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SOME RESULTS ON THE ANALYTIC REPRESENTATION INCLUDING THE CONVOLUTION IN THE L^p SPACES

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Abstract: In this paper we consider functions in the L^p spaces including the convolution of functions and validity of associative law of the convolution. Using the generalized Cauchy representation, we obtain some results concerning the analytic representation of convolution of functions and distributions. The boundary values representation has been studied for a long time ago and from different points of view. One of the first results is that if we consider three functions from L^1 spaces then their convolution has a Cauchy representation. We obtain several results regarding their convergence or the convergence of the sequence of their analytic representation.
We will prove result concerning the analytic representation of the convolution for two functions that are elements of the L^1 spaces and the third function is element of the L^p spaces, then their convolution has a Cauchy representation. Other results, using Fubini's Theorem, for sequence of functions from L^1 spaces and two functions from L^1 spaces, their convolution has a Cauchy representation, also has Cauchy representation the boundary value, illustrated with an

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example. Also we have stated and proved that if a sequence of functions of L^1 spaces, which converges to the function in $L^1(P_n)$ and a sequence of analytic functions which converges uniformly to the function on every compact subset of the real line, then the sequence of distributions converges to the distribution in D' sense.

Keywords: convolution; Cauchy representation; distribution.

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1. INTRODUCTION AND AUXILIARY FACTS

The term "distribution" was introduced by Laurent Schwartz in 1951, by analogy with a distribution of electrical charge. He extended and developed the idea of Sergei Sobolev who originated the generalized functions in his work on second-order hyperbolic partial differential equations, but did not achieve to formalize.

Distributions which also are known as generalized functions are objects that generalize the classical notion of functions in mathematical analysis, and make possible to differentiate functions whose derivatives do not exist in the classical sense. Distribution theory has an important role in applied mathematics, especially in the theory of partial differential equations, physics, engineering and many other areas.

We use the standard notation from the Schwartz distribution theory.

With $C^\infty(\mathbb{R}^n)$ is denoted the space of all infinitely differentiable functions on $\mathbb{R}^n$ and $C_0^\infty(\mathbb{R}^n)$ denotes the subspace of $C^\infty(\mathbb{R}^n)$ that consists of those functions of $C^\infty(\mathbb{R}^n)$ which have compact support.

Definition 1.1. The support of f is the closure of the set $x \in \Omega$, of points for which f is different from zero $(f(x) \neq 0)$, and is denoted by supp f.

With D, we denote the space of $C_0^\infty(\mathbb{R}^n)$ functions, called the set of test functions in which convergence is defined in the following way :

A sequence $\{\varphi_v\}$ of functions $\varphi_v \in D$ converges to $\varphi \in D$ in D as $v \to v_0$ if and only if there is a compact set $K \subset \mathbb{R}^n$ such that $\operatorname{supp}(\varphi_v) \subseteq K$ for each v, $\operatorname{supp}(v) \subseteq K$ and for every n-tiple k of nonnegative integers the sequence $\left\{ f^{(k)} \varphi_v(t) \right\}$ converges to $f^{(k)}\varphi(t)$ uniformly on K as $v \to v_0$.

Definition 1.2. A distribution T is continuous linear functional on D. Instead of writing $T(\varphi)$, it is conventional to write $< T, \varphi >$ for the value of T acting on a test function φ. The space of all distributions is denoted by D'.

Let φ be an element of one of the above function spaces D or S, and f be a function for which

(1) $$< T_f, \varphi >= \int_{\mathbb{R}^n} f(t)\varphi(t)dt, \quad \varphi \in D(\varphi \in S)$$

exists and is finite. Then T_f is regular distribution on D (or S) generated by f.

The boundary value representation has been studied for a long time and from different points of view. It is well known that every function $f \in L^1$ is a regular distribution and its analytic representation is, in fact, the Cauchy representation

(2) $$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} >.$$

This function is analytic in the complex plane except on the support of f and it holds

(3) $$\hat{f}(x+iy) - \hat{f}(x-iy) \to f(x)$$

as $y \to 0^+$ in D' sense,i.e.

(4) $$\lim_{y \to 0^+} < \hat{f}(x+iy) - \hat{f}(x-iy), \varphi(x) >=< f, \varphi >$$

for every $\varphi \in D$.

If $f, g \in L^1$ then

(5) $$\int_{\mathbb{R}} |f(x-y)g(y)| dy < \infty$$

for almost all x.

If $\|h\|_1 \le \|f\|_1 \|g\|_1$ and $g*f = f*g$, then h is called the **convolution** of f and g.

If $f \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$ for $1 \le p < \infty$, then $h = f*g$ belongs to L^p for $1 \le p < \infty$ is given in [5].

Lemma 1.1. Let $f, g \in L^1$. Then

(6) $$h(x) = (f*g)(x) = \int_{\mathbb{R}} f(t)g(x-t)dt$$

belongs to $L^1(\mathbb{R})$.

Lemma 1.2. Let $f_1, f_2, f_3 \in L^1$. Then

(7) $$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3).$$

Lemma 1.3. Let $f \in E'$ and $\varphi \in C^\infty$ be bounded function with each of its derivatives also bounded.

If

(8) $$f^*(z) = \frac{|y|}{\pi} < f(t), \frac{1}{|t-z|^2} >$$

then

(9) $$\int_{-\infty}^{\infty} f^*(x+i\varepsilon)\varphi(x)dx = < f(t), \varphi^*(t+i\varepsilon) >, \ \varepsilon > 0.$$

Theorem 1.1. Let f and φ be as in the above Lemma, then

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} [\hat{f}(x+i\varepsilon) - \hat{f}(x-i\varepsilon)]\varphi(x)dx = < f(x), \varphi(x) >.$$

Theorem 1.2. Let (f_n) be a sequence of functions of L^1 space which converges to the function f in L^1. Let (P_n) be a sequence of analytic functions which converges uniformly to the function P on every compact subset of the real line. Then the sequence of distributions

$(P_n f_n)$ converges to the distribution Pf in D' sense as $n \to \infty$ and $\hat{Pf}$ is analytic representation of the distribution Pf.

2. Main Results

Theorem 2.1. Let $f_1, f_2, f_3 \in L^1$ and $h = (f_1 * f_2) * f_3$. Then h has Cauchy representation

$$\hat{h}(z) = \frac{1}{2\pi i}\int \frac{h(t)}{t-z}dt =$$

$$= \int_{\mathbb{R}} (f_1 * f_2)(t)\, \hat{f_3}(z-t)dt = \tag{10}$$

$$= \int_{\mathbb{R}} f_3(t)\,(\widehat{f_1 * f_2})(z-t)dt$$

Proof.

Since $f_1, f_2, f_3 \in L^1$ from Lemma 1.1 implies that $f_1 * f_2 \in L^1$ and $h = (f_1 * f_2) * f_3 \in L^1$. Hence $\hat{h}(z)$ exists. Let $f_1 * f_2 = f$ implies that $f \in L^1$. Since $|f(t) f_3(u)|$ is integrable over the tOu-plane, then

$$\left|\frac{f(t) f_3(u)}{u+t-z}\right| \tag{11}$$

is integrable over the tOu- plane for $z = x+iy,\ \operatorname{Im} z \neq 0$.

Applying Fubini's theorem, we may change the order of integration and get that

$$2\pi i \int_{\mathbb{R}} (f_1 * f_2)(t)\, \hat{f_3}(z-t)dt =$$

$$= 2\pi i \int_{\mathbb{R}} f(t)\, \hat{f_3}(z-t)dt =$$

$$= \int_{\mathbb{R}} f(t)\Big[\int_{\mathbb{R}} \frac{f_3(u)}{u-(z-t)}du\Big]dt = \tag{12}$$

$$= \int_{\mathbb{R}} f_3(u) \int_{\mathbb{R}} \frac{f(t)}{t-(z-u)}du\, dt =$$

$$= 2\pi i \int_{\mathbb{R}} f_3(u)\, \hat{f}(z-u)du.$$

Also, we get,

$$\int_{\mathbb{R}}[\int_{\mathbb{R}}\frac{f(t)f_3(u)}{u-(z-t)}dt]du =$$

(13)
$$= \int_{\mathbb{R}}\frac{1}{u-z}[\int_{\mathbb{R}} f_3(u-t)f(t)dt]du =$$

$$= 2\pi i\,\hat{h}(z).$$

Using lemma 2.1 we can also state the following corollary:

Corollary 2.1. Let f_1, f_2, f_3 be in L^1 and let $h=(f_1 * f_2)* f_3$. Then h has Cauchy representation.

We will prove result concerning the analytic representation of the convolution $h = f * g$ for $f \in L^1, g \in L^p$.

Next we will prove result concerning the analytic representation of the convolution for $f_1, f_2 \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$.

Theorem 2.2. Let $f_1, f_2 \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $1 \le p < \infty$ and let $h=(f_1 * f_2)*g$. Then h has the Cauchy representation

(14)
$$\hat{h}(z) = \frac{1}{2\pi i}\int\frac{h(t)}{t-z}dt, z = x+iy, \operatorname{Im} z \neq 0 .$$

Proof.

Let $f = f_1 * f_2$, from Lemma 1.1 we have $f \in L^1$. For $\varphi \in D$,

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy)-\hat{h}(x-iy)]\varphi(x)dx =$$

$$\lim_{y\to 0^+}\int_{\mathbb{R}}\frac{1}{2\pi i}[\int_{\mathbb{R}}(\frac{((f_1 * f_2)*g)(t)}{t-z} -$$

(15)
$$-\frac{((f_1 * f_2)*g)(t)}{t-\overline{z}})dt]\varphi(x)dx =$$

$$\lim_{y\to 0^+}\int_{\mathbb{R}}\frac{1}{2\pi i}(\int_{\mathbb{R}}\int_{\mathbb{R}}[\frac{(f_1 * f_2)(u)g(t-u)du}{t-z} -$$

$$-\frac{(f_1 * f_2)(u)g(t-u)du}{t-\overline{z}}]dt)\varphi(x)dx$$

The above integrals exist by the Hölder inequality, hence applying Fubini's theorem, we may change the order of integration and get that

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy)-\hat{h}(x-iy)]\varphi(x)dx =$$
$$\lim_{y\to 0^+}\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{\varphi(x)}{t-z}-\frac{\varphi(x)}{t-\overline{z}}\right)dx\cdot \tag{16}$$
$$\cdot\int_{\mathbb{R}}(f_1 * f_2)(u)du\int_{\mathbb{R}}g(t-u)dt =$$
$$= \lim_{y\to 0^+}\frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)}{|t-z|^2}dx\cdot$$
$$\cdot\int_{\mathbb{R}}(f_1 * f_2)(u)du\int_{\mathbb{R}}g(t-u)dt.$$

Now by the Lemma 1.3, we get that

$$\frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)}{|t-z|^2}dx = \hat{\varphi}(t+iy) \tag{17}$$

and

$$\int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}g(t-u)\hat{\varphi}(t+iy)dt \tag{18}$$

converges to

$$\int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}g(t-u)\varphi(t)dt \tag{19}$$

as $y\to 0^+$.

Finally, with one more use of Fubini's theorem, we get

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy)-\hat{h}(x-iy)]\varphi(x)dx =$$
$$\int_{\mathbb{R}}(f_1 * f_2)(u)g(t-u)du\int_{\mathbb{R}}\varphi(t)dt = \tag{20}$$
$$= \int_{\mathbb{R}}((f_1 * f_2)*g)(t)\varphi(t)dt = <(f_1 * f_2)*g,\varphi>.$$

Theorem 2.3. Suppose that the sequence $\{l_n\}$ converges to l in L^1 sense and $f,g\in L^1$. Let

$h_n = l_n * (f * g)$. Then the sequence $\{h_n\}$ converges to $h = l * (f * g)$ in L^1. If $\hat{h}_n(z)$ is an analytic representation of every h_n for $n = 1, 2, 3, \ldots$, then $\hat{h}(z)$ is an analytic representation of h.

Proof.

From Lemma 1.1 we have $f * g \in L^1$ and $h = l * (f * g) \in L^1$.

Let us consider the difference

$$
\begin{aligned}
&\left| \int_{\mathbb{R}} h_n(x)dx - \int_{\mathbb{R}} h(x)dx \right| = \\
&= \left| \int_{\mathbb{R}} (l_n * (f * g))(x)dx - \int_{\mathbb{R}} (l * (f * g))(x)dx \right| \\
&= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [l_n(y)(f * g)(x - y)dy - \right. \\
&\qquad \left. - l(y)(f * g)(x - y)dy]dx \right| = \\
&= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [l_n(y) - l(y)](f * g)(x - y)dydx \right|
\end{aligned} \tag{21}
$$

By Fubini's Theorem, in the last integral, we may change the order of integration and get that

$$
\begin{aligned}
&\left| \int_{\mathbb{R}} h_n(x)dx - \int_{\mathbb{R}} h(x)dx \right| \leq \\
&\leq \int_{\mathbb{R}} |l_n(y) - l(y)| dy \int_{\mathbb{R}} |(f * g)(x - y)| dx
\end{aligned} \tag{22}
$$

Since $l_n \to l$ in L^1 and by Lebesgue's dominated convergence theorem we get that $h_n \to h$ in L^1.

that $\hat{h}(z)$ is analytic representation of h follow from the Theorem 2.2 in this paper.

Corollary 2.2. Let $f, g \in L^1(\mathbb{R})$. Suppose that the sequence $\{l_n\}$ converges to l in $L^p, 1 \leq p < \infty.$, and let $h = (f_1 * f_2) * l$. Then h has the Cauchy representation

$$
\hat{h}(z) = \frac{1}{2\pi i} \int \frac{h(t)}{t - z} dt, z = x + iy, \ \operatorname{Im} z \neq 0. \tag{23}
$$

Proof.

It is true that $f_1 * f_2 \in L^1$ and $h = (f_1 * f_2) * l \in L^p$.

Let us consider the difference

$$\left|\int_{\mathbb{R}} h_n(x)dx - \int_{\mathbb{R}} h(x)dx\right| =$$

$$= \left|\int_{\mathbb{R}} ((f_1 * f_2) * l_n))(x)dx - \int_{\mathbb{R}} ((f_1 * f_2) * l)(x)dx\right|$$

(24)
$$= \left|\int_{\mathbb{R}} \int_{\mathbb{R}} [(f_1 * f_2)(y)(l_n)(x-y)dy - \right.$$
$$\left. -(f_1 * f_2)(y)(l)(x-y)dy]dx\right| =$$

$$= \left|\int_{\mathbb{R}} \int_{\mathbb{R}} [f_1(y) - f_2(y)](l_n - l)(x-y)dydx\right|$$

In the last integral, we may change the order of integration (by Fubini's Theorem) and get that

(25)
$$\left|\int_{\mathbb{R}} h_n(x)dx - \int_{\mathbb{R}} h(x)dx\right| \le$$
$$\le \int_{\mathbb{R}} |f_1(y) - f_2(y)| dy \int_{\mathbb{R}} |(l_n - l)(x-y)| dx$$

Since $l_n \to l$ in L^p and by Lebesgue's dominated convergence theorem we get that $h_n \to h$ in L^p.

that $\hat{h}(z)$ is analytic representation of h follow from the Theorem 2.2 in this paper.

Example 2.1. Let (P_n) be the sequence of functions

(26)
$$P_n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

It is well known that the sequence (P_n) converges uniformly to the function $\sin x$ on any compact set. For any sequence of functions f_n of L^1 that converge to a function f in L^1, the sequence $(f_n P_n)$ converges in D' to $\sin x f(x)$ and its analytic representation is the function $\sin z f(z)$.

3. CONCLUSION

Using the generalized Cauchy representation, we obtain some new results concerning the analytic representation of convolution of functions.

Validity of associative law of convolution, using Fubini's Theorem, Lebesgue's dominated convergence theorem, considering the convergent sequences of functions and their analytic representation we state and proof results about the analytic representation of the boundary function, which can be used and in further new results.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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