) and X_0 is an initial guess sufficiently close to X*. Using the Taylor's series expansion of the function $f_k(\mathrm{x})$, we have

$$f_k(\mathrm{x}_0)+\frac{1}{1!}[\sum_{i=1}^{n}(x_i-x_i^{(0)})f_{k,i}(\mathrm{x}_0)]+\frac{1}{2!}[\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_i^{(0)})(x_j-x_j^{(0)})f_{k,ij}(\mathrm{x}_0)]+\ldots=0 \qquad (2)$$

where $k=1,2,\ldots,n,\ f_{k,i}=\dfrac{\partial f_k}{\partial x_i},\ f_{k,ij}=\dfrac{\partial^2 f_k}{\partial x_i \partial x_j}$ and $\mathrm{x}_0=[x_1^{(0)},x_2^{(0)},\ldots,x_n^{(0)}]^T$ is the initial approximation of Eq. (1). Matrices of first and second partial derivatives appearing in equation (2) are Jacobian J and Hessian matrix H respectively. In matrix notation

$$F(\mathrm{x}_0)+J(\mathrm{x}_0)[\mathrm{x}^{(1)}-\mathrm{x}_0]+\frac{1}{2!}\sum_{i=1}^{n}e_i\otimes[\mathrm{x}^{(1)}-\mathrm{x}_0]^T H_i(\mathrm{x}_0)[\mathrm{x}^{(1)}-\mathrm{x}_0]=0 \qquad (3)$$

from which we have

$$\left\{J(\mathrm{x}_0)+\frac{1}{2!}\sum_{i=1}^{n}e_i\otimes[\mathrm{x}^{(1)}-\mathrm{X}_0]^T H_i(\mathrm{x}_0)\right\}[\mathrm{x}^{(1)}-\mathrm{x}_0]=-F(\mathrm{x}_0)$$

$$\mathrm{x}^{(1)}=\mathrm{x}_0-\left\{J(\mathrm{x}_0)+\frac{1}{2!}\sum_{i=1}^{n}e_i\otimes[\mathrm{x}^{(1)}-\mathrm{x}_0]^T H_i(\mathrm{x}_0)\right\}^{-1}F(\mathrm{x}_0) \qquad (4)$$

where H_i is the Hessian matrix of the function f_i, $\otimes$ is the Kronecker product and e_i is a $n\times 1$ vector of zero except for a 1 in the position *i*. First two terms of the equation (3) gives the first approximation, as

$$\mathrm{x}^{(1)}=\mathrm{x}_0-J^{-1}(\mathrm{x}_0)F(\mathrm{x}_0) \qquad (5)$$

Substitution again of (5) into the right hand side of (4) gives the second approximation

$$\mathrm{x}^{(1)}=\mathrm{x}_0-\left\{J(\mathrm{x}_0)-\frac{1}{2!}\sum_{i=1}^{n}e_i\otimes[J^{-1}(\mathrm{x}_0)F(\mathrm{x}_0)]^T H_i(\mathrm{x}_0)\right\}^{-1}F(\mathrm{x}_0) \qquad (6)$$

This is extended Halley's method and this formulation allows us to suggest the following iterative methods for solving system of nonlinear equations (1).

Algorithm.1. For a given $\mathrm{x}^{(k)}=[x_1^{(k)},x_2^{(k)},...,x_n^{(k)}]^T$ calculate the approximation solution $\mathrm{x}^{(k+1)}=[x_1^{(k+1)},x_2^{(k+1)},...,x_n^{(k+1)}]^T$ for k =0, 1, 2, … by the iterative schemes

$$\mathrm{x}^{(k+1)}=\mathrm{x}^{(k)}-\left\{J(\mathrm{x}^{(k)})-\sum_{i=1}^{n}\frac{e_i}{2!}\otimes[J^{-1}(\mathrm{x}^{(k)})F(\mathrm{x}^{(k)})]^T H_i(\mathrm{x}^{(k)})\right\}^{-1}F(\mathrm{x}^{(k)}) \qquad (8)$$

we also remark that, if $f_{k,ij}=0, \forall k,i,j=1,2,...,n$, then Algorithm 2 reduces to the Newton method. That is, Algorithm 2 which is the generalized of Halley's method for solving system of nonlinear equations. Several authors have already studied the convergence of iteration (8) in Banach space setting (see [17] and etc.).

Again we can rewrite (3) as follows

$$\mathrm{x}^{(1)}=\mathrm{x}_0-J^{-1}(\mathrm{x}_0)F(\mathrm{x}_0)-\frac{1}{2!}J^{-1}(\mathrm{x}_0)\sum_{i=1}^{n}e_i\otimes[\mathrm{x}^{(1)}-\mathrm{x}_0]^T H_i(\mathrm{x}_0)[\mathrm{x}^{(1)}-\mathrm{x}_0] \qquad (9)$$

First two terms of the equation (9) gives

$$\mathrm{z}=\mathrm{x}^{(1)}-\mathrm{x}_0=-J^{-1}(\mathrm{x}_0)F(\mathrm{x}_0) \qquad (10)$$

Substitution again of (10) into the right hand side of (9) gives the second approximation

$$\mathrm{x}=\mathrm{x}_0-J^{-1}(\mathrm{x}_0)F(\mathrm{x}_0)-\frac{1}{2!}J^{-1}(\mathrm{x}_0)\sum_{i=1}^{n}e_i\otimes(\mathrm{z})^T H_i(\mathrm{x}_0)(\mathrm{z}) \qquad (11)$$

This formulation allows us to suggest the following iterative methods for solving system of nonlinear equations (1).

Algorithm.2. For a given $\mathrm{x}^{(k)}=[x_1^{(k)},x_2^{(k)},...,x_n^{(k)}]^T$ calculate the approximation solution $\mathrm{x}^{(k+1)}=[x_1^{(k+1)},x_2^{(k+1)},...,x_n^{(k+1)}]^T$ for k =0,1, 2, … by the iterative schemes

$$\mathrm{x}^{(k+1)}=\mathrm{x}^{(k)}+\mathrm{z}^{(k)}-J\left(\mathrm{x}^{(k)}\right)^{-1}\sum_{i=1}^{n}\frac{e_i}{2!}\otimes(\mathrm{z}^{(k)})^T H_i(\mathrm{x}^{(k)})(\mathrm{z}^{(k)}), \qquad (12)$$

Algorithm 2 is extended to Householder method.

Now using the technique of updating the solution, therefore, using Algorithm 1 as a predictor and Algorithm 2 as a corrector, we suggest and consider a new two-step iterative method for solving the nonlinear Eq. (1), which is the main motivation of this paper.

Algorithm.3. For a given $\mathrm{x}^{(k)} = [x_1^{(k)}, x_2^{(k)}, ..., x_n^{(k)}]^T$ calculate the approximation solution $\mathrm{x}^{(k+1)} = [x_1^{(k+1)}, x_2^{(k+1)}, ..., x_n^{(k+1)}]^T$ for k =0,1, 2, … by the iterative schemes

$$\mathrm{y}^{(k)} = \mathrm{x}^{(k)} - \left\{ J(\mathrm{x}^{(k)}) - \sum_{i=1}^{n} \frac{e_i}{2!} \otimes [\mathrm{z}^{(k)}]^T H_i(\mathrm{x}^{(k)}) \right\}^{-1} F(\mathrm{x}^{(k)}), \tag{13}$$

$$\mathrm{x}^{(k+1)} = \mathrm{y}^{(k)} + \bar{\mathrm{z}}^{(k)} - J\left(\mathrm{y}^{(k)}\right)^{-1} \sum_{i=1}^{n} \frac{e_i}{2!} \otimes (\bar{\mathrm{z}}^{(k)})^T H_i(\mathrm{y}^{(k)})(\bar{\mathrm{z}}^{(k)}), \tag{14}$$

where $\bar{\mathrm{z}}^{(k)} = -J^{-1}(\mathrm{y}^{(k)}) F(\mathrm{y}^{(k)})$.

Again, if we using classical Newton–Raphson method as a predictor and Algorithm 3 as a corrector, we suggest and consider a new three-step iterative method for solving the nonlinear Eq. (1), which is the anther main motivation of this paper.

Algorithm.4. For a given $\mathrm{x}^{(k)} = [x_1^{(k)}, x_2^{(k)}, ..., x_n^{(k)}]^T$ calculate the approximation solution $\mathrm{x}^{(k+1)} = [x_1^{(k+1)}, x_2^{(k+1)}, ..., x_n^{(k+1)}]^T$ for k =0,1, 2, … by the iterative schemes

$$\mathrm{w}^{(k)} = \mathrm{x}^{(k)} - \left[J(\mathrm{x}^{(k)}) \right]^{-1} F(\mathrm{x}^{(k)}), \tag{15}$$

$$\mathrm{y}^{(k)} = \mathrm{w}^{(k)} - \left\{ J(\mathrm{w}^{(k)}) - \sum_{i=1}^{n} \frac{e_i}{2!} \otimes [\hat{\mathrm{z}}^{(k)}]^T H_i(\mathrm{w}^{(k)}) \right\}^{-1} F(\mathrm{w}^{(k)}), \tag{16}$$

$$\mathrm{x}^{(k+1)} = \mathrm{y}^{(k)} + \bar{\mathrm{z}}^{(k)} - J\left(\mathrm{y}^{(k)}\right)^{-1} \sum_{i=1}^{n} \frac{e_i}{2!} \otimes (\bar{\mathrm{z}}^{(k)})^T H_i(\mathrm{y}^{(k)})(\bar{\mathrm{z}}^{(k)}), \tag{17}$$

where $\hat{\mathrm{z}}^{(k)} = -J^{-1}(\mathrm{w}^{(k)}) F(\mathrm{w}^{(k)})$.

4. Numerical results

We present some examples to illustrate the efficiency of our proposed methods. Here, numerical results are performed by Maple 15 with 2000 digits but only 14 digits are displayed. In Tables 1, 2 we list the results obtained by Newton–Raphson method (NM), Algorithm 2.1 (GHM), Algorithm 2.2 (MHIM), Algorithm 2.3 (HAM1) and the Algorithm 2.4 (HAM2) which are introduced in this present paper. The following stopping criteria is used for computer programs:

$$\| X^{(n+1)} - X^{(n)} \| + \| F(X^{(n)}) \| < 10^{-15}$$

and the computational order of convergence (COC) can be approximated using the formula,

$$\mathrm{COC} \approx \frac{\ln\left(\| X^{(n+1)} - X^{(n)} \| / \| X^{(n)} - X^{(n-1)} \|\right)}{\ln\left(\| X^{(n)} - X^{(n-1)} \| / \| X^{(n-1)} - X^{(n-2)} \|\right)}$$

Table 1 shows the number of iterations, the computational order of convergence (COC) , $\| X^{(n+1)} - X^{(n)} \|$ and norm of the function $F(X^{(n)})$ are also shown in Table 1 for various methods.

Case 1. In a case of one dimension, consider the following nonlinear functions [19], $f_1(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$ with $x_0 = -5$ and $f_2(x) = e^{x^2+7x-30} - 1$ with $x_0 = 10.$

Case 2. In a case two dimension, consider the following systems of nonlinear functions,

[14] $$F_3(\mathrm{x}) = \begin{cases} f_1(x,y) = x^2 - 10x + y^2 + 8 = 0 \\ f_2(x,y) = xy^2 + x - 10y + 8 = 0 \end{cases},$$

[15] $$F_4(\mathrm{x}) = \begin{cases} f_1(x,y) = x^4 y - xy + 2x - y - 1 = 0 \\ f_2(x,y) = ye^{-x} + x - y - e^{-1} = 0 \end{cases},$$

[18] $$F_5(\mathrm{x}) = \begin{cases} f_1(x,y) = x^3 + y^3 - 6x + 3 = 0 \\ f_2(x,y) = x^3 - y^3 - 6x + 2 = 0 \end{cases},$$

[2] $$F_6(\mathrm{x}) = \begin{cases} f_1(x,y) = x - y^2 + 3\ln(x) = 0 \\ f_2(x,y) = 2x^2 - xy - 5x + 1 = 0 \end{cases},$$

In the Table 1, we list results obtained by Algorithm 2.3 (HAM1) and the Algorithm 2.4 (HAM2) which are introduced in this present paper. As we see from this Tables, it is clear that, in most cases, the result obtained by HAM2 is very superior to that obtained by Newton–Raphson method (NM), Algorithm 2.1 (GHM), and Algorithm 2.2 (MHIM).

5. Conclusions

The high-order method continues to be an important subject of investigation. In our study we extend the standard iteration in order to obtain robust algorithms based on Halley and Householder iteration methods to construct new high-order iterative methods using predictor–corrector technique. These methods are applied for solving nonlinear system of equations. The numerical examples show that our methods are very effective and efficient. Moreover, our proposed methods provides highly accurate results in a less number of iterations as compared with Newton–Raphson method, the Halley's method and generalized Householder method.

Table 1.Numerical results for Examples 1-6

Methods & functions	IT	COC	$\|X^{(n+1)} - X^{(n)}\|$	$\|F(X^{(n)})\|$
F_1, x_0 =-5.				
NM	31	2.00001699276335	4.230034231235E-17	5.4582093344E-32
MHIM	13	5.95020085902909	1.584480277783E-19	2.734417295E-111
GHM	11	6.01193872688169	5.190546250140E-17	3.4207231673E-97
HAM1	10	9.21418518830165	2.556965663584E-52	3.909565243E-465
HAM2	8	17.8843219837502	1.642835572149E-42	2.070669602E-878
$F2$, x_0 =10				
NM	164	2.00001685431127	2.527034679455E-17	5.4599481518E-32
MHIM	59	5.86459887255759	4.699753089877E-19	2.285381559E-105
GHM	50	5.98881613728985	9.511771755061E-45	3.858940233E-260
HAM1	43	8.99372168109715	2.402506328285E-58	5.560127022E-513
HAM2	34	17.8550482936163	1.25507503943E-150	0
F_3, X_0=(8.6, 10)				
NM	42	2.00000021352583	1.833156409688E-27	1.4855318340E-53
MHIM	25	5.63074572982711	2.968044975474E-24	3.114395015E-143
GHM	12	5.98472490399457	8.240075047498E-56	8.763823533E-333
HAM1	11	8.60635450267381	1.084759742398E-35	1.054047428E-316
HAM2	8	12.8971510342434	9.039014113587E-34	2.204188854E-599

F_4, X_0=(0.7, 0.7)				
NM	11	2.00013006586688	6.686363653232E-24	1.8139394754E-46
MHIM	5	5.92910710799839	2.550632049191E-24	1.017264076E-140
GHM	5	6.0054279133138	1.061967377652E-40	1.462817790E-239
HAM1	4	9.28923248468268	4.120033065428E-40	2.165304576E-358
HAM2	4	17.8205244733424	5.16496532789E-149	1.01980390E-1999
F_5, X_0=(-1.5, -1.5)				
NM	8	2.00069292989550	1.187940161245E-18	1.1601643159E-35
MHIM	4	5.91604393232878	7.249182855124E-27	1.237030408E-157
GHM	4	5.95912810032605	3.291918383951E-34	4.536899810E-202
HAM1	4	9.02275372262172	7.18992241572E-105	2.108012223E-939
HAM2	3	19.5705094880727	1.475056758118E-46	7.972240274E-830
F_6, X_0=(5, 3.8)				
NM	7	1.99995400768303	1.240939797973E-29	1.1761745218E-58
MHIM	4	5.99911391602516	2.666474130888E-89	8.529933551E-535
GHM	3	5.74995970528599	1.267875270585E-16	4.8697241738E-99
HAM1	3	8.64339833802159	2.002270931566E-37	1.066565689E-335
HAM2	3	17.7573882427874	1.02458467213E-143	1.0e-1998

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