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FIXED POINT THEOREMS FOR MULTIVALUED CONTRACTIONS ON B-METRIC SPACES WITH GRAPH

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Abstract. In this paper, we consider multivalued maps defined on a complete b-metric space endowed with direct graph. We discuss some fixed point results for maps, called multivalued G_b-contraction, multivalued weak G_b-contraction and multivalued (G_b, φ)-contraction. Our results extend/generalize many pre-existing results in the literature.

Keywords: multivalued contraction; G_b-contraction; strong comparison function.

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1. Introduction

Czerwik [7, 8] had generalized the Banach's and Nadler's fixed point results in b-metric space. On the other hand, Jachymski [12] introduced the notion of the single valued G-contraction for which he obtained fixed point results. Inspired by this poineer work, many authors studied fixed point theorems with graph in metric spaces (see [1, 2, 4, 5, 16]) as well

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as in many other abstract spaces (see [3, 6, 11, 13, 15]). In particular, Dinevari and Frigon [10] extended some fixed point results of Jachymski [12] to multivalued map.

In this work, we introduce the notions of multivalued G_b-contraction, multivalued weak G_b-contraction and multivalued (G_b,φ)-contraction. We also establish some fixed point theorems for such contractions. Our results extend/generalize the various results in the literature.

2. Basic Concepts and Notations

Let (X,d) be a b-metric space with coefficient $s\geq 1$, and Δ be the diagonal of the Cartesian product $X\times X$. Let G be the directed graph such that $V(G)$ be its vertices coincides with X and $E(G)$ be its edges contains all loops. Suppose that G has no parallel edges. We identify G with the pair $(V(G),E(G))$.

A sequence $(x^i)_{i=0}^n$ of $n+1$ vertices is called an n-directed path from y to z if

$$x^0=y,\ x^n=z \text{ and } (x^{i-1},x^i)\in E(G) \text{ for } i=1,2,\cdots,n.$$

The subgraph of G consisting of all edges and vertices which are contained in some directed path beginning at y is denoted by G_y. So, $V(G_y)=[y]_G$. Where

$$[y]_G=\bigcup_{n\in\mathbb{N}}[y]_G^n.$$

and

$$[y]_G^n=\{z\in X:\ \text{there is an } n\text{- directed path from } y \text{ to } z\ \}.$$

Since $\Delta\subset E(G)$, we have $[y]_G^1\subset[y]_G^2\subset,\cdots,\subset[y]_G$. Now for $z_1\in[y]_G^n$ and $z_2\in[y]_G$, we define

$$q_n(y,z_1):=\inf\left\{\sum_{i=1}^n s^i d(x^{i-1},x^i):(x^i)_{i=0}^n \text{ is an n-directed path from y to } z_1\right\},$$

$$q(y,z_2):=\inf\Big\{\sum_{i=1}^n s^i d(x^{i-1},x^i):(x^i)_{i=0}^n \text{ is an n-directed path from y to } z_2 \text{ for some } n\in\mathbb{N}\Big\}.$$

Note that $q_{n+p}(y,z)\leq q_n(y,z)$ for all $p\in\mathbb{N}$ and

$$q(y,z):=\inf\{q_n(y,z):n\in\mathbb{N} \text{ such that } z\in[y]_G^n\} \text{ since } \Delta\subset E(G).$$

3. MULTIVALUED G-CONTRACTION IN b-METRIC SPACE

In this section we study the multivalued contraction in b-metric space endowed with graph and establish fixed point theorems for it. We start this section with the following definition.

Definition 3.1. *Let (X,d) be a b-metric space and $T: X \to X$ be a multivalued map with nonempty values. T is called a G_b-contraction if there exists $k \in (0,1)$ such that*

(C_{G_b}) *for all $(x,y) \in E(G)$ and all $u \in T(x)$, there exists $v \in T(y)$ such that $(u,v) \in E(G)$ and $d(u,v) \leq kd(x,y)$.*

Note that a G_b-contraction does not need to have closed value.

Remark 3.1. *Let (X,d) be a b-metric space and $T: X \to X$ be a multivalued map with nonempty closed values. T is said to be a multivalued contraction in b-metric space (see Czerwik [8]) if there exists $k \in [0, \frac{1}{s})$ such that $H(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. Where H is Hausdorff metric define by*

$$H(A,B) = \begin{cases} \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\} & \text{if the maximum exists,} \\ \infty & \text{otherwise.} \end{cases}$$

It is worthwhile to indicate that a multivalued contraction in b metric space is G_b-contraction (with $E(G) = X \times X$) but, the inverse in not true.

Remark 3.2. *Cristian and Gabriela* [9] *introduced the notion of a multivalued Ciric G-contraction by considering the following conditions.*

(i) *for all $(x,y) \in E(G)$ if $u \in T(x)$ and $v \in T(y)$ are such that $d(u,v) \leq \alpha d(x,y) + \beta$, for some positive real number β, then $(u,v) \in E(G)$;*

(ii) *for all $(x,y) \in E(G)$ there exists $\lambda \in (0, \frac{1}{s})$ such that*

$$H(Tx,Ty) \leq \lambda \max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{1}{2s}[D(x,Ty) + D(y,Tx)]\right\}.$$

If $\beta = 0$ and $\max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{1}{2s}[D(x,Ty) + D(y,Tx)]\right\} = d(x,y)$ then (i) and (ii) can be written as;

(iii) for all $(x,y) \in E(G)$ if $u \in T(x)$ and $v \in T(y)$ are such that $d(u,v) \leq \alpha d(x,y)$, then $(u,v) \in E(G)$;

(iv) for all $(x,y) \in E(G)$ there exists $\lambda \in (0, \frac{1}{s})$ such that $H(Tx,Ty) \leq \lambda d(x,y)$.

Note that if the conditions (iii) and (iv) hold, then (C_{G_b}) is also held with any $k \in (\lambda, \frac{1}{s})$. On the other hand, from (C_{G_b}) implies that for all $(x,y) \in E(G)$,

$$H_1(Tx,Ty) \leq kd(x,y),$$

where $H_1(A,B) := \sup\{d(a,B) : a \in A\}$ for $A,B \subset X$. But, in general, (C_{G_b}) does not implies (iv) as shown in the Example 3.1 and Example 3.2.

Example 3.1. *Let $Y = \mathbb{N} \cup \{0\}$ and $X = \left\{\frac{1}{2^r} : r \in Y\right\} \cup \{0\}$. Define $d : X \times X \to [0,\infty)$ by $d(x,y) = |x-y|^2$ for all $x,y \in X$, then (X,d) is b-metric space with $s = 2$. Consider the directed graph G such that $V(G) = X$ and*

$$E(G) = \Delta \cup \left\{ \left(\frac{1}{2^r}, 0\right), \left(\frac{1}{2^r}, \frac{1}{2^{r+1}}\right) : r \in Y \right\}.$$

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \left\{0, \frac{1}{2}, 1\right\} & \text{if } x = 0, \\ \left\{\frac{1}{2^{r+1}}, 1\right\} & \text{if } x = \frac{1}{2^r}, r \in \mathbb{N}, \\ \left\{\frac{1}{2}\right\} & \text{if } x = 1. \end{cases}$$

One can easily check that T is the multivalued G_b-contraction with the constant $k = \frac{1}{4}$. But is it not a multivalued contraction in the sense of Czerwik [8] *and Cristian Chifu and Gabriela Petrusel* [9]. *Indeed,*

$$H\left(T\left(\frac{1}{2^r}\right), T(0)\right) > d\left(\frac{1}{2^r}, 0\right) \text{ for all } r \geq 2,$$

and also the condition (iii) of the Remark 3.2 is not satisfied. Verily

$$(1,0) \in E(G),\ \frac{1}{2} \in T1,\ 1 \in T0,\ d\left(\frac{1}{2}, 1\right) < d(1,0),\ \text{but } \left(1, \frac{1}{2}\right) \notin E(G)$$

Example 3.2. *Let $Y=\mathbb{N}\cup\{0\}$ and $X=\left\{\frac{1}{2^r}:r\in Y\right\}\cup\{0\}$. Define $d:X\times X\to[0,\infty)$ by $d(x,y)=|x-y|^2$ for all $x,y\in X$, then (X,d) is b-metric space with $s=2$. Consider the undirected graph G such that $V(G)=X$ and*

$$E(G)=\Delta\cup\left\{\left(\frac{1}{2^r},0\right),\left(0,\frac{1}{2^r}\right),\left(\frac{1}{2^r},\frac{1}{2^{r+2}}\right)\left(\frac{1}{2^{r+2}},\frac{1}{2^r}\right):r\in Y\right\}.$$

Let $T:X\to X$ be defined by

$$Tx=\begin{cases}\{0\} & \text{if } x=0,\\ \left\{\frac{1}{2^{r+1}},\frac{1}{2^{r+2}}\right\} & \text{if } x=\frac{1}{2^r}, r\in Y.\end{cases}$$

Then T is the multivalued G_b-contraction with constant the $k=\frac{1}{4}$. But is it not a multivalued contraction in the sense of Cristian Chifu and Gabriela Petrusel [9]. *Indeed, the condition (iii) of the Remark 3.2 is not satisfied since*

$$\left(1,\frac{1}{4}\right)\in E(G),\ \frac{1}{4}\in T1,\ \frac{1}{8}\in T\frac{1}{4},\ d\left(\frac{1}{4},\frac{1}{8}\right)<d\left(1,\frac{1}{4}\right),\ \textit{but}\ \left(\frac{1}{4},\frac{1}{8}\right)\notin E(G).$$

The Example 3.2 shows that our definition of the G_b-contraction is more general than the Definition 3.2 of [9] if we take

$$\max\left\{d(x,y),D(x,Tx),D(y,Ty),\frac{1}{2s}\left[D(x,Ty)+D(y,Tx)\right]\right\}=d(x,y),$$

even if G is undirected.

Now we will state and prove the following lemma which will be useful in the sequel.

Lemma 3.1. *Let X be a complete b-metric space with coefficient $s\geq 1$ and $T:X\to X$ be a multivalued G_b-contraction with constant k, and let $\varepsilon>0$ and $n\in\mathbb{N}$. Then for every $x\in X$ and $y\in[x]_G^n$, we have*

(1) *for any $x_1\in Tx$, there exists $y_1\in Ty\cap[x_1]_G^n$ such that $q_n(x_1,y_1)\leq k\left(q_n(x,y)+\varepsilon\right)$;*

and inductively for all $j\in\mathbb{N}$,

(2) *for any $x_{j+1}\in Tx_j$, there exists $y_{j+1}\in Ty_j\cap[x_{j+1}]_G^n$ such that $q_n(x_{j+1},y_{j+1})$ $\leq k^{j+1}\left(q_n(x,y)+\varepsilon\right).$*

Proof. Let $(x)_{i=0}^{n}$ be an n-directed path from x to y such that

$$\sum_{i=1}^{n} s^i d(x^{i-1},x^i) \leq q_n(x,y)+\varepsilon.$$

Since T is a multivalued G_b-contraction, for any $x_1 \in Tx$, there exists $x_1^1 \in Tx^1$ such that

$$(x_1,x_1^1) \in E(G) \text{ and } d(x_1,x_1^1) \leq kd(x,x^1);$$

and recursively for $i = 2,3,\cdots N$, there exists $x_1^i \in Tx^1$ such that

$$(x_1^{i-1},x_1^i) \in E(G) \text{ and } d(x_1^{i-1},x_1^i) \leq kd(x^{i-1},x^i).$$

Hence, if we denote $y_1 = x_1^n$, we obtain

$$q_1(x_1,y_1) \leq \sum_{i=1}^{n} s^i d(x_1^{i-1},x_1^i) \leq k\sum_{i=1}^{n} s^i d(x^{i-1},x^i) \leq k(q_n(x,y)+\varepsilon).$$

Now, inductively for $j \geq 1$ and for all $x_{j+1} = x_{j+1}^0 \in Tx_j$, from $i = 1$ to n, there exists $x_{j+1}^i \in Tx_j^i$ such that

$$(x_{j+1}^{i-1},x_{j+1}^i) \in E(G) \text{ and } d(x_{j+1}^{i-1},x_{j+1}^i) \leq kd(x_j^{i-1},x_j^i).$$

Denote $y_{j+1} = x_{j+1}^n$, we get

$$\begin{aligned} q_n(x_{j+1},y_{j+1}) &\leq \sum_{i=1}^{n} s^i d(x_{j+1}^{i-1},x_{j+1}^i) \leq k\sum_{i=1}^{n} s^i d(x_j^{i-1},x_j^i) \\ &\leq k^{j+1}\sum_{i=1}^{n} s^i d(x^{i-1},x^i) \leq k^{j+1}(q_n(x,y)+\varepsilon). \end{aligned}$$

□

Definition 3.2. *Let $T : X \to X$ be a multivalued mapping.*

(1) *Let $n \in \mathbb{N}$. A sequence $(x_m)_{m\in\mathbb{N}}$ is said to be a G_n-Picard trajectory from x_0 if $x_m \in [x_{m-1}]_G^n \cap Tx_{m-1}$ for all $m \in \mathbb{N}$. The set of all G_n-Picard trajectories from x_0 is denoted by $F_n(T,G,x_0)$.*

(2) *A sequence $(x_m)_{m\in\mathbb{N}}$ is said to be a G-Picard trajectory from x_0 if $x_m \in [x_{m-1}]_G \cap Tx_{m-1}$ for all $m \in \mathbb{N}$. The set of all G-Picard trajectories from x_0 is denoted by $F(T,G,x_0)$.*

Definition 3.3. *Let $T : X \to X$ be a multivalued mapping.*

(1) *Let $n \in \mathbb{N}$. T is said to be G_n-Picard continuous from x_0 if the limit of any convergent sequence $(x_m)_{m\in\mathbb{N}} \in F_n(T,G,x_0)$ is a fixed point of T.*

(2) *T is said to be G-Picard continuous from x_0 if the limit of any convergent sequence $(x_m)_{m\in\mathbb{N}} \in F(T,G,x_0)$ is a fixed point of T.*

Remark 3.3. *For every $n \in \mathbb{N}$, if T is G-Picard continuous from x_0 then it is G_n-Picard continuous from x_0. Note that if T has a closed graph then T is G-Picard continuous from x_0.*

Theorem 3.1. *Let (X,d) be a complete b-metric space and $T : X \to X$ be a multivalued G_b-contraction. Assume there is $n \in \mathbb{N}$ such that*

(1) *there exists $x_0 \in X$ such that $[x_0]_G^n \cap Tx_0 \neq \emptyset$;*

(2) *T is G_n-Picard continuous from x_0.*

Then there exists a G_n-Picard trajectory $(x_m)_{m\in\mathbb{N}}$ converging to a fixed point of T.

Proof. Let $x_1 \in [x_0]_G^n \cap Tx_0$. By Lemma 3.1, for every $\varepsilon > 0$, there exists $x_2 \in Tx_1 \cap [x_1]_G^n$ such that

$$d(x_1,x_2) \leq q_n(x_1,x_2) \leq k\,(q_n(x_0,x_1)+\varepsilon)$$

and for $m \geq 2$, there exists $x_{m+1} \in Tx_m \cap [x_m]_G^n$ such that

$$d(x_m,x_{m+1}) \leq q_n(x_m,x_{m+1}) \leq k^m\,(q_n(x_0,x_1)+\varepsilon)\,.$$

Therefore, $(x_m)_{m\in\mathbb{N}}$ is a G_n- Picard trajectory and, for $r \geq 1$,

$$\begin{aligned} d(x_m,x_{m+r}) &\leq \sum_{i=0}^{r-1} s^{i+1} d(x_{m+i},x_{m+i+1}) \\ &\leq \sum_{i=0}^{r-1} k^{m+i} s^{i+1}\,(q_n(x_0,x_1)+\varepsilon) \\ &\leq \frac{sk^m}{1-sk}\,(q_n(x_0,x_1)+\varepsilon)\,. \end{aligned}$$

Hence we have $\lim_{n\to\infty} d(x_m,x_{m+r}) = 0$. So, $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence. Since T is a G_n-Picard continious from x_0, the limit of $(x_m)_{m\in\mathbb{N}}$ is a fixed point of T. □

Corollary 3.1. *Let (X,d) be a complete b-metric space and $T : X \to CL(X)$ be a multivalued G_b-contraction. Assume there is $n \in \mathbb{N}$ such that*

(i) *there exists* $x_0 \in X$ *such that* $[x_0]_G^n \cap Tx_0 \neq \emptyset$;

(ii) *for every* $(x_m)_{m\in\mathbb{N}} \in F_n(T,G,x_0)$ *such that* $x_m \to x$, *there exists* $(m_j)_{j\in\mathbb{N}}$ *such that* $(x_{m_j},x) \in E(G)$ *for* $j \in \mathbb{N}$.

Then there exists a G_n*-Picard trajectory* $(x_m)_{m\in\mathbb{N}}$ *converging to a fixed point of* T.

Proof. Let $(x_m)_{m\in\mathbb{N}} \in F_n(T,G,x_0)$ be such that $x_m \to x$. By (ii) there exists $(m_j)_{j\in\mathbb{N}}$ such that $(x_{m_j},x) \in E(G)$ for $j \in \mathbb{N}$. Since T is a multivalued G_b-contraction, for $j \in \mathbb{N}$, there exists $y_{m_j+1} \in Tx$ such that $(x_{m_j+1},y_{m_j+1}) \in E(G)$ and $d(x_{m_j+1},y_{m_j+1}) \leq kd(x_{m_j},y_{m_j})$. By using triangular inequality we obtain

$$\begin{aligned} d(y_{m_j+1},x) &\leq s[d(y_{m_j+1},x_{m_j+1}) + d(x_{m_j+1},x)] \\ &\leq s[kd(y_{m_j},x_{m_j}) + d(x_{m_j+1},x)]. \end{aligned}$$

Thus, $y_{m_j} \to x$ and $x \in Tx$. Since Tx is closed so, T is G_n- Picard continuous. The conclusion follows from the Theorem 3.1. □

Corollary 3.2 (Theorem 5, [8]). *Let* (X,d) *be a complete b-metric space and* $T : X \to CL(X)$ *be a multivalued contraction which satisfies the inequality*

$$H(Tx,Ty) \leq kd(x,y) \textit{ for all } x,y \in X \textit{ and } 0 \leq k < s^{-1}.$$

Then T *has a fixed point.*

Proof. The result follows form Corollary 3.1 by taking $E(G) = X \times X$ and $n = 1$. □

Corollary 3.3. *Let* (X,d) *be a complete b-metric space and* $f : X \to X$ *be a single valued* G_b*-contraction such that for some* $x_0 \in X$, $fx_0 \in [x_0]_G$. *Assume that one of the following conditions holds:*

(a) *if* $f^m x_0 \to x$ *then* $fx = x$;

(b) *if* $f^m x_0 \to x$ *then there exists a subsequence* $(f^{m_j} x_0)_{j\in\mathbb{N}}$ *with* $(f^{m_j} x_0,x) \in E(G)$ *for* $j \in \mathbb{N}$.

Then f *has a fixed point.*

Corollary 3.4. *Let* (X,d) *be a complete b-metric space and* $f : X \to X$ *be a single valued* G_b*-contraction. Assume that one of the following conditions holds:*

(a) *for all* $x,y \in X$ *and any sequence* $(m_j)_{j\in\mathbb{N}}$ *of positive integers such that* $f^{m_j}x \to y$ *and* $(f^{m_j}x, f^{m_{j+1}}x) \in E(G)$ *for all* $j \in \mathbb{N}$*, one has* $f(f^{m_j}x) \to fy$*;*

(b) *for any sequence* $(m_j)_{j\in\mathbb{N}}$ *such that* $x_m \to x$ *and* $(x_m, x_{m+1}) \in E(G)$ *for* $m \in \mathbb{N}$ *then there exists a subsequence* $(x_{m_j})_{j\in\mathbb{N}}$ *with* $(x_{m_j}, x) \in E(G)$ *for* $j \in \mathbb{N}$.

If there exists $x_0 \in X$ *such that* $(x_0, fx_0) \in E(G)$*, then there exists* $x \in X$ *such that* $fx = x$.

The Corollaries 3.3 and 3.4 are the generalization of the results of Jachymski [12] for single valued maps in b-metric space.

Corollary 3.5. *Let* (X,d) *be a complete b-metric space and* $T : X \to CL(X)$ *be a multivalued map. Assume that*

(a) *there exists* $0 < k < s^{-1}$ *such that*

$$H(Tx,Ty) \leq kd(x,y) \text{ for all } (x,y) \in E(G);$$

(b) *for each* $(x,y) \in E(G)$*, each* $u \in Tx, v \in Ty$ *satisfying* $d(u,v) \leq \lambda d(x,y)$ *for some* $\lambda \in (0,1)$*, we have* $(u,v) \in E(G)$*;*

(c) *for any sequence* $(m_j)_{j\in\mathbb{N}}$ *such that* $x_m \to x$ *and* $(x_m, x_{m+1}) \in E(G)$ *for* $m \in \mathbb{N}$*, then there exists a subsequence* $(x_{m_j})_{j\in\mathbb{N}}$ *with* $(x_{m_j}, x) \in E(G)$ *for* $j \in \mathbb{N}$.

If there exists $x_0, x_1 \in X$ *such that* $x_1 \in [x_0]_G^1 \cap Tx_0$*, then there exists* $x \in X$ *such that* $x \in Tx$.

Proof. The result follows form Theorem 3.1 by taking $n = 1$. ☐

It is worthwhile to point out that if $p < n$ then $F_p(T,G,x_0) \subset F_n(T,G,x_0)$, since $[x]_G^p \subset [x]_G^n$. Thus, if T is G_n-Picard continuous then T is G_p-Picard continuous.

Now we esteblish the following result.

Theorem 3.2. *Let* (X,d) *be a complete b-metric space and* $T : X \to X$ *be a multivalued* G_b*-contraction and let* x_0 *be such that* $[x_0]_G^n \cap Tx_0 \neq \emptyset$*. Assume that one of the following holds:*

(1) *T has closed values and, for every* $(x_m)_{m\in\mathbb{N}} \in F(T,G,x_0)$ *such that* $x_m \to x$*, there exists* $(m_j)_{j\in\mathbb{N}}$ *such that* $(x_{m_j}, x) \in E(G)$ *for* $j \in \mathbb{N}$.

(2) *T is G-Picard continuous from* x_0.

Then there exists $n \in \mathbb{N}$ *and a* G_n*-Picard trajectory* $(x_m)_{m\in\mathbb{N}}$ *converging to a fixed point of* T.

By using Theorem 3.2, we can generalized the fixed theorem of Nadler [14] for (ε,λ)-uniformly locally contractive multivalued maps with the setting of b-metric space as following.

Corollary 3.6. *Let (X,d) be a complete b-metric space and $T: X \to CL(X)$ be a multivalued contraction. Suppose that there exists $\varepsilon > 0$ such that*

(1) *there exists $\lambda \in (0,1)$ such that for all $x,y \in X$ and $d(x,y) < \varepsilon$ implies that $H(Tx,Ty) \leq \lambda d(x,y)$;*

(2) *there exist x_0 and $x' \in Tx_0$ such that there is a finite set of points $\{x_0,\cdots,x_n\} \subset X$ such that $x' = x_n,$ and $d(x_{i-1},x_i) < \varepsilon$ for all $i = 1,2,\cdots,n$.*

Then T has a fixed point.

Proof. Consider the graph G with $V(G) = X$ and $E(G) = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}$. One can easily check that T is a multivalued G_b-contraction with a contractive constant k such that $\lambda < k < s^{-1}$. Also from (ii), $x' \in Tx_0 \cap [x_0]_G$, thus the conclusion follows from Thorem 3.2. □

4. Weak G_b-Contraction

In this section we introduce the notion of a weak G_b- contraction which is the generalization of a G_b- contraction. We also discuss the relation between a weak G_b- contraction and a G_b-contraction and also we establish some fixed point results for the weak G_b- contraction. We start this section by the following definition.

Definition 4.1. *Let (X,d) be a b-metric space and Y is subset of X. A multivalued map $T: Y \to N(X)$ is said to be a weak G_b- contraction if there exists $k \in (0,s^{-1})$ such that for all $x,y \in Y$ with $y \in [x]_G$, and all $u \in Tx$, there exists $v \in Ty$ such that $v \in [u]_G$ and $q(u.v) \leq kq(x,y)$.*

Remark 4.1. *Every G_b- contraction is a weak G_b- contraction but the converse not necessarily true. Indeed , for $x \in X$ and $y \in [x]_G$, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $[x]_G^n$ and*

$$q_n(x,y) \leq q(x,y) + \varepsilon.$$

From Lemma 3.1, for $u \in Tx$, there exists $v \in [u]_G^n \cap Ty$ such that

$$q_n(u,v) \leq k(q_n(x,y)+\varepsilon) \leq k(q(x,y)+2\varepsilon).$$

So, $v \in [u]$ and $q(u,v) \leq k(q(x,y)+2\varepsilon)$. Since ε is arbitrary, we obtain the conclusion.

Note that if there is a n-directed path from x to y, there may be no n-directed path from $u \in Tx$ to any element of Ty. The following example shows that a weak G_b-contraction is not necessarily a G_b-contraction.

Example 4.1. *Let $X = \left\{\frac{1}{2^r} : r \in \mathbb{N} \cup \{0\}\right\} \cup \{0\}$ with respect to b-metric $d(x,y) = |x-y|^2$. Consider the graph G such that $V(G) = X$ and*

$$E(G) = \left\{\left(\frac{1}{2^r}, \frac{1}{2^{r+1}}\right) : r \in \mathbb{N} \cup \{0\}\right\} \cup \left\{\left(\frac{1}{2^{r-1}}, 0\right) : n \text{ is odd }\right\} \cup \Delta.$$

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \left\{0, \frac{1}{2}, 1\right\} & \text{if } x = 0, \\ \left\{\frac{1}{2^{r+2}}, 1\right\} & \text{if } x = \frac{1}{2^r} \text{ with } n \text{ odd}, \\ \left\{\frac{1}{2^{r+1}}, 1\right\} & \text{if } x = \frac{1}{2^r} \text{ with } n \text{ even}, \\ \left\{\frac{1}{2}\right\} & \text{if } x = 1. \end{cases}$$

A simple calculation show that T is a multivalued weak G_b-contraction with constant $k = \frac{5}{16}$, but it is not a G_b-contraction. Because, for $x = \frac{1}{4}, y = \frac{1}{8}$ and $u = \frac{1}{8} \in Tx$, there is no $v \in Ty$ such that $\left(\frac{1}{8}, v\right) \in E(G)$.

Now we establish a fixed point theorem for a multivalued weak G_b-contraction.

Theorem 4.1. *Let (X,d) be a complete b-metric space and $T : X \to X$ be a multivalued weak G_b-contraction and let x_0 be such that $[x_0]_G \cap Tx_0 \neq \emptyset$. Assume that one of the following hold:*

(1) *T is G-Picard continuous from x_0;*

(2) *T has closed values and, for every $(x_m)_{m\in\mathbb{N}} \in F(T,G,x_0)$ such that $x_m \to x$, there exists $(m_j)_{j\in\mathbb{N}}$ such that $q(x_{m_j},x) \to 0$ and $x \in [x_{m_j}]_G$ for $j \in \mathbb{N}$.*

Then there exists a G-Picard trajectory $(x_m)_{m\in\mathbb{N}}$ converging to a fixed point of T.

Proof. First consider the condition (i) is satisfied. Let $x_1 \in [x_0]_G \cap Tx_0$ Since T is a weak G_b-contraction, there exists $(x_m)_{m\in\mathbb{N}}$ a G-Picard trajectory such that

$$q(x_m,x_{m+1}) \leq kq(x_{m-1},x_m) \leq k^m q(x_0,x_1) \text{ for all } m \in \mathbb{N}.$$

Since $k \leq \frac{1}{s}$ and $d(x_m, x_{m+1}) \leq$ so, $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence which converges to some $x \in X$. From the fact that T is G-Picard continuous from x_0, we deduce that $x \in Tx$. Now if (ii) is satisfied, there exists $(x_j)_{j\in\mathbb{N}}$ such that $x \in [x_{m_j}]_G$. As T is weak G_b-contraction, there exists $y_{m_j} \in [x]_G \cap Tx$ such that for all $j \in \mathbb{N}$, $q(x_{m_j+1}, y_{m_j+1}) \leq kq(x_{m_j}, x)$. Hence,

$$\begin{aligned} d(x, y_{m_j+1}) &\leq s[d(x, x_{m_j+1}) + d(x_{m_j+1}, y_{m_j+1})] \\ &\leq s[d(x, x_{m_j+1}) + q(x_{m_j+1}, y_{m_j+1})] \\ &\leq s[d(x, x_{m_j+1}) + kd(x_{m_j}, x)] \to 0. \end{aligned}$$

Since T has closed values so, x is fixed point of T. □

It is worthwhile to point out that Theorem 4.1 still holds if we replace the assumption that T is a weak G_b-contraction by the following condition.

$(wC_{G_b})'$ for all $x \in X$ and $u \in Tx \cap [x]_G$, there exists $v \in Tu \cap [u]_G$ such that $q(u, v) \leq kq(x, u)$.

Corollary 4.1. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a single valued map such that for some $x_0 \in X$, $fx_0 \in [x_0]_G$. Assume that there exists constant k such that $0 < k < s^{-1}$ and for any $x, y \in X$ with $y \in [x]_G$, we have $fy \in [fx]_G$ and $q(fy, fy) \leq kq(x, y)$. Moreover, suppose that one of the following conditions holds:*

(a) *if $f^m x_0 \to x$ then $fx = x$;*
(b) *if $f^m x_0 \to x$ then there exists a subsequence $(f^{m_j} x_0)_{j\in\mathbb{N}}$ with $q(f^{m_j} x_0, x) \to 0$ and $f^{m_j} x_0 \in [x]_G$ for $j \in \mathbb{N}$.*

Then f has a fixed point.

Corollary 4.2. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to CL(X)$. Suppose that*

(a) *there exists $k \in \left(0, \frac{1}{s}\right)$ such that $H(Tx, Ty) \leq kd(x, y)$ for all $x, y \in E(G)$;*
(b) *for each $x \in X$, if $u \in [x]_G \cap Tx$, then for every $v \in Tu$, there exists an n-directed path $u = u^0, u^1, \cdots, u^n = v$ from u to v such that,*

$$d(u, v) = \sum_{i=1}^{n} s^i d(u^{i-1}, u^i);$$

(c) *T has closed graph.*

If there exists $x_0, x_1 \in X$ *such that* $x_1 \in [x_0]_G^1 \cap Tx_0$, *then* T *has a fixed point.*

Remark 4.2. *Note that Theorem 3.2 is corollary of the Theorem 4.1. In the case of single valued maps, the Corollary 4.2 is the generalization of the Corollary 3.3.*

Theorem 4.2. *Let* $T : X \to X$ *be a multivalued weak* G_b*-contraction and* $x_0 \in X$. *Then for every* $y_0 \in [x_0]_G$,

$$Fix(T,x_0) \subset Fix(T,y_0).$$

Where

$$Fix(T,x_0) = \{x : \exists (x_m)_{m\in\mathbb{N}} \text{ such that } x_m \to x \in Tx \text{ and } x_m \in Tx_{m-1} \forall m \in \mathbb{N}\}.$$

Proof. Let $x \in Fix(T,x_0)$ and $(x_m)_{m\in\mathbb{N}}$ such that $x_n \to x \in Tx$ and $x_m \in Tx_{m-1}$ for all $m \in \mathbb{N}$. Let $y_0 \in [x_0]_G$. So, there exists $(y_m)_{m\in\mathbb{N}}$ such that, for every $m \in \mathbb{N}, y_m \in Ty_{m-1}, y_m \in [x_m]_G$, and

$$d(x_m,y_m) \le q(x_m,y_m) \le k^m q(x_0,y_0).$$

Therefore, $y_m \to x \in Tx$ and hence, $x \in Fix(T,y_0)$. □

Theorem 4.3. *Let* $T : X \to X$ *be a multivalued weak* G_b*-contraction and* $x_0 \in X$ *is such that* G_{x_0} *is connected. Then for every* $y_0 \in G_{x_0}$,

$$Fix_G(T,x_0) = Fix_G(T,y_0).$$

Where

$$Fix_G(T,x_0) = \{x : \exists (x_m)_{m\in\mathbb{N}} \in F(T,G,x_0) \text{ such that } x_m \to x \in Tx\}.$$

Moreover, if $[x_0]_G \cap Tx_0 \neq \emptyset$ *and* T *is G-Picard continuous from* x_0, *then* $Fix_G(T,x_0) \neq \emptyset$.

Proof. Let $x \in Fix_G(T,x_0)$ and $(x_m)_{m\in\mathbb{N}} \in F(T,G,x_0)$ such that $x_m \to x$. Let $y_0 \in G_{x_0}$. By the proof of the previous theorem, we obtain $(y_m)_{m\in\mathbb{N}}$ such that, $y_m \to x, y_m \in Ty_{m-1}$ and $y_m \in [x_m]_G$, for every $m \in \mathbb{N}$. Since G_{x_0} is connected, we deduce that $y_m \in [y_{m-1}]_G$ for all $n \in \mathbb{N}$. Hence $(y_m)_{m\in\mathbb{N}} \in F(T,G,y_0)$ and $x \in Fix_G(T,y_0)$. Also, since G_{x_0} is connected, for $y_0 \in G_{x_0}$, we have $G_{x_0} = G_{y_0}$. Interchanging x_0 and y_0 in previous argument, we have $Fix_G(T,x_0) = Fix_G(T,y_0)$.

In the end, if $[x_0]_G \cap Tx_0 \neq \emptyset$ and T is G-Picard continuous from x_0, one can deduce from the Theorem 4.1 that $Fix_G(T,x_0) \neq \emptyset$. □

Corollary 4.3. *Let $T : X \to X$ be a multivalued weak G_b-contraction and G is connected. Then*

$$Fix(T) = Fix_G(T, y_0).$$

Where,

$$Fix(T) = \{x : x \in Tx\}.$$

Moreover, if T is G-Picard continuous from x_0, then $Fix(T) \neq \emptyset$.

Remark 4.3. *The results similar to the Corollary 4.3 and Theorem 4.3 can be stated by replacing the condition on the G-Picard continuity of T by the condition (2) of Theorem 4.1.*

5. Strong b-Comparison Functions and Related Fixed Point Results

In this section, we introduce the notion of new type of a comparison functions, we call these type of functions the strong b-comparison functions. We also obtain fixed point results for set values maps by using such functions.

Definition 5.1. *Let (X,d) be a b-metric space with a coefficient $s \geq 1$. A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a strong b-comparison function if the following holds:*

(i) *φ is increasing function;*

(ii) $\varphi(0) = 0$*;*

(iii) $\sum_{m=1}^{\infty} s^m \varphi^m(t)$ *converges for all $t > 0$.*

Note that for $s = 1$, the notion of a strong b-comparison function coincides with a strong comparison function [10, Definition 6.1]. Let (X,d) be a b-metric space with coefficient $s \geq 1$. Define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \alpha t$; for all $t \in [0,\infty)$ with $\alpha \in (0, s^{-1})$ is a strong b-comparison function.

Definition 5.2. *One says that a multivalued mapping $T : X \to X$ with non empty values is a (G_b, φ)-contraction if*

> *for all $(x,y) \in E(G)$ and all $u \in Tx$, there exists $v \in Ty$ such that $(u,v) \in E(G)$ and $d(u,v) \leq \varphi(d(x,y))$.*

Where $\varphi : [0,\infty) \to [0,\infty)$ is a strong b-comparison function.

Now we establish the fixed point results for multivalued maps satisfying a (G_b,φ)-contraction condition.

Theorem 5.1. *Let (X,d) be a complete b-metric space and $T: X\to X$ be a multivalued (G_b,φ)-contraction, where $\varphi:[0,\infty)\to[0,\infty)$ is a strong b-comparison function. Let x_0 be such that $[x_0]_G^n\cap Tx_0\neq\emptyset$ for $n\in\mathbb{N}$. Assume that one of the following hold:*

(1) *T is G_n-Picard continuous from x_0;*

(2) *T has closed values and, for every $(x_m)_{m\in\mathbb{N}}\in F_n(T,G,x_0)$ such that $x_m\to x$, there exists $(m_j)_{j\in\mathbb{N}}$ such that $(x_{m_j},x)\in E(G)$ for $j\in\mathbb{N}$.*

Then there exists G_n-Picard trajectory $(x_m)_{m\in\mathbb{N}}$ converging to a fixed point of T.

Proof. Let $x_1\in[x_0]_G^n\cap Tx_0$ and $x_0=x^0,x^1,\cdots,x^n=x_1$ is n- directed path between x_0 and x_1. Since T is a (G_b,φ)-contraction and from Lemma 3.1 we yield $(x_m)_{m\in\mathbb{N}}\in F_n(T,G,x_0)$ such that

$$d(x_m,x_{m+1})\leq q_n(x_m,x_{m+1})\leq\sum_{i=1}^{m}s^m\varphi^m(d(x_0,x_1))\text{ for all }m\in\mathbb{N}.$$

Since φ is a strong comparison function so, $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence which converges to some $x\in X$. From the fact that T is a G-Picard continuous from x_0, we deduce that $x\in Tx$. Now if the condition (ii) is satisfied, then by the fact that T is a (G_b,φ)-contraction, there exists $(n_j)_{j\in\mathbb{N}}$ and $(y_{m_j})_{j\in\mathbb{N}}$ such that

$$(x_{m_j},x)\in E(G),(x_{m_j+1},y_{m_j+1})\in E(G),y_{m_j+1}\in Tx$$

and

$$d(x_{m_j+1},y_{m_j+1})\leq\varphi(d(x_{m_j},x))\text{ for all }j\in\mathbb{N}.$$

Since φ is strong comparison function so, $d(x_{m_j+1},y_{m_j+1})\to 0$ therefore, $y_{m_j+1}\to x$. Since T has closed values so, x is fixed point of T. □

By using the concepts of a weak G_b- contraction and a (G_b,φ)-contraction we introduce the notion of a weak (G_b,φ)-contraction as following.

Definition 5.3. *A multivalued mapping $T: X\to X$ with non empty values is said to be a weak (G_b,φ)-contraction if* *for all* $(x,y) \in E(G)$ *and all* $u \in Tx$*, there exists* $v \in Ty$ *such that* $(u,v) \in E(G)$ *and* $q(u,v) \leq \varphi(q(x,y))$.

Where $\varphi : [0,\infty) \to [0,\infty)$ *is a strong b-comparison function.*

Theorem 5.2. *Let* (X,d) *be a complete b-metric space and* $T : X \to X$ *be a multivalued weak* (G_b,φ)*-contraction, where* $\varphi : [0,\infty) \to [0,\infty)$ *is a strong b-comparison function. Let* $x_0 \in X$ *be such that* $[x_0]_G \cap Tx_0 \neq \emptyset$*. Assume that one of the following hold:*

(1) T *is* G_n*-Picard continuous from* x_0*;*

(2) T *has closed values and, for every* $(x_m)_{m\in\mathbb{N}} \in F(T,G,x_0)$ *such that* $x_m \to x$*, there exists* $(m_j)_{j\in\mathbb{N}}$ *such that* $q(x_{m_j},x) \to 0$ *and* $x \in [x_{m_j}]_G$ *for* $j \in \mathbb{N}$.

Then there exists a G-Picard trajectory $(x_m)_{m\in\mathbb{N}}$ *converging to a fixed point of* T.

Proof. The conclusion follows from the arguments used in the proof of Theorem 4.1 and 5.1. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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