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COMPOSITION OPERATORS IN SOME FUNCTION SPACES OF HYPERBOLIC TYPE

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Abstract. In this paper, we introduce natural metrics in the weighted hyperbolic (α,ω)-Bloch class and $F^*(p,q,s;\omega)$ classes. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, Lipschitz continuous and boundedness of the composition operator C_ϕ acting from the hyperbolic (α,ω)-Bloch class to the classes $F^*(p,q,s;\omega)$ are characterized by conditions depending on an analytic self-map $\phi:\mathbb{D}\to\mathbb{D}$.

Keywords: hyperbolic classes, composition operators, Lipschitz continuous, (α,ω)-Bloch space.

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1. Introduction

Let $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ be the open unit disc of the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ it's boundary. Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ and let $B(\mathbb{D})$ be the subset of $\mathcal{H}(\mathbb{D})$ consisting of those $f\in\mathcal{H}(\mathbb{D})$ for which $|f(z)|<1$ for all $z\in\mathbb{D}$. Also, $dA(z)$ be the normalized area measure on $\mathbb{D}$ so that $A(\mathbb{D})\equiv 1$.

Let the Green's function of $\mathbb{D}$ be defined as $g(z,a)=\log\frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z)=\frac{a-z}{1-\bar{a}z}$, for $z,a\in\mathbb{D}$ is the Möbius transformation related to the point $a\in\mathbb{D}$.

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If (X,d) is a metric space, we denote the open and closed balls with center x and radius $r>0$ by $B(x,r) := \{y \in X : d(y,x) < r\}$ and $\bar{B}(x,r) := \{y \in X : d(x,y) \le r\}$, respectively.

Hyperbolic function classes are usually defined by using either the hyperbolic derivative $f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ of $f \in B(\mathbb{D})$, or the hyperbolic distance $\rho(f(z),0) := \frac{1}{2}\log\left(\frac{1+|f(z)|}{1-|f(z)|}\right)$ between $f(z)$ and zero.

A function $f \in B(\mathbb{D})$ is said to belong to the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ if

$$\|f\|_{\mathcal{B}^*_\alpha} = \sup_{z\in\mathbb{D}} f^*(z)(1-|z|^2)^\alpha < \infty.$$

The little hyperbolic Bloch-type class $\mathcal{B}^*_{\alpha,0}$ consists of all $f \in \mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1} f^*(z)(1-|z|^2)^\alpha = 0.$$

The usual α-Bloch spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ are defined as the sets of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty,$$

and

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2)^\alpha = 0,$$

respectively.

It is obvious that $\mathcal{B}^*_\alpha$ is not a linear space since the sum of two functions in $B(\mathbb{D})$ does not necessarily belong to $B(\mathbb{D})$. From [5, 16, 17], we have the following:
For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbb{D}$ is said to belong to the $\alpha-$weighted Bloch space $\mathcal{B}^\alpha_\omega$ (see [16, 17]) if

$$\|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| < \infty.$$

Also, for a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbb{D}$ is said to belong to the little weighted Bloch space $\mathcal{B}^\alpha_{\omega,0}$ if

$$\|f\|_{\mathcal{B}^\alpha_{\omega,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| = 0.$$

Throughout this paper and for some techniques we consider the case of $\omega \not\equiv 0$.

Now, we give the following definitions.

Definition 1.1. For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. A function $f \in B(\mathbb{D})$ is said to belong to the (α,ω)-weighted hyperbolic Bloch space $\mathcal{B}^*_{\omega,\alpha}$ if

$$\|f\|_{\mathcal{B}^*_{\omega,\alpha}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^\alpha}{\omega(1-|z|)}(f^*(z)) < \infty.$$

Also, for a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. A function $f \in B(\mathbb{D})$ is said to belong to the little weighted hyperbolic Bloch space $\mathcal{B}^*_{\omega,\alpha,0}$ if

$$\|f\|_{\mathcal{B}^*_{\omega,\alpha}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha}{\omega(1-|z|)}(f'(z)) = 0.$$

We now turn to consider hyperbolic $F(p,q,s;\omega)$ type classes, which will be called $F^*(p,q,s;\omega)$. For $0 < p, s < \infty$, $-2 < q < \infty$, the hyperbolic class $F^*(p,q,s;\omega)$ consists of those functions $f \in B(\mathbb{D})$ for which

$$\|f\|^p_{F^*(p,q,s;\omega)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z) < \infty.$$

Moreover, we say that $f \in F^*(p,q,s)$ belongs to the class $F^*_0(p,q,s)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z) = 0.$$

The usual general Besov-type spaces $F(p,q,s;\omega)$ (defined using the conventional derivative f' instead of f^*) and their norms are denoted by the same symbols but with f'.

Yamashita was probably the first one considered systematically hyperbolic function classes. He introduced and studied hyperbolic Hardy, BMOA and Dirichlet classes in [21, 22, 23] and others. More recently, Smith studied inner functions in the hyperbolic little Bloch-class [18], and the hyperbolic counterparts of the Q_p spaces were studied by Li in [12] and Li et. al. in [13]. Further, hyperbolic Q_p classes and composition operators studied by Pérez-González et. al. in [15].

In this paper, we study the hyperbolic α-Bloch classes $\mathcal{B}^*_{\omega\alpha}$ and the general hyperbolic $F(p,q,s;\omega)$ type classes. We will also give some results to characterize Lipschitz continuous and compact composition operators mapping from the hyperbolic (α,ω)-Bloch class $\mathcal{B}^*_{\omega\alpha}$ to $F^*(p,q,s;\omega)$ class by conditions depending on the symbol ϕ only.

Note that the general hyperbolic $F(p,q,s;1)$ type classes include the class of so-called Q_p^* classes and the class of (hyperbolic) Besov class B_p^*. Thus, the results are generalizations of the recent results of Pérez-González, Rät tyä and Taskinen [15].

For any holomorphic self-map ϕ of $\mathbb{D}$. The symbol ϕ induces a linear composition operator $C_\phi(f) = f \circ \phi$ from $\mathcal{H}(\mathbb{D})$ or $B(\mathbb{D})$ into itself. The study of composition operator C_ϕ acting on spaces of analytic functions has engaged many analysts for many years (see e.g. [2, 3, 10, 11, 13, 14, 19, 24] and others).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \le C\|f\|_X$ for all maps $f \in X$. By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y contained in $B(\mathbb{D})$ or $\mathcal{H}(\mathbb{D})$, $T : X \to Y$ is compact if and only if for each bounded sequence $\{x_n\} \in X$, the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to a function $f \in Y$.

Definition 1.2. A composition operator $C_\phi : \mathcal{B}^*_{\omega\alpha} \to F^*(p,q,s;\omega)$ is said to be bounded, if there is a positive constant C such that $\|C_\phi f\|_{F^*(p,q,s;\omega)} \le C\|f\|_{\mathcal{B}^*_{\omega\alpha}}$ for all $f \in \mathcal{B}^*_{\omega\alpha}$.

Definition 1.3. A composition operator $C_\phi : \mathcal{B}^*_{\omega\alpha} \to F^*(p,q,s;\omega)$ is said to be compact, if it maps any ball in $\mathcal{B}^*_{\omega\alpha}$ onto a precompact set in $F^*(p,q,s;\omega)$.

The following lemma follows by standard arguments similar to those outline in [19]. Hence we omit the proof.

Lemma 1.1. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 < p, s < \infty$, $-1 < q < \infty$ and $0 < \alpha < \infty$. Then $C_\phi : \mathcal{B}^*_{\omega\alpha} \to F^*(p,q,s;\omega)$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{B}^*_{\omega\alpha}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\lim_{n\to\infty} \|C_\phi f_n\|_{F^*(p,q,s;\omega)} = 0$.*

The following lemma can be found in [20], Theorem 2.1.1.

Lemma 1.2. *Let $0 < \alpha < \infty$. Then there are two holomorphic maps $f, g : \mathbb{D} \to \mathbb{C}$ such that $|f'(z)| + |g'(z)| \approx (1 - |z|^2)^{-\alpha}$, $z \in \mathbb{D}$.*

2. Natural metrics

In this section, we introduce natural metrics on the hyperbolic $(\alpha;\omega)$-Bloch classes $\mathcal{B}^*_{\omega,\alpha}$ and the classes $F^*(p,q,s;\omega)$.

Let $0<p,s<\infty, -2<q<\infty$ and $0<\alpha<1$. First, we can find a natural metric in $\mathcal{B}^*_{\omega,\alpha}$ by defining

$$d(f,g;\mathcal{B}^*_{\omega,\alpha}) := d_{\mathcal{B}^*_{\omega,\alpha}}(f,g) + \|f-g\|_{\mathcal{B}_{\omega,\alpha}} + |f(0)-g(0)|, \tag{1}$$

where

$$d_{\mathcal{B}^*_{\omega,\alpha}}(f,g) := \sup_{z\in\mathbb{D}}\left|\frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2}\right|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)},$$

for $f,g\in\mathcal{B}^*_{\omega\alpha}$. The presence of the conventional $(\alpha;\omega)$-Bloch-norm here perhaps unexpected. It is motivated as the example see [15] (see [15], Example in Section 7). It shows the phenomenon that, though trivially $d_{\mathcal{B}^*_{\omega,\alpha}}(f,0)\geq\|f\|_{\mathcal{B}_{\omega,\alpha}}$ for all $f\in\mathcal{B}^*_{\omega,\alpha}$, the same does no more hold for the differences of two functions: there does not even exist a constant $C>0$ such that

$$\sup_{z\in\mathbb{D}}\left|\frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2}\right|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)} \geq C\|f-g\|_{\mathcal{B}_{\omega,\alpha}}$$

hold for all $f,g\in\mathcal{B}^*_{\omega,\alpha}, 0<\alpha<1$. For $f,g\in F^*(p,q,s;\omega)$, define their distance by

$$d(f,g;F^*(p,q,s;\omega)) := d_{F^*}(f,g) + \|f-g\|_{F(p,q,s;\omega)} + |f(0)-g(0)|,$$

where

$$d_{F^*}(f,g) := \left(\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2}\right|^p\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z)\right)^{\frac{1}{p}}.$$

The following characterization of complete metric space $d(.,.;\mathcal{B}^*_{\omega,\alpha})$ can be proved as the corresponding result in [15].

Proposition 2.1. The class $\mathcal{B}^*_{\omega,\alpha}$ equipped with the metric $d(.,.;\mathcal{B}^*_{\omega,\alpha})$ is a complete metric space. Moreover, $\mathcal{B}^*_{\omega,\alpha,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_{\omega,\alpha}$.

Now we are in a position to prove the following proposition.

Proposition 2.2. The class $F^*(p,q,s;\omega)$ equipped with the metric $d(.,.;F^*(p,q,s;\omega))$ is a complete metric space. Moreover, $F_0^*(p,q,s;\omega)$ is a closed (and therefore complete) subspace of $F^*(p,q,s;\omega)$.

Proof. For $f,g,h \in F^*(p,q,s;\omega)$, we have

- $d(f,g;F^*(p,q,s;\omega)) \geq 0$,
- $d(f,f;F^*(p,q,s;\omega)) = 0$,
- $d(f,g;F^*(p,q,s;\omega)) = 0$ implies $f = g$.
- $d(f,g;F^*(p,q,s;\omega)) = d(g,f;F^*(p,q,s;\omega))$,
- $d(f,h;F^*(p,q,s;\omega)) \leq d(f,g;F^*(p,q,s;\omega)) + d(g,h;F^*(p,q,s;\omega))$,

Hence, d is metric on $F^*(p,q,s;\omega)$. For the completeness proof, let $(f_n)_{n=0}^{\infty}$ be a Cauchy sequence in the metric space $F^*(p,q,s;\omega)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$, for all $n,m > N$. Since $f_n \in B(\mathbb{D})$ such that f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Let $m > N$ and $0 < r < 1$. In view of Fatou's lemma, we find that

$$
\begin{aligned}
&\int_{\mathbb{D}(0,r)} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{f'_m(z)}{1-|f_m(z)|^2} \right|^p \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z) \\
= &\int_{\mathbb{D}(0,r)} \lim_{n\to\infty} \left| \frac{f'_n(z)}{1-|f_n(z)|^2} - \frac{f'_m(z)}{1-|f_m(z)|^2} \right|^p \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z) \\
\leq &\lim_{n\to\infty} \int_{\mathbb{D}} \left| \frac{f'_n(z)}{1-|f_n(z)|^2} - \frac{f'_m(z)}{1-|f_m(z)|^2} \right|^p \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z) \leq \varepsilon^p.
\end{aligned}
$$

Letting $r \to 1^-$, we obtain that

$$
\int_{\mathbb{D}} (f^*(z))^p \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z) \leq 2^p \varepsilon^p + 2^p \int_{\mathbb{D}} (f_m^*(z))^p \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z). \quad (2)
$$

This yields $\|f\|^p_{F^*(p,q,s;\omega)} \leq 2^p\varepsilon^p + 2^p \|f_m\|^p_{F^*(p,q,s;\omega)}$, thus $f \in F^*(p,q,s;\omega)$. We also find that $f_n \to f$ with respect to the metric of $F^*(p,q,s;\omega)$. The second part of the assertion follows by (1).

3. Lipschitz continuity

For $0 < p, s < \infty, -2 < q < \infty$ and $0 < \alpha < \infty$. We define the following notations:

$$\Phi_\phi(p,q,s,a;\omega) = \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z)$$

and

$$\Omega_{\phi,r}(p,q,s,a;\omega) = \int_{|\phi|\geq r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z).$$

Theorem 3.1. *Let $\omega : (0,1] \to (0,\infty)$ and assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 \leq p < \infty$, $0 \leq s \leq 1$, $-1 < q < \infty$ and $0 < \alpha \leq 1$. Then the following are equivalent:*

(i) $C_\phi : \mathcal{B}^*_{\omega,\alpha} \to F^*(p,q,s;\omega)$ *is bounded;*

(ii) $C_\phi : \mathcal{B}^*_{\omega,\alpha} \to F^*(p,q,s;\omega)$ *is Lipschitz continuous;*

(iii) $\sup_{a\in\mathbb{D}} \Phi_\phi(p,q,s,a;\omega) < \infty$.

Proof. First, assume that (i) holds, then there exists a constant C such that

$$\|C_\phi f\|_{F^*(p,q,s;\omega)} \leq C\|f\|_{\mathcal{B}^*_{\omega,\alpha}}, \quad \text{for all } f \in \mathcal{B}^*_{\omega,\alpha}.$$

For given $f \in \mathcal{B}^*_{\omega,\alpha}$, the function $f_t(z) = f(tz)$, where $0 < t < 1$, belongs to $\mathcal{B}^*_{\omega,\alpha}$ with the property $\|f_t\|_{\mathcal{B}^*_{\omega,\alpha}} \leq \|f\|_{\mathcal{B}^*_{\omega,\alpha}}$. Let f, g be the functions from Lemma 1.1, such that

$$\frac{\omega(1-|z|)}{(1-|z|^2)^\alpha} \leq f^*(z) + g^*(z),$$

for all $z \in \mathbb{D}$. It follows that

$$\frac{|\phi'(z)|}{(1-|\phi(z)|)^\alpha} \leq (f\circ\phi)^*(z) + (g\circ\phi)^*(z).$$

Thus, we have

$$\begin{aligned} &\int_{\mathbb{D}} \frac{|t\phi'(z)|^p}{(1-|t\phi(z)|^2)^{\alpha p}} \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z) \\ \leq\ & 2^p \int_{\mathbb{D}} \Big[\big((f\circ t\phi)^*(z)\big)^p + \big((g\circ t\phi)^*(z)\big)^p\Big] \frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)} dA(z) \\ \leq\ & 2^p \|C_\phi\|^p \big(\|f\|^p_{\mathcal{B}^*_{\omega,\alpha}} + \|g\|^p_{\mathcal{B}^*_{\omega,\alpha}}\big). \end{aligned}$$

This estimate together with the Fatou's lemma implies (iii).

Conversely, assuming that (iii) holds and that $f \in \mathcal{B}^*_{\omega,\alpha}$, we see that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\big((f\circ\phi)^*(z)\big)^p\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z)) \\
=\ &\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\big(f^*(\phi(z))\big)^p|\phi'(z)|^p\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z) \\
\leq\ &\|f\|^p_{\mathcal{B}^*_\alpha}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z).
\end{aligned}$$

Hence, it follows that (i) holds.

(ii)$\Longleftrightarrow$(iii). Assume first that $C_\phi : \mathcal{B}^*_{\omega,\alpha} \to F^*(p,q,s;\omega)$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(f\circ\phi, g\circ\phi; F^*(p,q,s;\omega)) \leq Cd(f,g;\mathcal{B}^*_{\omega,\alpha}),\quad \text{for all } f,g\in\mathcal{B}^*_{\omega,\alpha}.$$

Taking $g = 0$, we find

$$\|f\circ\phi\|_{F^*(p,q,s;\omega)} \leq C\big(\|f\|_{\mathcal{B}^*_{\omega,\alpha}} + \|f\|_{\mathcal{B}_{\omega,\alpha}} + |f(0)|\big),\quad \text{for all } f\in\mathcal{B}^*_{\omega,\alpha}. \tag{3}$$

The assertion (iii) for $\alpha = 1$ follows by choosing $f(z) = z$ in (3). If $0 < \alpha < 1$, then

$$\begin{aligned}
|f(z)| &= \left|\int_0^z f'(s)ds + f(0)\right| \leq \|f\|_{\mathcal{B}_{\omega,\alpha}}\int_0^{|z|}\frac{dx}{(1-x^2)^\alpha} + |f(0)| \\
&\leq \frac{\|f\|_{\mathcal{B}_{\omega,\alpha}}}{(1-\alpha)} + |f(0)|,
\end{aligned}$$

which yields

$$\big|f(\phi(0)) - g(\phi(0))\big| \ \leq\ \frac{\|f-g\|_{\mathcal{B}_{\omega,\alpha}}}{(1-\alpha)} + |f(0)-g(0)|.$$

Moreover, Lemma 1.1. implies the existence of $f,g\in\mathcal{B}^*_{\omega,\alpha}$ such that

$$\big(f'(z)+g'(z)\big)(1-|z|^2)^\alpha \geq C > 0,\quad \text{for all } z\in\mathbb{D}. \tag{4}$$

Combining (3) and (4), we obtain

$$\begin{aligned}
&\|f\|_{\mathcal{B}^*_{\omega,\alpha}} + \|g\|_{\mathcal{B}^*_{\omega,\alpha}} + \|f\|_{\mathcal{B}_{\omega,\alpha}} + \|g\|_{\mathcal{B}_{\omega,\alpha}} + |f(0)| + |g(0)| \\
\geq\ &C\int_{\mathbb{D}}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z) \\
\geq\ &C\ \Phi_\phi(\alpha,p,q,s,a;\omega),
\end{aligned}$$

for which the assertion (iii) follows.

Assume now that (iii) is satisfied, we have

$$\begin{aligned}
&d(f\circ\phi, g\circ\phi; F^*(p,q,s;\omega))\\
=\ & d_{F^*}(f\circ\phi, g\circ\phi) + \|f\circ\phi - g\circ\phi\|_{F(p,q,s;\omega)} + \big|f(\phi(0)) - g(\phi(0))\big|\\
\le\ & d_{\mathcal{B}^*_{\omega,\alpha}}(f,g)\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z)\right)^{\frac{1}{p}}\\
&+\|f-g\|_{\mathcal{B}_{\omega,\alpha}}\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}\frac{(1-|z|^2)^q g^s(z,a)}{\omega(1-|z|)}dA(z)\right)^{\frac{1}{p}}\\
&+\frac{\|f-g\|_{\mathcal{B}_{\omega,\alpha}}}{(1-\alpha)} + |f(0)-g(0)|\\
\le\ & C' d(f,g;\mathcal{B}^*_{\omega,\alpha}).
\end{aligned}$$

Thus $C_\phi : \mathcal{B}^*_{\omega,\alpha} \to F^*(p,q,s;\omega)$ is Lipschitz continuous and the proof is completed.

Remark 3.1. Theorem 3.1 shows that $C_\phi : \mathcal{B}^*_{\omega,\alpha} \to F^*(p,q,s;\omega)$ is bounded if and only if it is Lipschitz-continuous, that is, if there exists a positive constant C such that

$$d(f\circ\phi, g\circ\phi; F^*(p,q,s;\omega)) \le C d(f,g;\mathcal{B}^*_{\omega,\alpha}), \ \text{ for all } f,g\in\mathcal{B}^*_{\omega,\alpha}.$$

By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. So, our result for composition operators in hyperbolic spaces is the correct and natural generalization of the linear operator theory.

The following observation is sometimes useful.

Proposition 3.1. Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 < p, s < \infty$, $-1 < q < \infty$ and $0 < \alpha < \infty$. If $C_\phi : \mathcal{B}^*_{\omega,\alpha} \to F^*(p,q,s;\omega)$ is compact, it maps closed balls onto compact sets.

Proof. If $B \subset \mathcal{B}^*_{\omega,\alpha}$ is a closed ball and $g \in F^*(p,q,s;\omega)$ belongs to the closure of $C_\phi(B)$, we can find a sequence $(f_n)_{n=1}^\infty \subset B$ such that $f_n\circ\phi$ converges to $g \in F^*(p,q,s;\omega)$ as $n\to\infty$. But $(f_n)_{n=1}^\infty$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^\infty$ converging uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. As in earlier arguments of Proposition 2.1 in [15], we get a positive estimate which shows that f must belong to the closed ball B. On the other hand, also the sequence $(f_{n_j}\circ\phi)_{j=1}^\infty$ converges uniformly on compact subsets to an analytic function, which is $g \in F^*(p, q, s; \omega)$. We get $g = f \circ \phi$, i.e. g belongs to $C_\phi(B)$. Thus, this set is closed and also compact.

Remark 3.2. It is still an open problem to extend the results of this paper by using generalized hyperbolic derivative as defined in [1].

Remark 3.3. It is still an open problem to extend the results of this paper to the classes $Q_K(p, q)$ and $Q_{K,\omega}(p, q)$ of hyperbolic functions. For some studies on analytic or meromorphic $Q_{K,\omega}(p, q)$ and $Q_K(p, q)$ classes, we refer to [4, 5, 6, 7, 8, 9, 16, 17].

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