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FOUR NEW OPERATIONS OF GRAPHS AND ZAGREB INDICES

J. BURAGOHAIN[1], A. MAHANTA[2], A. BHARALI[3,*]

[1]Department of Mathematics, Pragjyotish College, Guwahati 781009, India
[2]Department of Mathematics, Dergaon Kamal Dowerah College, Golaghat 785614, India
[3]Department of Mathematics, Dibrugarh University, Dibrugarh 786004, India

Abstract. Indulal and Balakrishnan (2016) have put forward a new operation of graphs called Indu-Bala product. In this paper we propose four new operations of graphs based on Indu-Bala product of graphs. We also establish explicit formulas for Zagreb indices of the four newly proposed operations of graphs.

Keywords: degree of vertex; Zagreb indices; operations of graphs; Indu-Bala product.

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1. Introduction

Let G be a graph and the set of vertices and edges of G be denoted as $V(G)$ and $E(G)$ respectively. Throughout the paper we consider only simple finite graphs. The degree of a vertex $u \in V(G)$ is denoted by $d_G(u)$, if their is no confusion we simply write it as $d(u)$. Two vertices u and v are called adjacent if there is an edge connecting them. The connecting edge is usually denoted by uv. Any unexplained graph theoretic notions and symbols may be found in [15].

*Corresponding author
E-mail address: a.bharali@dibru.ac.in
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Topological indices are the numerical values which are associated with a graph structure. These graph invariants are utilized for modeling information of molecules in structural chemistry and biology. Over the years many topological indices have been proposed and studied based on degree, distance and other parameters of graph. Some of them may be found in [6, 8]. Historically Zagreb indices can be considered as the first degree-based topological indices, which came into picture during the study of total π-electron energy of alternant hydrocarbons by *Gutman* and *Trinajstić* in 1972 [10]. But these indices are recognized as topological indices much later (almost after 30 years, due to their completely different purpose of utility). Since these indices were coined, various studies related to different aspects of these indices are reported; for detail see the papers [5, 3, 9, 12, 17] and the references therein.

The first and second Zagreb indices of a graph G are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)), \ \ M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The concept of the first general Zagreb index of a graph was introduced by Li and Zheng in [16]. For a graph G this index is defined as

$$M_1^{\alpha}(G) = \sum_{v \in V(G)} d(v)^{\alpha}, \tag{1.1}$$

where α is an arbitrary real number and $\alpha \neq 0, 1$. Clearly, for $\alpha = 2$, $M_1^{\alpha}(G)$ is the first Zagreb index i.e., $M_1^2(G) = M_1(G)$. The above equation (1.1) can also be written as

$$M_1^{\alpha}(G) = \sum_{uv \in E(G)} (d(u)^{\alpha-1} + d(v)^{\alpha-1}).$$

In a molecular graph we consider atoms as vertices and the bonds between them as edges. But the intermolecular forces do not only exist between the atoms of a molecule but also between the atoms and bonds, so one should also take into account the relations (forces) between the edges and vertices in addition to the relations between vertices. The four related operations of a graph G viz., $S(G)$, $R(G)$, $Q(G)$ and $T(G)$ can capture these relations. For a connected graph G, the four related graphs are as follows:

- $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.

- $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.
- $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.
- $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

More detail on these operations may be found in [2]. Different operations of graphs based on these four related graphs were defined and studied in connection with Wiener indices [7] and Zagreb indices [4, 5, 13, 14]. In this paper we propose four new operations of graphs based on Indu-Bala product of graphs [11] and also establish explicit expressions for the Zagreb indices of these newly defined graph products. The study of Zagreb indices of Indu-Bala products of graphs are also found in the literature [1]. The rest of the paper is organized as follows. In section 2 we define the four new operations of graphs. In section 3 expressions for the Zagreb indices of the four new graph products are presented.

2. THE NEW F-SUMS OF GRAPHS

Here, we introduce four new operations of graphs based on the Indu-Bala product of two connected graphs G_1 and G_2: The join $G = G_1 + G_2$ of graphs G_1 and G_2 is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. Let $F = \{S,R,Q,T\}$. The F-sum of G_1 and G_2, denoted by $G_1 \nabla_F G_2$, is defined by $F(G_1)\nabla G_2$, where ∇ is the Indu-Bala product of graphs. The Indu-Bala product $G_1 \nabla G_2$ of graphs G_1 and G_2 is obtained from two disjoint copies of the join $G_1 + G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2. So $V(G_1\nabla_F G_2) = V(G_1) \cup E(G_1) \cup V(G_2)$ and $E(G_1\nabla_F G_2) = E(F(G_1)\nabla G_2) \setminus E^*$, where $E^* = \{uv | u \in V(F(G_1)) \setminus V(G_1), v \in V(G_2)\}$.

As an example, $P_3\nabla_S P_4$, $P_3\nabla_R P_4$, $P_3\nabla_Q P_4$ and $P_3\nabla_T P_4$ are shown in figure 1.

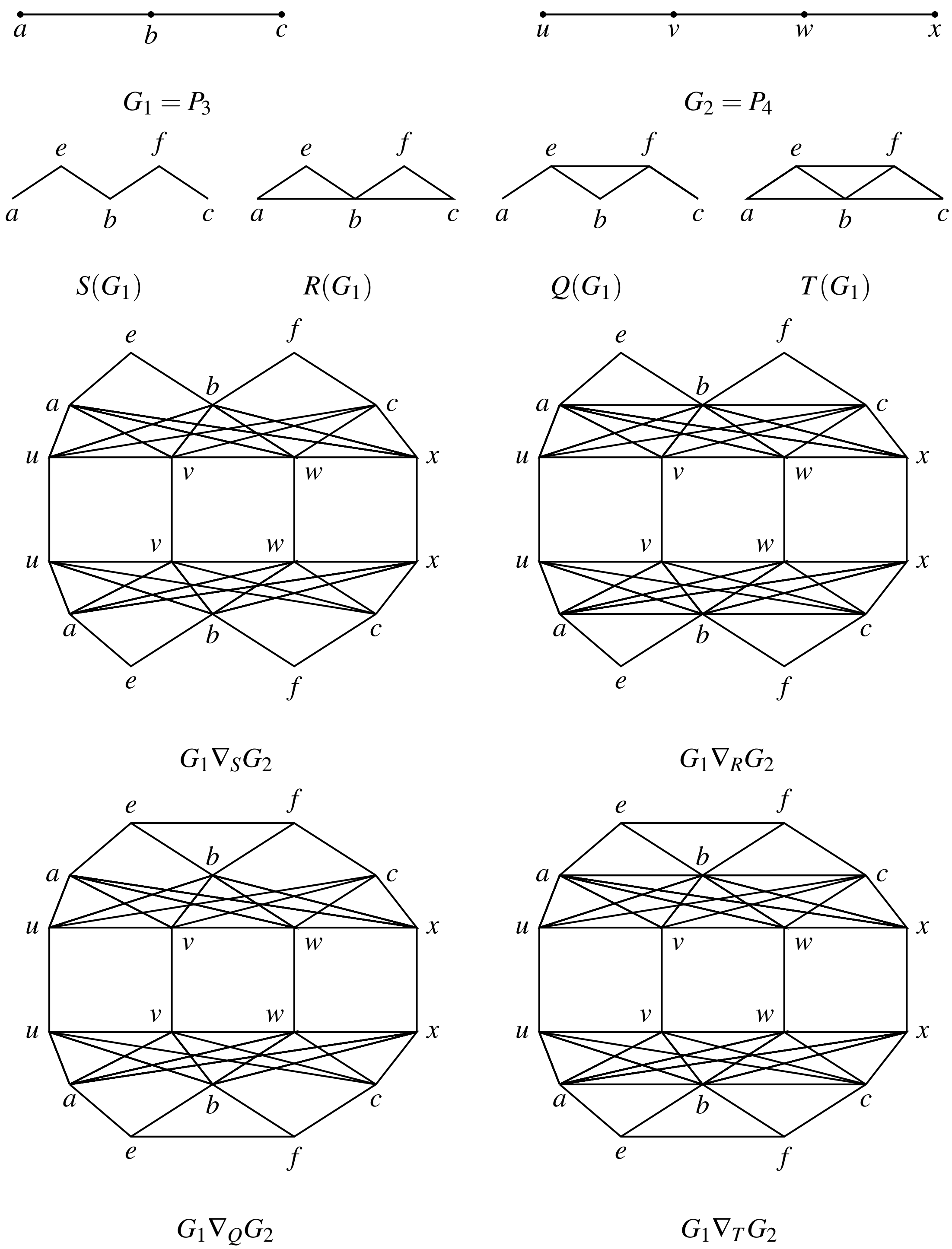

FIGURE 1. The four new operations of graphs based on Indu-Bala product

Lemma 2.1. *Let G_1, G_2 be two graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, where $i = 1,2$. Then,*

(a)

$$d_{G_1\nabla_S G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 \quad if \quad u \in V(G_1) \\ 2 \quad if \quad u \in V(S(G_1)) \setminus V(G_1) \\ d_{G_2}(u) + n_1 + 1 \quad if \quad u \in V(G_2). \end{cases}$$

(b)

$$d_{G_1\nabla_R G_2}(u) = \begin{cases} 2d_{G_1}(u) + n_2 \quad if \quad u \in V(G_1) \\ 2 \quad if \quad u \in V(R(G_1)) \setminus V(G_1) \\ d_{G_2}(u) + n_1 + 1 \quad if \quad u \in V(G_2). \end{cases}$$

(c)

$$d_{G_1\nabla_Q G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 \quad if \quad u \in V(G_1) \\ d_{G_1}(w) + d_{G_1}(w') \quad if \quad u \in V(Q(G_1)) \setminus V(G_1) \\ d_{G_2}(u) + n_1 + 1 \quad if \quad u \in V(G_2), \end{cases}$$

where in the second case u is inserted into the edge $ww' \in E(G_1)$.

(d)

$$d_{G_1\nabla_T G_2}(u) = \begin{cases} 2d_{G_1}(u) + n_2 \quad if \quad u \in V(G_1) \\ d_{G_1}(w) + d_{G_1}(w') \quad if \quad u \in V(T(G_1)) \setminus V(G_1) \\ d_{G_2}(u) + n_1 + 1 \quad if \quad u \in V(G_2), \end{cases}$$

where in the second case u is inserted into the edge $ww' \in E(G_1)$.

3. Main Results

In this section, we put forward the first and second Zagreb index of the new F-sums of graphs.

3.1. Zagreb indices of $G_1\nabla_S G_2$.

Theorem 3.1. *Let G_1 and G_2 be two graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$ where $i = 1,2$. Then*

$$M_1(G_1 \nabla_S G_2) = 2\Big[M_1(G_1) + M_1(G_2) + n_1 n_2(n_1 + n_2 + 2) + 4(m_1 + m_2) + 4n_1 m_2 + n_2\Big].$$

Proof:

$$M_1(G_1 \nabla_S G_2) = \sum_{u \in V(G_1 \nabla_S G_2)} d^2(u)$$

$$= 2\Big[\sum_{u \in V(G_1)} d^2(u) + \sum_{u \in V(G_2)} d^2(u) + \sum_{u \in V(S(G_1)) \setminus V(G_1)} d^2(u)\Big] \tag{3.2}$$

Case I:

$$\sum_{u \in V(G_1)} d^2(u) = \sum_{u \in V(G_1)} (d_{G_1}(u) + n_2)^2$$

$$= \sum_{u \in V(G_1)} \Big[d^2_{G_1}(u) + 2n_2 d_{G_1}(u) + n_2^2\Big]$$

$$= M_1(G_1) + 4n_2 m_1 + n_1 n_2^2 \tag{3.3}$$

Case II:

$$\sum_{u \in V(G_2)} d^2(u) = \sum_{u \in V(G_2)} (d_{G_2}(u) + n_1 + 1)^2$$

$$= \sum_{u \in V(G_2)} \big[d^2_{G_2}(u) + (n_1 + 1)^2 + 2(n_1 + 1) d_{G_2}(u)\big]$$

$$= M_1(G_2) + n_2(n_1 + 1)^2 + 4m_2(n_1 + 1) \tag{3.4}$$

Case III:

$$\sum_{u \in V(S(G_1)) \setminus V(G_1)} d^2(u) = \sum_{u \in V(S(G_1)) \setminus V(G_1)} (2)^2 = 4m_1 \tag{3.5}$$

Now, putting (3.3), (3.4) and (3.5) in (3.2) we get the result. □

Theorem 3.2. *Let G_1 and G_2 be two graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, where $i = 1,2$. Then*

$$\begin{aligned} M_2(G_1\nabla_S G_2) =& M_1(G_2)+(n_1+1)\{n_2(n_1+1)+4m_2\}+2[2M_1(G_1)+(n_1+1)M_1(G_2) \\ &+M_2(G_2)+m_2(n_1+1)^2+n_2(n_1+1)(2m_1+n_1n_2) \\ &+2m_2(2m_1+n_1n_2)+4m_1n_2]. \end{aligned}$$

Proof:

$$M_2(G_1\nabla_S G_2) = \sum_{uv\in E(G_1\nabla_S G_2)} d(u)d(v)$$

The edges of $G_1\nabla_S G_2$ can be classified into four categories depending on the end vertices of an edge.

Case I: $uv \in E(G_1\nabla_S G_2)$ *s.t.* uv is an edge in G_2.

$$\begin{aligned} \sum_{uv\in E(G_2)} d(u)d(v) &= \sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1+1)(d_{G_2}(v)+n_1+1) \\ &= \sum_{uv\in E(G_2)} \Big[d_{G_2}(u)d_{G_2}(v)+(n_1+1)(d_{G_2}(u)+d_{G_2}(v)) \\ &\quad +(n_1+1)^2\Big] \\ &= M_2(G_2)+(n_1+1)M_1(G_2)+m_2(n_1+1)^2 \end{aligned} \tag{3.6}$$

Case II: $uv \in E(G_1\nabla_S G_2)$ *s.t.* $u \in V(G_2)$ *and* $v \in V(G_1)$.

$$\begin{aligned} \sum_{uv\in E(G_1\nabla_S G_2)} d(u)d(v) &= \sum_{uv\in E(G_1\nabla_S G_2)} (d_{G_2}(u)+n_1+1)(d_{G_1}(v)+n_2) \\ &= \sum_{uv\in E(G_1\nabla_S G_2)} \Big[d_{G_2}(u)d_{G_1}(v)+n_2d_{G_2}(u)+(n_1+1)d_{G_1}(v) \\ &\quad +n_2(n_1+1)\Big] \\ &= \sum_{u\in V(G_2)}\sum_{v\in V(G_1)} d_{G_2}(u)d_{G_1}(v)+n_2(n_12m_2)+(n_1+1)(n_22m_1) \\ &\quad +n_2(n_1+1)(n_1n_2) \\ &= 4m_1m_2+2n_1n_2m_2+(n_1+1)(n_22m_1)+n_1n_2^2(n_1+1) \end{aligned} \tag{3.7}$$

Case III: *$uv \in E(G_1 \nabla_S G_2)$ s.t. uv is an edge in $S(G_1)$.*

$$\begin{aligned}\sum_{uv \in E(S(G_1))} d(u)d(v) &= \sum_{uv \in E(S(G_1))} (d_{G_1}(u) + n_2)(2) \\ &= 2\Big[\sum_{uv \in E(S(G_1))} d_{G_1}(u) + \sum_{uv \in E(S(G_1))} n_2\Big] \\ &= 2\Big[\sum_{u \in V(G_1)} d^2_{G_1}(u) + 2m_1n_2\Big] \\ &= 2\Big[M_1(G_1) + 2m_1n_2\Big] \qquad (3.8)\end{aligned}$$

Case IV: *$uu \in E(G_1 \nabla_S G_2)$ s.t. $u \in V(G_2)$.*

$$\begin{aligned}\sum_{uu \in E(G_1 \nabla_S G_2)} d(u)d(u) &= \sum_{uu \in E(G_1 \nabla_S G_2)} (d_{G_2}(u) + n_1 + 1)^2 \\ &= \sum_{u \in V(G_2)} \Big[d^2_{G_2}(u) + 2d_{G_2}(u)(n_1+1) + (n_1+1)^2\Big] \\ &= M_1(G_2) + 4m_2(n_1+1) + n_2(n_1+1)^2 \qquad (3.9)\end{aligned}$$

From the graph $G_1 \nabla_S G_2$, it is clear that we have to consider the Cases I-III two times and Case IV only once. Now combining all the five cases we get the result. □

3.2. Zagreb indices of $G_1 \nabla_R G_2$.

Theorem 3.3. *Let G_1 and G_2 be two graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, where $i = 1, 2$. Then*

$$\begin{aligned}M_1(G_1 \nabla_R G_2) =& 2\Big[4M_1(G_1) + M_1(G_2) + n_1n_2(n_1+n_2+2) \\ &+ 4m_2(n_1+1) + n_2(8m_1+1) + 4m_1\Big].\end{aligned}$$

Proof:

$$\begin{aligned}M_1(G_1 \nabla_R G_2) &= \sum_{u \in V(G_1 \nabla_R G_2)} d^2(u) \\ &= 2\Big[\sum_{u \in V(G_1)} d^2(u) + \sum_{u \in V(G_2)} d^2(u) + \sum_{u \in V(R(G_1)) \setminus V(G_1)} d^2(u)\Big]. \qquad (3.10)\end{aligned}$$

Case I:

$$\sum_{u\in V(G_1)} d^2(u) = \sum_{u\in V(G_1)} (2d_{G_1}(u)+n_2)^2$$
$$= \sum_{u\in V(G_1)} \Big[4d^2_{G_1}(u)+4n_2 d_{G_1}(u)+n_2^2\Big]$$
$$=4M_1(G_1)+8n_2m_1+n_1n_2^2 \tag{3.11}$$

Case II:

$$\sum_{u\in V(G_2)} d^2(u) = \sum_{u\in V(G_2)} (d_{G_2}(u)+n_1+1)^2$$
$$=M_1(G_2)+n_2(n_1+1)^2+4m_2(n_1+1) \tag{3.12}$$

Case III:

$$\sum_{u\in V(S(G_1))\setminus V(G_1)} d^2(u) = \sum_{u\in V(S(G_1))\setminus V(G_1)} (2)^2 = 4m_1 \tag{3.13}$$

Now, putting (3.11), (3.12) and (3.13) in (3.10) we get the result. □

Theorem 3.4. *Let G_1 and G_2 be two graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, where $i = 1,2$. Then*

$$\begin{aligned}M_2(G_1\nabla_R G_2) =& M_1(G_2)+4m_2(n_1+1)+n_2(n_1+1)^2+2\Big[2(n_2+2)M_1(G_1)\\ &+(n_1+1)M_1(G_2)+4M_2(G_1)+M_2(G_2)+m_2(n_1+1)^2\\ &+n_2(n_1+1)(4m_1+n_1n_2)+2m_2(4m_1+n_1n_2)+4m_1n_2+m_1n_2^2\Big].\end{aligned}$$

Proof:

$$M_2(G_1\nabla_R G_2) = \sum_{uv\in E(G_1\nabla_R G_2)} d(u)d(v)$$

There are five types of edges in $G_1\nabla_R G_2$, depending on the end vertices of an edge.

Case I: $uv \in E(G_1\nabla_R G_2)$ *s.t.* uv is an edge in G_2.

$$\sum_{uv\in E(G_2)} d(u)d(v) = \sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1+1)(d_{G_2}(v)+n_1+1)$$
$$=M_2(G_2)+(n_1+1)M_1(G_2)+m_2(n_1+1)^2 \tag{3.14}$$

Case II: $uv \in E(G_1\nabla_R G_2)$ $s.t.$ $u \in V(G_2)$ and $v \in V(G_1)$.

$$\begin{aligned}\sum_{uv\in E(G_1\nabla_R G_2)} d(u)d(v) &= \sum_{uv\in E(G_1\nabla_R G_2)} (d_{G_2}(u)+n_1+1)(2d_{G_1}(v)+n_2)\\
&= \sum_{uv\in E(G_1\nabla_R G_2)} \Big[2d_{G_2}(u)d_{G_1}(v)+n_2 d_{G_2}(u)+2(n_1+1)d_{G_1}(v)\\
&\quad +n_2(n_1+1)\Big]\\
&=2(2m_2)(2m_1)+n_2(n_1 2m_2)+2(n_1+1)(n_2 2m_1)\\
&\quad +n_2(n_1+1)(n_1 n_2)\\
&=8m_1m_2+2n_1n_2m_2+4n_2m_1(n_1+1)+n_1n_2^2(n_1+1) \qquad (3.15)\end{aligned}$$

Case III: $uv \in E(G_1\nabla_R G_2)$ $s.t.$ $u \in V(G_1)$ and $v \in V(R(G_1))\backslash V(G_1)$.

$$\begin{aligned}\sum_{uv\in E(R(G_1))\backslash E(G_1)} d(u)d(v) &= \sum_{uv\in E(R(G_1))\backslash E(G_1)} (2d_{G_1}(u)+n_2)(2)\\
&=2\Big[2\sum_{uv\in E(R(G_1))\backslash E(G_1)} d_{G_1}(u)+\sum_{uv\in E(R(G_1))\backslash E(G_1)} n_2\Big]\\
&=2\Big[2\sum_{u\in E(G_1)} d^2_{G_1}(u)+n_2(2m_1)\Big]\\
&=2(2M_1(G_1)+2m_1n_2) \qquad (3.16)\end{aligned}$$

Case IV: $uv \in E(G_1\nabla_R G_2)$ $s.t.$ uv is an edge in G_1.

$$\begin{aligned}\sum_{uv\in E(G_1)} d(u)d(v) &= \sum_{uv\in E(G_1)} (2d_{G_1}(u)+n_2)(2d_{G_1}(v)+n_2)\\
&=4\sum_{uv\in E(G_1)} d_{G_1}(u)d_{G_1}(v)+2n_2\sum_{uv\in E(G_1)}(d_{G_1}(u)+d_{G_1}(v))+\sum_{uv\in E(G_1)} n_2^2\\
&=4M_2(G_1)+2n_2M_1(G_1)+m_1n_2^2 \qquad (3.17)\end{aligned}$$

Case V: $uu \in E(G_1\nabla_R G_2)$ $s.t.$ $u \in V(G_2)$.

$$\begin{aligned}\sum_{uu\in E(G_1\nabla_R G_2)} d(u)d(u) &= \sum_{uu\in E(G_1\nabla_R G_2)} (d_{G_2}(u)+n_1+1)^2\\
&=M_1(G_2)+4m_2(n_1+1)+n_2(n_1+1)^2 \qquad (3.18)\end{aligned}$$

We have to consider the Cases I-IV two times and Case V only once. Now, combining all the six cases we get the result. □

3.3. Zagreb indices of $G_1\nabla_Q G_2$.

Theorem 3.5. *Let G_1 and G_2 be two graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, where $i = 1,2$. Then*

$$M_1(G_1\nabla_Q G_2) = 2\Big[M_1(G_1) + M_1(G_2) + 2M_2(G_1) + M_1^3(G_1) + n_1 n_2(n_1+n_2+2) + n_2(4m_1+1) + 4m_2(n_1+1)\Big].$$

Proof:

$$M_1(G_1\nabla_Q G_2) = 2\Big[\sum_{u\in V(G_1\nabla_Q G_2)} d^2(u)\Big]$$

$$= 2\Big[\sum_{u\in V(G_1)} d^2(u) + \sum_{u\in V(G_2)} d^2(u) + \sum_{u\in V(Q(G_1))\setminus V(G_1)} d^2(u)\Big] \tag{3.19}$$

Case I:

$$\sum_{u\in V(G_1)} d^2(u) = \sum_{u\in V(G_1)} (d_{G_1}(u) + n_2)^2$$

$$= M_1(G_1) + 4n_2 m_1 + n_1 n_2^2 \tag{3.20}$$

Case II:

$$\sum_{u\in V(G_2)} d^2(u) = \sum_{u\in V(G_2)} (d_{G_2}(u) + n_1 + 1)^2$$

$$= M_1(G_2) + n_2(n_1+1)^2 + 4m_2(n_1+1) \tag{3.21}$$

Case III: Let u be inserted in $ww' \in E(G_1)$

$$\sum_{u\in V(Q(G_1))\setminus V(G_1)} d^2(u) = \sum_{ww'\in E(G_1)} (d_{G_1}(w) + d_{G_1}(w'))^2$$

$$= \sum_{ww'\in E(G_1)} (d^2_{G_1}(w) + d^2_{G_1}(w')) + 2\sum_{ww'\in E(G_1)} d_{G_1}(w) d_{G_1}(w')$$

$$= M_1^3(G_1) + 2M_2(G_1) \tag{3.22}$$

Now, putting (3.20), (3.21) and (3.22) in (3.19) we get the result. □

Theorem 3.6. *Let G_1 and G_2 be two graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, where $i = 1,2$. Then*

$$\begin{aligned} M_2(G_1\nabla_Q G_2) =& M_1(G_2) + 4m_2(n_1+1) + n_2(n_1+1)^2 + 2\big[2n_2M_1(G_1) + (n_1+1)M_1(G_2) \\ &+ M_2(G_2) + \frac{1}{2}(M_1^3(G_1) + M_1^4(G_1)) + (n_1+1)\{m_2(n_1+1) \\ &+ n_2(2m_1+n_1n_2)\} + 2m_2(2m_1+n_1n_2) + \sum_{u,w\in V(G_1)} \gamma_{uw} d_{G_1}(u) d_{G_1}(w) \\ &+ \sum_{u\in V(G_1)} d_{G_1}^2(u) \sum_{\substack{v\in V(G_1),\\ uv\in E(G_1)}} d_{G_1}(v)\big], \end{aligned}$$

where γ_{uw} is the number of common neighbors of $u,w \in V(G_1)$.

Proof:

$$M_2(G_1\nabla_Q G_2) = \sum_{uv\in E(G_1\nabla_Q G_2)} d(u)d(v)$$

Case I: $uu \in E(G_1\nabla_Q G_2)$ *s.t.* $u \in G_2$

$$\begin{aligned} \sum_{uu\in E(G_1\nabla_Q G_2)} d(u)d(u) =& \sum_{uu\in E(G_1\nabla_Q G_2)} (d_{G_2}(u) + n_1 + 1)^2 \\ =& M_1(G_2) + 4m_2(n_1+1) + n_2(n_1+1)^2 \end{aligned} \tag{3.23}$$

Case II: $uv \in E(G_1\nabla_Q G_2)$ *s.t.* uv is an edge in G_2.

$$\begin{aligned} \sum_{uv\in E(G_1\nabla_Q G_2)} d(u)d(v) =& \sum_{uv\in E(G_2)} (d_{G_2}(u) + n_1 + 1)(d_{G_2}(v) + n_1 + 1) \\ =& M_2(G_2) + (n_1+1)M_1(G_2) + m_2(n_1+1)^2 \end{aligned} \tag{3.24}$$

Case III: $uv \in E(G_1\nabla_Q G_2)$ *s.t.* $u \in V(G_2)$ and $v \in V(G_1)$.

$$\begin{aligned} \sum_{uv\in E(G_1\nabla_Q G_2)} d(u)d(v) =& \sum_{uv\in E(G_1\nabla_Q G_2)} (d_{G_2}(u) + n_1 + 1)(d_{G_1}(v) + n_2) \\ =& 4m_1m_2 + 2n_1n_2m_2 + 2n_2m_1(n_1+1) + n_1n_2^2(n_1+1) \end{aligned} \tag{3.25}$$

Case IV: $ue \in E(G_1\nabla_Q G_2)$ $s.t.$ $u \in V(G_1)$ and $e \in V(Q(G_1))\backslash V(G_1)$. Let e be inserted in $uv \in E(G_1)$.

$$\begin{aligned}
\sum_{ue\in E(G_1\nabla_Q G_2)} d(u)d(e) &= \sum_{uv\in E(G_1)} \Big[(d_{G_1}(u)+n_2)(d_{G_1}(u)+d_{G_1}(v)) \\
&\quad + (d_{G_1}(u)+d_{G_1}(v))(d_{G_1}(v)+n_2)\Big] \\
&= \sum_{uv\in E(G_1)} \Big[(d_{G_1}(u)+d_{G_1}(v))(d_{G_1}(u)+d_{G_1}(v)+2n_2)\Big] \\
&= \sum_{uv\in E(G_1)} (d^2_{G_1}(u)+d^2_{G_1}(v)) + 2\sum_{uv\in E(G_1)} d_{G_1}(u)d_{G_1}(v) \\
&\quad + 2n_2 \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)) \\
&= M_1^3(G_1) + 2M_2(G_1) + 2n_2M_1(G_1) \qquad (3.26)
\end{aligned}$$

Case V: $ef \in E(G_1\nabla_Q G_2)$ $s.t.$ $e, f \in V(Q(G_1))\backslash V(G_1)$.

Let e be inserted in $uv \in E(G_1)$ and f be inserted in $vw \in E(G_1)$. γ_{uw} be the number of common neighbors of u *and* w in G_1.

$$\begin{aligned}
\sum_{ef\in E(G_1\nabla_Q G_2)} d(e)d(f) &= \sum_{uv,vw\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))(d_{G_1}(v)+d_{G_1}(w)) \\
&= \sum_{uv,vw\in E(G_1)} \Big[d^2_{G_1}(v) + d_{G_1}(u)d_{G_1}(w) + d_{G_1}(v)\Big(d_{G_1}(u)+d_{G_1}(w)\Big)\Big] \\
&= \sum_{u\in V(G_1)} \begin{pmatrix} d_{G_1}(u) \\ 2 \end{pmatrix} d^2_{G_1}(u) + \sum_{u,w\in V(G_1)} \gamma_{uw} d_{G_1}(u)d_{G_1}(w) \\
&\quad + \sum_{v\in V(G_1)} d_{G_1}(v)(d_{G_1}(v)-1) \sum_{u\in V(G_1), uv\in E(G_1)} d(u) \\
&= \frac{1}{2}(M_1^4(G_1) - M_1^3(G_1)) - 2M_2(G_1) + \sum_{u,w\in V(G_1)} \gamma_{uw} d_{G_1}(u)d_{G_1}(w) \\
&\quad + \sum_{v\in V(G_1)} d^2_{G_1}(v) \sum_{\substack{u\in V(G_1),\\ uv\in E(G_1)}} d_{G_1}(u)
\end{aligned}$$

Clearly, we have to consider case I only once and the Cases II-V two times. Now, by combining all the cases we get the result. □

3.4. Zagreb indices of $G_1\nabla_T G_2$. We can see that the degree of a vertex in $G_1\nabla_T G_2$ can be written in terms of degree of a vertex in $G_1\nabla_R G_2$ or $G_1\nabla_Q G_2$, which is as follows.

$$d_{G_1\nabla_T G_2}(u)=\begin{cases} d_{G_1\nabla_R G_2}(u) & \text{if } u\in V(G_1)\\ d_{G_1\nabla_Q G_2}(u) & \text{if } u\in V(G_2)\\ d_{G_1\nabla_Q G_2}(u) & \text{if } u\in V(G_1\nabla_T G_2)\backslash(V(G_1)\cup V(G_2)).\end{cases}$$

Now, based on these relations we can easily obtain the Zagreb indices of $G_1\nabla_T G_2$ and hence the following theorems are proposed.

Theorem 3.7. *Let G_1 and G_2 be two graphs with $|V(G_i)|=n_i$ and $|E(G_i)|=m_i$, where $i=1,2$. Then*

$$\begin{aligned}M_1(G_1\nabla_T G_2)=&2\Big[4M_1(G_1)+M_1(G_2)+2M_2(G_1)+M_1^3(G_1)\\&+n_1n_2(n_1+n_2+2)+n_2(8m_1+1)+4m_2(n_1+1)\Big].\end{aligned}$$

Theorem 3.8. *Let G_1 and G_2 be two graphs with $|V(G_i)|=n_i$ and $|E(G_i)|=m_i$, where $i=1,2$. Then*

$$\begin{aligned}M_2(G_1\nabla_T G_2)=&M_1(G_2)+4m_2(n_1+1)+n_2(n_1+1)^2+2\big[4n_2M_1(G_1)+(n_1+1)M_1(G_2)\\&+6M_2(G_1)+M_2(G_2)+\frac{1}{2}(3M_1^3(G_1)+M_1^4(G_1))+(n_1+1)\{m_2(n_1+1)\\&+n_2(2m_1+n_1n_2)\}+2m_2(2m_1+n_1n_2)+n_2^2m_1\\&+\sum_{u,w\in V(G_1)}\gamma_{uw}d_{G_1}(u)d_{G_1}(w)+\sum_{u\in V(G_1)}d_{G_1}^2(u)\sum_{\substack{v\in V(G_1),\\ uv\in E(G_1)}}d_{G_1}(v)\big],\end{aligned}$$

where γ_{uw} is the number of common neighbors of $u,w\in V(G_1)$.

4. Conclusions

In this paper we define four new operations of graphs based on Indu-Bala product of graphs. Then we also establish expressions for the first and second Zagreb indices of the operations of graphs. In future work we will study the adjacency spectrum of the four new operations and other topological indices like forgotten topological index, hyper- Zagreb index will also be studied in connection with these new operations of graphs

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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