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THE EXISTENCE FOR MULTIPLE POSITIVE SOLUTIONS OF NONLINEAR MAPPING EQUATIONS IN BANACH SPACES

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Abstract. In this paper, under some suitable boundary conditions, some existence problems for multiple positive solutions of strict set contractive set-valued mapping equation are studied by using quasi-derivative of multi-valued mapping in Banach spaces.

Keywords: Set-valued mapping equation, Quasi-derivative, Multiple positive solution, Fixed point index, Boundary condition.

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1. Introduction

It is an important tool for studying the solutions of single-valued or multi-valued mapping equation to use differentiability and derivate of mapping. The early Frechet derivate[1] was widely used for the solutions and the eigenvectors of single-valued mapping equation. The concept of semi-derivate was introduced by Petryshyn[2] in 1988

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and was used to study the positive solutions of single-valued strict set contractive mapping equation. In 1991, Yang [3] introduced the concept of weak demi-derivative, which abolished the restriction conditions on semi-derivative, such as boundedness, continuity, monotonousness. By using weak demi-derivative, Yang studied the corresponding problems of solutions, positive solutions and multiple positive solutions of 1-set contractive mapping equation. To investigate the existence for positive solutions of multi-valued strict set contractive mapping equation, in 1995, Yang [4] introduced the concept of quasi-derivative for multi-valued mapping, which extended the concept of weak semi-derivative of single-valued mapping. Motivated by the work of Yang, in this paper, we study the existence problems for multiple positive solutions and eigenvectors of the following set-valued mapping equation

$$\theta \in T(x) - x, \tag{1.1}$$

by using quasi-derivative of multi-valued mapping under some suitable boundary conditions. These works are very interesting in theory and applications. The results presented in this paper improve and extend the corresponding results in [2-8].

2. Preliminaries

For the sake of convenience, throughout of this paper, we assume that X is a Banach space with a cone K, $K^0 = K \setminus \{\theta\} \neq \emptyset$, $K^\infty = K \cup \{\infty\}$. Setting $I = [0,1]$, $C_K = C \cap K$, " $\leq$ " denotes the order on X induced by K. $C(X), cf(X), B(X), K(X)$ denote the families of nonempty closed subsets, closed convex subsets, bounded subsets, compact subsets of X, respectively. The boundary and closure of C relative to K are denoted by ∂C_K and $\bar{C}_K$, respectively. Denoting $\Omega^r = \{x \in X; \|x\| < r\}$. $S((1.1), E)$ and $S^+((1.1), E)$ denote the set of all solutions and set of positive solutions for equation (1.1) in E, respectively.

We first recall some definitions and some known results.

Let $C \subset X$, $T : C \to 2^X$

(i) T is said to be a positively homogeneous mapping if $T(\lambda x) = \lambda T(x)$ $(\lambda > 0, x \in C)$.

(ii) For $u \in X$, mapping $T^u : C - u \to 2^X$ is called the u-parallel transformation of T, if

$$T^u(x') = T(u + x') - u \quad (x' \in C - u)$$

(iii) $x \in C$ is said to be a u-eigenvector of T if, $x - u$ is eigenvector of T^u, i.e.,

$$tx' \in T^u(x') \quad (x' = x - u \neq \theta)$$

for some number t, where t is said to be the corresponding u-eigenvalue of T.

Definition 2.1.[4] Let $T : K \to cf(X)$.

(i) T is said to be quasi-differentiable at θ along K if $T(\theta) \in B(X)$ and there exists a positively homogeneous and upper semi-continuous (in short $u.s.c$) mapping $T'_\theta : K \to cf(X)$ such that

$$T(x) = T(\theta) + T'_\theta(x) + w(\theta, x) \ \ (x \in K), \tag{2.1}$$

where $w(\theta, \cdot) : K \to 2^X$ satisfies

$$\limsup_{x \in K, \|x\| \to 0} \{\frac{\|y\|}{\|x\|}; y \in w(\theta, x)\} = 0. \tag{2.2}$$

where T'_θ is said to be a quasi-derivative of T at θ along K.

(ii) T is said to be quasi-differentiable at ∞ along K if $T(\theta) \in B(X)$ and there exists a positively homogeneous mapping $T'_\infty : K \to cf(X)$ such that

$$T(x) = T'_\infty(x) + w(\infty, x) \ (x \in K), \tag{2.3}$$

where $w(\infty, \cdot) : K \to 2^X$satisfies

$$\limsup_{x \in K, \|x\| \to \infty} \{\frac{\|y\|}{\|x\|}; y \in w(\infty, x)\} = 0. \tag{2.4}$$

where T'_∞ is said to be a quasi-derivative of T at ∞ along K.

Definition 2.2. Let $u \in K$, C be a open subset of X. $g : \bar{C}_K \to 2^X$ is said to satisfy boundary condition $(MS; \partial C_K, u)$ if there exists a positive homogeneous function $l : X \to [0, +\infty)$ such that

$$l^{-1}(0) = \{\theta\}, \ \ l(y - x) \neq l(y - u) - l(x - u), \tag{2.5}$$

where $x \in \partial C_K$, and either $y \in g(x) \setminus \bar{C}_K$ as $u \in C_K$ or $y \in g(x)$ as $u \notin C_K$.

Let U be a bounded open subset of X with $U \cap K \neq \emptyset$, $T : U_K \to cf(K)$ be a *u.s.c* strict set contractive mapping with $x \notin T(x)$ ($x \in \partial U_K$). The fixed point index $i_K(T, U)$ is well defined and has the following properties[2,7]

(i) If $i_K(T, U) \neq 0$, then equation (1.1) has a solution in U_K;

(ii) For mapping $\hat{x}_0$ with constant value x_0, if $x_0 \in U_K$ then $i_K(\hat{x}_0, U) = 1$;

(iii) Let U_1, U_2 be two open and bounded subsets of X such that $U_1 \cup U_2 \subset U_K$ and $U_1 \cap U_2 = \emptyset$, if $x \neq T(x)$ for $x \in \partial U_{1K} \cup \partial U_{2K}$, then $i_K(T, U_1 \cup U_2) = i_K(T, U_1) + i_K(T, U_2)$

(iv) Let $H : [0, 1] \times \bar{U}_K \to cf(K)$ be a *u.s.c* mapping and $H_I(\cdot) : \bar{U}_K \to cf(K)$. If $H_I(x) = \cup_{t \in I} H(t, x)$ is a strict set contractive mapping and for all $(t, x) \in [0, 1] \times \partial U_K$, $x \notin H(t, x)$, then $i_K(H(1, \cdot), U) = i_K(H(0, \cdot), U)$.

Lemma 2.1.[4] Suppose that $T : K \to cf(X)$ is a *u.s.c* strict set contractive mapping, $a = \theta$ or $a = \infty$, T'_a is the quasi-derivate of T at a and 1 is not eigenvalue of T'_a. Then

(i) The solution set $S((1.1), E)$ of the equation (1.1) in E is compact, where $E = K$ when $a = \infty$, or $E \in B(K) \cap C(K)$ when $a = \theta$.

(ii) Let $T(K) \subset K$, and let $\theta \in T(\theta)$ when $a = \theta$. Then there exists $\rho > 0$ such that, if $a = \infty$, then

$$i_K(T, \Omega^r_K) = i_K(T'_\infty, \Omega^r_K) \;\; (\forall r > \rho), \tag{2.6}$$

if $a = \theta$, then either there exists a positive solution x of equation (1.1) satisfying $\|x\| < r$ for any $r \in (0, \rho)$, or

$$i_K(T, \Omega^r_K) = i_K(T'_\theta, \Omega^r_K) \;\; (\forall r \in (0, \rho)). \tag{2.7}$$

Definition 2.3. Let $T : K \to 2^X$, $L \subset K$. T is said to be quasi-increasing along L if, for any $x, y \in L$ and any real numbers β and γ satisfying the following conditions

(i) $x > y$, (i.e., $x - y \in K^0$);

(ii) $\gamma x, \beta y \in L$;

(iii) $\gamma x (\geq) T(x)$ and $T(y) (\geq) \beta y$, the relation $\gamma x \geq \beta y$ holds.

Denote $E_V(T, A, u, 1)$ and $E_V(T, A, u, 1+)$ are the sets of all (u)-eigenvectors of T in A with respect to (u)-eigenvalues$= 1$ or > 1, respectively. For $A \in 2^X$ and $a \in X$, notice the fact that $a \geq b$ (or $b \geq a$) for some $b \in A$ by $a(\geq)A$ (or $A(\geq)a$)

Setting

$F(K, -) = \{T : K \to cf(X)$,there exist $\rho > 0$ and $u \in K$ such that $E_V(T, \partial\Omega_K^\rho, u, 1) \cup E_V(T, \partial\Omega_K^\rho, u, 1+) = \emptyset\}$;

$F(K, +) = \{T : K \to cf(X)$,there exist $\rho > 0, \eta > 1$ and $u \in K^0$ such that $E_V(T, \partial\Omega_K^\rho, \theta, 1) = \emptyset, T(u)(\geq)\eta u$ and T is quasi-increasing along $L(u)\}$

where

$$L(u) = \{x \in K; x \geq \lambda u for some \lambda > 0\}. \tag{2.8}$$

Lemma 2.2.[4] Suppose that $T : K \to cf(X)$ is a $u.s.c$ positively homogeneous strict set contractive mapping.

(i) If $T \in F(K, -)$, then $i_K(T, \Omega_K^r) = 1$ $(r > 0)$.

(ii) If $T \in F(K, +)$, then $i_K(T, \Omega_K^r) = 0$ $(r > 0)$.

3. Main results

Lemma 3.1. Let U be a open subset of X and $T : U_K \to cf(K)$ be a $u.s.c$ strict set contractive mapping. Then the following propositions hold.

(i) If U is a convex subset of X, $x \notin T(x)$ $(x \in \partial U_K)$ and there exists a $u \in U_K$ such that T satisfies boundary condition $(MS; \partial U_K, u)$, then $i_K(T, U) = 1$.

(ii) If there exists $u \in K \setminus \bar{U}_K$ such that T satisfies boundary condition $(MS; \partial U_K, u)$, then $i_K(T, U) = 0$.

Proof. Define $H : I \times \bar{U}_K \to cf(K)$ as follows

$$H(t, x) = (1 - t)u + tT(x) \ \ ((t, x) \in I \times \bar{U}_K). \tag{3.1}$$

If condition (i) holds, then

$$x \notin H(t, x) \ \ ((t, x) \in I \times \partial U_K). \tag{3.2}$$

Otherwise, there exist $x_0 \in \partial U_K$ and $t_0 \in I$ such that $x_0 \in H(t_0, x_0)$. Let $y_0 \in T(x_0)$ such that $x_0 = (1 - t_0)u + t_0 y_0$. Obviously, $t_0 \neq 0, 1$, and so $y_0 \notin \bar{U}_K$. Suppose that $l : X \to [0, +\infty)$ is a positive homogeneous function satisfying (2.5). We have $l(y_0 - x_0) = l[(y_0 - u) - (x_0 - u)] = l[(\frac{1}{t_0} - 1)(x_0 - u)] = (\frac{1}{t_0} - 1)l(x_0 - u) = \frac{1}{t_0}l(x_0 - u) - l(x_0 - u) = l(y_0 - u) - l(x_0 - u)$, which is a contradiction. Since $u \in U_K$, we have

$$i_K(T, U) = i_K(H(1, \cdot), U) = i_K(H(0, \cdot), U) = i_K(\hat{u}, U) = 1$$

If condition (ii) holds, then (3.2) can be verified similarly. Since $u \notin U_K$, it follows that $i_K(T, U) = i_K(\hat{u}, U) = 0$. This completes the proof.

Theorem 3.1. Suppose that $T : K \to cf(K)$ is a $u.s.c$ strict set contractive mapping such that T has a quasi-derivative T'_a at $a = \theta, \infty$ and $\theta \in T(\theta)$. If the following two conditions are satisfied

(i) $T'_a \in F(K, +)$ $(a = \theta, \infty)$;

(ii) there exist bounded convex open neighborhood Ω of θ and $u \in \Omega_K$ such that $x \notin T(x)$ as $x \in \partial\Omega_K$ and T satisfies boundary condition $(MS; \partial\Omega_K, u)$.

then the equation (1.1) has at least two positive solutions in K.

Proof. It follows from condition (ii) and Lemma 3.1 that $i_K(T, \Omega_K) = 1$.

For $a = \infty$, it follows from Lemma 2.1 that there exists $\rho_1 > 0$ such that $i_K(T, \Omega^r_K) = i_K(T'_\infty, \Omega^r_K)$ ($\forall r > \rho_1$).

For $a = \theta$, there exists $\rho_2 > 0$ such that either (I) there exists $x_r \in \Omega^r_K \setminus \{\theta\}$ such that $x_r \in T(x_r)$ or (II) $i_K(T, \Omega^r_K) = i_K(T'_\theta, \Omega^r_K)$ $(\forall r \in (0, \rho_2))$.

For the case of (II), from Lemma 2.2, we may take $R > r > 0$ such that $\bar{\Omega}^r \subset \Omega \subset \bar{\Omega} \subset \Omega^R$ and $i_K(T, \Omega^r_K) = i_K(T'_\theta, \Omega^r_K) = 0$ and $i_K(T, \Omega^R_K) = i_K(T'_\infty, \Omega^R_K) = 0$. Setting $A = \Omega^R_K \setminus \bar{\Omega}_K$, $B = \Omega \setminus \bar{\Omega}^r_K$, respectively. Then $i_K(T, A) = -1$ and $i_K(T, B) = 1$. Therefore, equation (1.1) has two solutions x_1, x_2, which one is in A, another is in B. Obviously, x_1 and x_2 are positive solutions of the equation (1.1) in K.

For the case of (I), equation (1.1) has also two positive solutions, one is x_r in Ω^r_K, another is in $\Omega_K \setminus \bar{\Omega}^r_K$. This completes the proof.

Theorem 3.2. Let T be the same as in Theorem 3.1. If the following two conditions are satisfied

(i) $T'_a \in F(K,-)$ $(a = \theta, \infty)$;

(ii) there exist bounded convex open neighborhood Ω of θ and $u \in K \setminus \bar{\Omega}_K$ such that T satisfies boundary condition $(MS; \partial\Omega_K, u)$.

then equation (1.1) has at least two positive solutions in K.

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1, we omit it.

Theorem 3.3. Let $T : K \to cf(K)$ be a $u.s.c$ strict set contractive mapping such that T has a quasi-derivative T'_a at $a = \theta$, ∞ and $T'_\theta \in F(K,-), T'_\infty \in F(K,+)$ and $\theta \in T(\theta)$. Suppose that T satisfies the boundary condition $(MS; \partial\Omega_K, u)$ and the equation (1.1) does not have any solutions on $\partial\Omega_K$. If there exist $d > c > 0$, bounded convex open neighborhood Ω of θ, $u \in \Omega^d_K$ and a continuous concave function $\varphi : K \to R^+ = [0, +\infty)$ such that

(i) $\varphi(u) > c$ and $\varphi(x) \leq \|x\|$ $(x \in \Omega_K,)$;

(ii) $\Omega^d \subset \Omega$

(iii) If $\varphi(x) = c$ then $\varphi(y) > c$ where either $x \in \bar{\Omega}^d_K$ and $y \in T(x)$ or $x \in \bar{\Omega}_K \setminus \bar{\Omega}^d_K$, $y \in T(x)$ and $\|y\| > d$.

then equation (1.1) has at least four solutions in K where at least three of them are positive solutions.

Where φ is said to be concave function if, for any $x, y \in K$ and $t \in [0,1]$, $\varphi(tx + (1 - t)y) \geq t\varphi(x) + (1-t)\varphi(y)$.

In order to prove Theorem 3.3, we need the following lemma.

Lemma 3.2. Let T be a $u.s.c$ strict set contractive mapping from K to $cf(x)$ such that T has a quasi-derivative T'_a at $a = \theta$ or ∞. And let θ is a isolated solution of equation (1.1) (that means the θ is a solution of equation (1.1) and there exists a neighborhood N of θ such that no solution is in $N \setminus \{\theta\}$). Then

(i) If $T'_a \in F(K,-)$, there exists $\rho > 0$ such that $i_K(T, \Omega^r_K) = 1$;

(ii) If $T'_a \in F(K,+)$, there exists $\rho > 0$ such that $i_K(T, \Omega_K^r) = 0$,

where r satisfies

$$r > \rho\,(a = \infty) \quad or \quad r \in (0, \rho)\,(a = \theta). \tag{3.3}$$

Proof. It follows from Lemma 2.1 that there exists $\rho > 0$ such that $i_K(T, \Omega_K^r) = i_K(T'_a, \Omega_K^r)$ where r satisfies (3.3). From Lemma 2.2, if $T'_a \in F(K,-)$, then $i_K(T, \Omega_K^r) = i_K(T'_a, \Omega_K^r) = 1$; if $T'_a \in F(K,+)$, then $i_K(T, \Omega_K^r) = i_K(T'_a, \Omega_K^r) = 0$. This completes the proof.

The proof of Theorem 3.3.

If θ is not isolated solution of equation (1.1), then the conclusions of Theorem 3.3 is true automatically . If θ is a isolated solution of equation (1.1), from Lemma 3.2, there exists $\rho > 0$ such that

$$i_K(T, \Omega_K^r) = 1 \;\; (a = \infty); \tag{3.4}$$

$$i_K(T, \Omega_K^r) = 0 \;\; (a = \theta). \tag{3.5}$$

where the r satisfies (3.3). Therefore, we may take $R > c > r > 0$ such that $\bar{\Omega}_K^r \subset \Omega_K \subset \bar{\Omega}_K \subset \Omega_K^R$ and $i_K(T, \Omega_K^r) = 1$, $i_K(T, \Omega_K^R) = 0$.

On the other hand, it follows from Lemma 3.1 that $i_K(T, \Omega_K) = 1$. Setting

$$D = \{x \in \Omega_K; \varphi(x) > c\}. \tag{3.6}$$

Clearly, $D \neq \emptyset$ (because $u \in D$) and $\theta \notin D$. In addition, it is easy to show that D is convex open subset with respect to K and $D \cap \Omega^r = \emptyset$.

Define $H : \bar{D} \times I \to cf(K)$ by

$$H(x,t) = (1-t)u + tT(x) \;\; ((x,t) \in \bar{D} \times I). \tag{3.7}$$

Now we prove

$$x \notin H(x,t) \;\; (x \in \partial D_K, t \in I). \tag{3.8}$$

Otherwise, there exist $x_0 \in \partial D_K$ and $t_0 \in I$ such that $x_0 = (1-t_0)u + t_0 T(x_0)$. Let $y_0 \in T(x_0)$ such that $x_0 = (1-t_0)u + t_0 y_0$. If $x_0 \in \partial \Omega_K$ then $x_0 \notin T(x_0)$, thus $t_0 \neq 0, 1$. Setting $m = \frac{1}{t_0} > 1$, we have $m(x_0 - u) = y_0 - u$. Therefore, for any positive homogeneous function $l : X \to [0, +\infty)$ satisfying (2.5), we have $l(y_0 - x_0) = l[(m-1)(x_0 - u)] =$

$l(y_0-u)-l(x_0-u)$. This contradicts with the boundary condition $(MS;\partial\Omega_K,u)$. Hence, $x_0\in\Omega_K$ and so $\varphi(x_0)=c$.

On the other hand, if $x_0\in\bar\Omega^d_K$, then $\varphi(y_0)>c$, thus, $\varphi(x_0)\geq(1-t_0)\varphi(u)+t_0\varphi(y_0)>c$, which is a contradiction; if $x_0\notin\bar\Omega^d_K$, then $y_0\notin\bar\Omega^d_K$, and $\varphi(x_0)>c$. This also is a contradiction. Hence, (3.8) is true. Therefore, $i_K(T,D_K)=i_K(H(\cdot,1),D_K)=i_K(H(\cdot,0),D_K)=1$. It implies that there exists $x_1\in D_K$, which is a solution of equation (1.1).

Putting $A=\Omega^R_K\setminus\bar\Omega_K$, $B=\Omega_K\setminus(\bar\Omega^r_K\cup\bar D_K)$. Obviously, $i_K(T,A)=-1$ and $i_K(T,B)=-1$. Hence, there exist $x_2\in A$ and $x_3\in B$ which are solutions of equation (1.1).

In addition, it follows from $i_K(T,\Omega^r_K)=1$ that there exists $x_4\in\Omega^r_K$, which is solution of equation (1.1). This completes the proof.

Now, we further study the multiple positive solutions problems under other boundary condition.

Let $u\in K^0$, Ω be an open subset of X, $T:K\to cf(X)$.

(i) T is said to satisfy boundary condition $(XC,\partial\Omega_K)$ if, $y\not\geq x$ $(x\in\partial\Omega_K, y\in T(x)\setminus\Omega_K)$.

(ii) T is said to satisfy boundary condition $(QX,\partial\Omega_K,u)$ if, $y\not\leq x$ $(x\in\partial\Omega_K\cap L(u)$, $y\in T(x))$, where $L(u)$ is defined by (2.8).

Lemma 3.3. Let Ω be a bounded convex open neighborhood of θ, $u\in K^0$ and $T:\bar\Omega_K\to cf(K)$ be a $u.s.c$ strict set contractive mapping with $x\notin T(x)$ $(x\in\partial\Omega_K)$.

(i) If T satisfies boundary condition $(XC,\partial\Omega_K)$, then $i_K(T,\Omega_K)=1$;

(ii) If T satisfies boundary condition $(QX,\partial\Omega_K,u)$, then $i_K(T,\Omega_K)=0$.

Proof. If T satisfies boundary condition $(XC,\partial\Omega_K)$, we define $H:\Omega_K\times I\to cf(K)$ by $H(x,t)=tT(x)$. It is easy to see that for each $t\in I$, $H(\cdot,t)$ is a $u.s.c$ strict set contractive mapping. Now we verify that

$$x\notin H(x,t)\ (x\in\partial\Omega_K,t\in I). \tag{3.9}$$

Otherwise, there exist $x_0 \in \partial\Omega_K$ and $t_0 \in I$ such that $x_0 \in t_0T(x_0)$. Thus we may take $y_0 \in T(x_0)$ such that $x_0 = t_0y_0$. Clearly, $t_0 \neq 0, 1$. Setting $m = \frac{1}{t_0} > 1$, we have $y_0 = mx_0 \geq x_0$ and $y_0 \in T(x_0) \setminus \bar{\Omega}_K$. This is contradiction with boundary condition $(XC, \partial\Omega_K)$. It implies that (3.9) is true. Thus, $i_K(T, \Omega_K) = i_K(\hat{\theta}, \Omega_K) = 1$.

If T satisfies boundary condition $(QX, \partial\Omega_K, u)$, we can prove that $x - \beta u \notin T(x)$ for $x \in \partial\Omega_K \cap L(u), \beta > 0$. Otherwise, there exist $x \in \partial\Omega_K \cap L(u), \beta > 0$ such that $x - \beta u \notin T(x)$. Taking $y \in T(x)$ such that $x - \beta u = y$, i.e., $x = \beta u + y > y$. This is contradiction with boundary condition $(QX, \partial\Omega_K, u)$. It follows from Lemma 4 of [7] that $i_K(T, \Omega_K) = 0$. This completes the proof.

Theorem 3.4. Suppose that $T : K \to cf(K)$ is a $u.s.c$ strict set contractive mapping and $\theta \in T(\theta)$ and T has a quasi-derivative T'_a at $a = \theta, \infty$. If one of the following conditions is satisfied

(i) $T'_a \in F(K, -)$ and there exist a bounded convex open neighborhood Ω of θ and $u \in K^0$ such that T satisfies boundary condition $(QX, \partial\Omega_K, u)$

(ii) $T'_a \in F(K, +)$ and there exists a bounded convex open neighborhood Ω of θ such that T satisfies boundary condition $(XC, \partial\Omega_K)$.

then the equation (1.1) has positive solutions.

Proof. Without loss of generality, we assume that θ is isolated solution of equation (1.1) and $x \in \partial\Omega_K$ such that $x \notin T(x)$.

If the condition (i) is satisfied, it follows from Lemma 3.2 that there exists $\rho > 0$ such that $i_K(T, \Omega^r) = i_K(T'_a, \Omega^r) = 1$ as r satisfies (3.3), where $a = \theta$ or ∞. We may take $r > 0$ such that either $\overline{\Omega}^r_K \subset \Omega_K$ as $a = \theta$ or $\overline{\Omega}_K \subset \Omega^r_K$ as $a = \infty$ and $i_K(T, \Omega^r_K) = 1$. On the other hand, it follows from (ii) of Lemma 3.3 that $i_K(T, \Omega_K) = 0$. Letting $A = \Omega_K \setminus \overline{\Omega}^r_K$ as $a = \theta$ or $A = \Omega^r_K \setminus \overline{\Omega}_K$ as $a = \infty$. We have

$$i_K(T, A) = \{ \begin{array}{ll} -1 & (a = \theta) \\ 1 & (a = \infty) \end{array}$$

This implies that the equation (1.1) has positive solutions in A.

Similarly, it can be proved that there exist positive solutions of equation (1.1) when the condition (ii) is satisfied. The proof is completed.

Theorem 3.5. Suppose that $T : K \to cf(K)$ is a $u.s.c$ strict set contractive mapping such that $\theta \in T(\theta)$ and T has a quasi-derivative T'_a at $a = \theta, \infty$. If there exists a bounded convex open neighborhood Ω of θ such that $x \notin T(x))$ as $x \in \partial\Omega_K$ and one of the following conditions is satisfied

(i) $T'_a \in F(K, -)$ and there exists $u \in K^0$ such that T satisfies boundary condition $(QX, \partial\Omega_K, u)$

(ii) $T'_a \in F(K, +)$ and T satisfies boundary condition $(XC, \partial\Omega_K)$.

then the equation (1.1) has at least two positive solutions.

Proof. Without loss of generality, we assume that θ is isolated solution of equation (1.1) in K. If condition (i) is satisfied, by Lemma 2.1, we may take $R > r > 0$ such that $\bar{\Omega}^r_K \subset \Omega_K \subset \bar{\Omega}_K \subset \Omega^R_K$ and $i_K(T, \Omega^r_K) = i_K(T, \Omega^R_K) = 1$. On the other hand, it follows from Lemma 3.3 that $i_K(T, \Omega_K) = 0$. This implies that $i_K(T, \Omega_K \setminus \bar{\Omega}^r_K) = -1$ and $i_K(T, \Omega^R_K \setminus \bar{\Omega}) = 1$. Therefore, there exist $x_1 \in \Omega_K \setminus \bar{\Omega}^r_K$ and $x_2 \in \Omega^R_K \setminus \bar{\Omega}$ which are two positive solutions of equation (1.1). Similarly, we can prove that there exist two positive solutions of equation (1.1) when the condition (ii) is satisfied. The proof is completed.

Theorem 3.6. Let $R > r > 0$ and $T : \Omega^R_K \to cf(K)$ be a $u.s.c$ strict set contractive mapping which satisfies boundary conditions $(XC, \partial\Omega^R_K)$ and $(XC, \partial\Omega^r_K)$. If there exist $u \in \bar{\Omega}^R_K$, $c > 0$ and a concave continuous function $\varphi : K \to R^+$ with $\varphi(u) > 0$, and the following conditions are satisfied

(i) When $x \in \bar{\Omega}^R_K$ with $\varphi(x) \geq c$, $\varphi(y) > c$ for any $y \in T(x)$.

(ii) $\varphi(x) \leq c$ as $x \in \bar{\Omega}^r_K$

(iii) $E_V(T, \partial\Omega^R_K, u, +1) = \emptyset$.

then equation (1.1) has at least two positive solutions in $\bar{\Omega}^R_K \setminus \Omega^r_K$ and if equation (1.1) does not have solutions in $\partial\Omega^r_K$, then equation (1.1) has at least three solutions in $\bar{\Omega}^R_K$ where at least two of them are positive solutions.

Proof. 1. Denote $B = \bar{\Omega}^R$. Define $F_u : K \to cf(K)$ by $F_u(x) = T(\hat{B}_u(x))$, where $\hat{B}_u$ is radial retraction mapping of B with respected to u, i.e.,

$$\hat{B}_u(x) = \{ \begin{array}{ll} x & x \in B \\ h^{-1}(x-u)x + (1 - h^{-1}(x-u))u & x \notin B \end{array}$$

where h is Minkowskii function of $B - u$. Clearly, F_u is a $u.s.c$ strictly set contractive mapping and $F_u(x) = T(x)$ as $x \in B_K$.

Now we show that F_u does not have fixed points in $K \setminus B$ when $EV(F_u, \partial B, u, 1+) = \emptyset$. Otherwise, there exists $x_0 \in K \setminus B$ such that $x_0 \in F_u(x_0) = T(\hat{B}_u(x_0))$. Setting $\bar{x} = \hat{B}_u(x_0) = \lambda_0 x_0 + (1 - \lambda_0)u$ ($\lambda_0 = h^{-1}(x_0 - u) < 1$). Thus, $\bar{x} \in \partial B_K$ and $\bar{x} = \hat{B}_u(\bar{x})$, and so $F_u(\bar{x}) = T(\bar{x})$. On the other hand, since $x_0 = \frac{1}{\lambda_0}[\bar{x} + (\lambda_0 - 1)u]$, $\frac{1}{\lambda_0}(\bar{x} - u) \in F_u(\bar{x}) - u$. This is a contradiction with $E_V(F_u, \partial B, u, 1+) = \emptyset$.

2. Taking $l > max\{R, S\}$, where $S = Sup\{\|y\|; x \in B_K, y \in T(x)\}$. Suppose that F is a restriction of F_u on $\bar{\Omega}^l_K$. We can prove that $F : \bar{\Omega}^l_K \to \bar{\Omega}^l_K$ is a $u.s.c$ strictly set contractive mapping such that $x \notin F(x)$ as $x \in \partial\Omega^l_K$. In fact, it is easy to see that $F : \bar{\Omega}^l_K \to \bar{\Omega}^l_K$ is a $u.s.c$ strictly set contractive mapping. If there exists $x \in \partial\Omega^l_K$ such that $x \in F(x) = T(\bar{x})$ where $\bar{x} = \hat{B}_u(x) = h^{-1}(x-u)x + (1-h^{-1}(x-u))u$, then $\bar{x} \in \partial B_K = \partial\Omega^R_K$ and $h(x-u)(\bar{x}-u) \in T(\bar{x}) - u$. Taking $y \in T(\bar{x})$ such that $h(x-u)(\bar{x}-u) = y - u$, then $y \notin \bar{\Omega}^R_K$ and $y - u > \bar{x} - u$. This contradicts with the condition that T satisfies condition $(XC, \partial\Omega^R_K)$. It follows that $i_K(F, \Omega^l_K) = 1$.

3. Denote $A = \{x \in \bar{\Omega}^l_K; \varphi(x) > c\}$. It follows from $u \in A$ that $A \neq \emptyset$. Obviously, A is an open subset with respect to K and $A \cap \bar{\Omega}^r_K = \emptyset$. Define $H : A \times I \to cf(K)$ by $H(x,t) = tu + (1-t)F(x)$. It is easy to prove that $x \notin H(x,t)$ as $x \in \partial A, t \in I$. Otherwise, there exist $x_0 \in \partial A$, $y_0 \in F(x_0)$ and $t_0 \in I$ such that $x_0 = t_0 u + (1 - t_0)y_0$. This implies that $\varphi(x_0) \geq t_0\varphi(u) + (1-t_0)\varphi(y_0) > c$. It contradicts with $\varphi(x_0) = c$. Thus, $i_k(F, A) = 1$, and so there exists $x_1 \in A$ such that $x_1 \in F(x_1)$.

4. If equation (1.1) has solution $x_1' \in \partial\Omega^r_K$, then x_1 and x_1' are positive fixed points of F, thus the conclusions of Theorem are true. Now we assume that equation (1.1) does not have any solutions in $\partial\Omega^r$. Thus, we have $i_K(F, A) = 1$, and so there exists $x_2 \in \Omega^r_K$ such that $x_2 \in T(x_2)$.

5. Denote $U = \Omega_K^l \setminus (\bar{\Omega}_K^r \cup \bar{A})$. Since $i_K(T, \Omega_K^r) = 1$ and $A \cap \Omega_K^r = \emptyset$, we have $i_K(F, U) = -1$. Hence, there exists $x_3 \in U$ such that $x_3 \in T(x_3)$. It follows from $E_V(F, \partial B, u.1+) = \emptyset$ that $x_1, x_3 \in B_K$.

In conclusion, x_1, x_2 and x_3 are solutions of equation(1.1), where at least x_1 and x_3 are positive solutions. This completes the proof.

Theorem 3.7. Let $T : K \to cf(K)$ be a $u.s.c$ strict set contractive mapping such that T has quasi-derivative T'_a at $a = \theta, \infty$, $T'_\theta \in F(K, -), T'_\infty \in F(K, +)$ and $\theta \in T(\theta)$. Suppose that there exist $d > c > 0$ and a bounded convex open neighborhood Ω of θ and $u \in \Omega^d \setminus \Omega$. If T satisfies boundary conditions $(XC, \partial\Omega_K)$, equation (1.1) does not have any solutions on $\partial\Omega_K$ and there exists a concave continuous function $\varphi : K \to R^+$ such that the following conditions are satisfied

(i) $\varphi(u) > c$ and $\varphi(x) \leq \|x\|$ as $x \in \Omega_K$;

(ii) $\varphi(y) > c$ as $x \in \Phi_1 \cup \Phi_2, y \in T(x)$ and $\|y\| > d$,

where $\Phi_1 = \{x \in \bar{\Omega}_K^d; \varphi(x) = c\}$ and $\Phi_2 = \{x \in \bar{\Omega}_K \setminus \Omega_K^d; \varphi(x) = c\}$, respectively. Then equation (1.1) has at least four solutions in K, where at least three of them are positive solutions.

Proof. Without loss generality, we assume that θ is a isolated solution of equation (1.1) in K.

1. By Lemma 3.2, there exists $\rho > 0$ such that

$$i_K(T, \Omega_K^r) = \{ \begin{array}{l} 0 \ \ (a = \infty) \\ 1 \ \ (a = \theta) \end{array}$$

where r satisfies (3.3). Thus, we may take $R > c > r > 0$ such that $\bar{\Omega}^r \subset \Omega \subset \bar{\Omega} \subset \Omega^R$ and $i_K(T, \Omega_K^r) = 1, i_K(T, \Omega_K^R) = 0$. Hence, there exists $x_0 \in \Omega_K^r$ such that it is a solution of equation (1.1)

2. By Lemma 3.3, we have $i_K(T, \Omega_K) = 1$. Denote $D = \{x \in \Omega_K; \varphi(x) > c\}$. It is easy to see that D is nonempty convex subset of K and $\theta \notin D$. Define $H : \bar{D} \times I \to cf(K)$ as follows

$$H(x, t) = (1 - t)u + tT(x)$$

Similarly, we can prove (3.8) holds. Therefore, $i_K(T, D) = 1$, and equation (1.1) has a solution $x_1 \in D$.

3. Denote $A = \Omega^R \setminus \bar{\Omega}K$ and $B = \Omega_K \setminus (\bar{\Omega}_K^r \cup D)$. It is easy to see that $i_K(T, A) = -1$ and $i_K(T, B) = -1$. Therefore, equation (1.1) has two solutions $x_2 \in A$ and $x_3 \in B$.

In conclusion, x_0, x_1, x_2, x_3 are four solutions of equation (1.1) where at least x_1, x_2 and x_3 are positive solutions. The proof is completed.

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