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ON r-FUZZY ℓ-OPEN SETS AND CONTINUITY OF FUZZY MULTIFUNCTIONS VIA FUZZY IDEALS

I. M. TAHA[1,2,*]

[1]Department of Basic Sciences, Higher Institute of Engineering and Technology, Menoufia, Egypt

[2]Department of Mathematics, Faculty of Science, Sohag University, Egypt

Abstract. In this paper, the concepts of r-fuzzy ℓ-open, r-fuzzy semi-ℓ-open, r-fuzzy pre-ℓ-open, r-fuzzy α-ℓ-open and r-fuzzy β-ℓ-open sets are introduced in a fuzzy ideal topological space (X,τ,ℓ) based on the sense of Šostak. Also, the relations of these sets with each other are investigated with the help of examples. Moreover, the concepts of fuzzy upper (resp. lower) ℓ-continuous, almost ℓ-continuous and weakly ℓ-continuous multifunctions are introduced and some properties of these multifunctions along with their mutual relationships are specified.

Keywords: Fuzzy ideal topological space, fuzzy multifunction, r-fuzzy ℓ-open set, r-fuzzy α-ℓ-open set, fuzzy upper (lower) ℓ-continuous, almost ℓ-continuous, weakly ℓ-continuous, compactness.

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1. Introduction and Preliminaries

The theory of fuzzy sets provides a framework for mathematical modeling of those real world situations, which involve an element of uncertainty, imprecision, or vagueness in their description. Since its inception thirty years ago by Zadeh [14], this theory has found wide applications

*Corresponding author
E-mail address: imtaha2010@yahoo.com
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in engineering, economics, information sciences, medicine, etc.; for details the reader is referred to [6, 15]. A fuzzy multifunction is a fuzzy set valued function [2, 7, 12, 13]. Fuzzy multifunctions arise in many applications, for instance, the budget multifunction occurs in economic theory, noncooperative games, artificial intelligence, and decision theory. The biggest difference between fuzzy functions and fuzzy multifunctions has to do with the definition of an inverse image. For a fuzzy multifunction there are two types of inverses. These two definitions of the inverse then leads to two definitions of continuity. Ramadan and Abd El-latif [9], introduced and studied the concepts of fuzzy upper and lower almost continuous, weakly continuous and almost weakly continuous multifunctions where the domain of these functions is a classical topological space with their values as arbitrary fuzzy sets in fuzzy topological space. Al-shami and Noiri [3, 4], defined and studied new generalization of open set.

In this work, a new forms of sets called r-fuzzy ℓ-open, r-fuzzy semi-ℓ-open, r-fuzzy pre-ℓ-open, r-fuzzy α-ℓ-open and r-fuzzy β-ℓ-open sets are introduced on a fuzzy ideal topological space (X,τ,ℓ) in Šostak sense. Also, the relations of these sets with each other are investigated with the help of examples. Moreover, the concepts of fuzzy upper (resp. lower) ℓ-continuous, almost ℓ-continuous and weakly ℓ-continuous multifunctions are introduced and some interesting properties of them are specified. Throughout this paper, X refers to an initial universe. The family of all fuzzy sets in X is denoted by I^X and for $\lambda \in I^X$, $\lambda^c(x) = 1-\lambda(x)$ for all $x \in X$ (where $I=[0,1]$ and $I_\circ = (0,1]$). For $t \in I$, $\underline{t}(x) = t$ for all $x \in X$. The fuzzy difference between two fuzzy sets [11] $\lambda,\mu \in I^X$ defined as follows:

$$\lambda \overline{\wedge} \mu = \begin{cases} \underline{0}, & \text{if} \quad \lambda \leq \mu, \\ \lambda \wedge \mu^c, & \text{otherwise.} \end{cases}$$

All other notations are standard notations of fuzzy set theory. Now, we recall that a fuzzy idea ℓ on X [8], is a map $\ell : I^X \longrightarrow I$ that satisfies the following conditions: (i) $\forall\, \lambda,\mu \in I^X$ and $\lambda \leq \mu$ $\Rightarrow \ell(\mu) \leq \ell(\lambda)$. (ii) $\forall\, \lambda,\mu \in I^X \Rightarrow \ell(\lambda \vee \mu) \geq \ell(\lambda) \wedge \ell(\mu)$. Also, ℓ is called proper if $\ell(\underline{1}) = 0$ and there exists $\mu \in I^X$ such that $\ell(\mu) > 0$. The simplest fuzzy ideals on X, ℓ_0 and ℓ_1 defined as follows:

$$\ell_0(\lambda)=\begin{cases}1, & \text{if} \quad \lambda=\underline{0},\\ 0, & \text{otherwise}\end{cases} \qquad \text{and} \qquad \ell_1(\lambda)=1 \ \forall \ \lambda\in I^X.$$

If ℓ^1 and ℓ^2 are fuzzy ideals on X, we say that ℓ^1 is finer than ℓ^2 (ℓ^2 is coarser than ℓ^1), denoted by $\ell^2\leq\ell^1$, iff $\ell^2(\lambda)\leq\ell^1(\lambda)\ \forall\ \lambda\in I^X$.

Let (X,τ) be a fuzzy topological space in Šostak sense [10], the closure and the interior of any fuzzy set $\lambda\in I^X$ denoted by $C_\tau(\lambda,r)$ and $I_\tau(\lambda,r)$. Let (X,τ,ℓ) be a fuzzy ideal topological space, $\lambda\in I^X$ and $r\in I_\circ$, then the r-fuzzy local function [11] λ^*_r of λ defined as follows: $\lambda^*_r=\bigwedge\{\mu\in I^X\ :\ \ell(\lambda\ \overline{\wedge}\ \mu)\geq r,\ \tau(\mu^c)\geq r\}$. If we take $\ell=\ell_0$, for each $\lambda\in I^X$ we have $\lambda^*_r=\bigwedge\{\mu\in I^X\ :\ \lambda\leq\mu,\ \tau(\mu^c)\geq r\}=C_\tau(\lambda,r)$. Also, if we take $\ell=\ell_1$ (resp. $\ell(\lambda)\geq r$), for each $\lambda\in I^X$ we have $\lambda^*_r=\underline{0}$. Moreover, we define an operator $C^*_\tau: I^X\times I_\circ\to I^X$ as follows: $C^*_\tau(\lambda,r)=\lambda\ \vee\ \lambda^*_r$.

A mapping $F:X\multimap Y$ is called a fuzzy multifunction [5] iff $F(x)\in I^Y$ for each $x\in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y)=G_F(x,y)$ for any $(x,y)\in X\times Y$. Also, F is Normalized iff for each $x\in X$, there exists $y_0\in Y$ such that $G_F(x,y_0)=1$ and F is Crisp iff $G_F(x,y)=1$ for each $x\in X$ and $y\in Y$. The upper inverse $F^u(\mu)$, the lower inverse $F^l(\mu)$ of $\mu\in I^Y$ and the image $F(\lambda)$ of $\lambda\in I^X$ are defined as follows: $F^u(\mu)(x)=\bigwedge_{y\in Y}[G^c_F(x,y)\vee\mu(y)]$, $F^l(\mu)(x)=\bigvee_{y\in Y}[G_F(x,y)\wedge\mu(y)]$ and $F(\lambda)(y)=\bigvee_{x\in X}[G_F(x,y)\wedge\lambda(x)]$. All definitions and properties of image, lower and upper are found in [1] .

2. On r-Fuzzy ℓ-Open and r-Fuzzy α-ℓ-Open Sets

Definition 2.1. Let (X,τ,ℓ) be a fuzzy ideal topological space, $\lambda\in I^X$ and $r\in I_\circ$. Then, λ is said to be:

(1) r-fuzzy ℓ-open iff $\lambda\leq I_\tau(\lambda^*_r,r)$.

(2) r-fuzzy semi-ℓ-open iff $\lambda\leq C^*_\tau(I_\tau(\lambda,r),r)$.

(3) r-fuzzy pre-ℓ-open iff $\lambda\leq I_\tau(C^*_\tau(\lambda,r),r)$.

(4) r-fuzzy α-ℓ-open iff $\lambda\leq I_\tau(C^*_\tau(I_\tau(\lambda,r),r),r)$.

(5) r-fuzzy β-ℓ-open iff $\lambda \leq C_\tau(I_\tau(C^*_\tau(\lambda,r),r),r)$.

The following implications hold:

$$\begin{array}{ccccc} & r\text{-fuzzy open} \Rightarrow & r\text{-fuzzy } \alpha\text{-}\ell\text{-open} & \Rightarrow & r\text{-fuzzy semi-}\ell\text{-open} \\ & & \Downarrow & & \Downarrow \\ r\text{-fuzzy } \ell\text{-open} \Rightarrow & & r\text{-fuzzy pre-}\ell\text{-open} & \Rightarrow & r\text{-fuzzy } \beta\text{-}\ell\text{-open} \end{array}$$

In general the converses are not true.

Remark 2.2. (1) r-fuzzy ℓ-open and r-fuzzy open $[\lambda \leq I_\tau(\lambda,r)]$ are independent notions as shown by Example 2.3.

(2) r-fuzzy semi-ℓ-open and r-fuzzy pre-ℓ-open are independent notions as shown by Example 2.4 and Example 2.5.

Example 2.3. Define $\tau_1,\tau_2,\ell^1,\ell^2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.2}, \\ \frac{3}{4}, & \text{if} \quad \lambda = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\ell^1(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{2}{3}, & \text{if} \quad \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell^2(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.2}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, (1) In (X,τ_1,ℓ^1), $\underline{0.2}$ is $\frac{1}{2}$-fuzzy open set but it is not $\frac{1}{2}$-fuzzy ℓ-open.

(2) In (X,τ_2,ℓ^2), $\underline{0.2}$ is $\frac{1}{2}$-fuzzy ℓ-open set but it is not $\frac{1}{2}$-fuzzy open.

Example 2.4. Define $\tau, \ell : I^X \longrightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}, \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.8}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{3}{4}, & \text{if } \underline{0} < \nu < \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, (1) $\underline{0.5}$ is $\frac{1}{2}$-fuzzy pre-ℓ-open set but it is not $\frac{1}{2}$-fuzzy semi-ℓ-open.

(2) $\underline{0.5}$ is $\frac{1}{2}$-fuzzy β-ℓ-open set but it is not $\frac{1}{2}$-fuzzy semi-ℓ-open.

(3) $\underline{0.5}$ is $\frac{1}{2}$-fuzzy pre-ℓ-open set but it is not $\frac{1}{2}$-fuzzy α-ℓ-open.

Example 2.5. Let $X = \{x, y, z\}$ be a set and $\mu_1, \mu_2 \in I^X$ defined as follows: $\mu_1 = \{\frac{x}{0.3}, \frac{y}{0.4}, \frac{z}{0.8}\}$ and $\mu_2 = \{\frac{x}{0.2}, \frac{y}{0.3}, \frac{z}{0.2}\}$. Define $\tau, \ell : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_2, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{2}{3}, & \text{if } \underline{0} < \nu < \underline{0.2}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, (1) μ_1 is $\frac{1}{2}$-fuzzy semi-ℓ-open set but it is neither $\frac{1}{2}$-fuzzy pre-ℓ-open nor $\frac{1}{2}$-fuzzy α-ℓ-open.

(2) μ_1 is $\frac{1}{2}$-fuzzy β-ℓ-open set but it is not $\frac{1}{2}$-fuzzy pre-ℓ-open.

Example 2.6. Let $X = \{x, y, z, w\}$ be a set and $\mu_1, \mu_2 \in I^X$ defined as follows:
$\mu_1 = \{\frac{x}{0.9}, \frac{y}{0.9}, \frac{z}{0.5}, \frac{w}{0.5}\}$ and $\mu_2 = \{\frac{x}{0.9}, \frac{y}{0.9}, \frac{z}{0.9}, \frac{w}{0.5}\}$.
Define $\tau, \ell : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, μ_2 is $\frac{1}{2}$-fuzzy α-ℓ-open set but it is not $\frac{1}{2}$-fuzzy open.

Corollary 2.7. Let (X,τ,ℓ) be a fuzzy ideal topological space and $r\in I_\circ$,

(1) Every r-fuzzy pre-ℓ-open set is r-fuzzy preopen.

(2) Every r-fuzzy semi-ℓ-open set is r-fuzzy semi-open.

(3) Every r-fuzzy α-ℓ-open set is r-fuzzy α-open.

(4) Every r-fuzzy β-ℓ-open set is r-fuzzy β-open.

In general the converses are not true.

Example 2.8. Define $\tau,\ell : I^X \longrightarrow I$ as follows:

$$\tau(\lambda)=\begin{cases} 1, & \text{if} \quad \lambda\in\{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda=\underline{0.3}, \\ \frac{3}{4}, & \text{if} \quad \lambda=\underline{0.7}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu)=\begin{cases} 1, & \text{if} \quad \nu=\underline{0}, \\ \frac{3}{4}, & \text{if} \quad \underline{0}<\nu<\underline{0.6}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\underline{0.6}$ is $\frac{1}{2}$-fuzzy preopen set but it is not $\frac{1}{2}$-fuzzy pre-ℓ-open.

Example 2.9. Define $\tau,\ell : I^X \longrightarrow I$ as follows:

$$\tau(\lambda)=\begin{cases} 1, & \text{if} \quad \lambda\in\{\underline{0},\underline{1}\}, \\ \frac{3}{4}, & \text{if} \quad \lambda=\underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu)=\begin{cases} 1, & \text{if} \quad \nu=\underline{0}, \\ \frac{2}{3}, & \text{if} \quad \underline{0}<\nu<\underline{0.5}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\underline{0.5}$ is $\frac{1}{2}$-fuzzy semi-open set but it is not $\frac{1}{2}$-fuzzy semi-ℓ-open.

Example 2.10. Let $X=\{x,y,z,w\}$ be a set and $\mu_1,\mu_2\in I^X$ defined as follows:
$\mu_1=\{\frac{x}{0.9},\frac{y}{0.9},\frac{z}{0.5},\frac{w}{0.5}\}$ and $\mu_2=\{\frac{x}{0.9},\frac{y}{0.9},\frac{z}{0.9},\frac{w}{0.5}\}$.
Define $\tau,\ell : I^X \to I$ as follows:

$$\tau(\lambda)=\begin{cases} 1, & \text{if} \quad \lambda\in\{\underline{0},\underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \lambda=\mu_1, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu)=\begin{cases} 1, & \text{if} \quad \nu=\underline{0}, \\ \frac{2}{3}, & \text{if} \quad \underline{0}<\nu<\underline{0.9}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, μ_2 is $\frac{1}{3}$-fuzzy α-open set but it is not $\frac{1}{3}$-fuzzy α-ℓ-open.

Example 2.11. Define $\tau, \ell : I^X \longrightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.2}, \\ \frac{3}{4}, & \text{if} \quad \lambda = \underline{0.8}, \\ 0, & \text{otherwise,} \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.8}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\underline{0.5}$ is $\frac{1}{2}$-fuzzy β-open set but it is not $\frac{1}{2}$-fuzzy β-ℓ-open.

Remark 2.12. The complement of r-fuzzy ℓ-open (resp. r-fuzzy semi-ℓ-open, r-fuzzy pre-ℓ-open, r-fuzzy α-ℓ-open and r-fuzzy β-ℓ-open) set is said to be r-fuzzy ℓ-closed (resp. r-fuzzy semi-ℓ-closed, r-fuzzy pre-ℓ-closed, r-fuzzy α-ℓ-closed and r-fuzzy β-ℓ-closed).

Corollary 2.13. Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. If we take $\ell = \ell_0$, we have

(1) r-fuzzy ℓ-open, r-fuzzy pre-ℓ-open and r-fuzzy preopen are equivalent.

(2) r-fuzzy semi-ℓ-open and r-fuzzy semi-open are equivalent.

(3) r-fuzzy α-ℓ-open and r-fuzzy α-open are equivalent.

(4) r-fuzzy β-ℓ-open and r-fuzzy β-open are equivalent.

Lemma 2.14. Let (X, τ, ℓ) be a fuzzy ideal topological space. Then any union of r-fuzzy ℓ-open (resp. any intersection of r-fuzzy ℓ-closed) sets is r-fuzzy ℓ-open (resp. r-fuzzy ℓ-closed).

Proof. Let $\{\lambda_j \in I^X : \lambda_j \text{ is } r\text{-fuzzy } \ell\text{-open}\}_{j \in J}$. Then, $\lambda_j \leq I_\tau((\lambda_j)^*_r, r)$ for each $j \in J$ and hence, $\bigvee_{j \in J} \lambda_j \leq \bigvee_{j \in J} I_\tau((\lambda_j)^*_r, r) \leq I_\tau(\bigvee_{j \in J} (\lambda_j)^*_r, r) \leq I_\tau((\bigvee_{j \in J} \lambda_j)^*_r, r)$. Other case is similarly proved.

Proposition 2.15. Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. The following statements are equivalent,

(1) λ is r-fuzzy α-ℓ-open.

(2) λ is r-fuzzy semi-ℓ-open and r-fuzzy pre-ℓ-open.

Proof. (1) $\Rightarrow$ (2) Let λ be r-fuzzy α-ℓ-open, then $\lambda \leq I_\tau(C^*_\tau(I_\tau(\lambda,r),r),r) \leq C^*_\tau(I_\tau(\lambda,r),r)$. This shows that λ is r-fuzzy semi-ℓ-open. Moreover,

$$\lambda \leq I_\tau(C^*_\tau(I_\tau(\lambda,r),r),r) = I_\tau(I_\tau(\lambda,r) \vee (I_\tau(\lambda,r))^*_r, r) \leq I_\tau(\lambda \vee \lambda^*_r, r).$$

Therefore, λ is r-fuzzy pre-ℓ-open.

(2) $\Rightarrow$ (1) Let λ be r-fuzzy pre-ℓ-open and r-fuzzy semi-ℓ-open. Then,

$$\lambda \leq I_\tau(C^*_\tau(\lambda,r),r) \leq I_\tau(C^*_\tau(C^*_\tau(I_\tau(\lambda,r),r),r),r) = I_\tau(C^*_\tau(I_\tau(\lambda,r),r),r).$$

This shows that λ is r-fuzzy α-ℓ-open.

Definition 2.16. Let (X,τ,ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. Then λ is said to be r-fuzzy $*$-dense-in-itself (resp. r-fuzzy $*$-perfect) if $\lambda \leq \lambda^*_r$ (resp. $\lambda = \lambda^*_r$).

Proposition 2.17. Let (X,τ,ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. The following statements are equivalent,

(1) λ is r-fuzzy ℓ-open.

(2) λ is r-fuzzy pre-ℓ-open and r-fuzzy $*$-dense-in-itself.

Proof. (1) $\Rightarrow$ (2) Let λ be r-fuzzy ℓ-open, then $\lambda \leq I_\tau(\lambda^*_r, r) \leq \lambda^*_r$. This shows that λ is r-fuzzy $*$-dense-in-itself. Moreover,

$$\lambda \leq I_\tau(\lambda^*_r, r) \leq I_\tau(\lambda \vee \lambda^*_r, r) = I_\tau(C^*_\tau(\lambda,r),r).$$

Therefore, λ is r-fuzzy pre-ℓ-open.

(2) $\Rightarrow$ (1) Let λ be r-fuzzy pre-ℓ-open and r-fuzzy $*$-dense-in-itself. Then,

$$\lambda \leq I_\tau(C^*_\tau(\lambda,r),r) = I_\tau(\lambda \vee \lambda^*_r, r) = I_\tau(\lambda^*_r, r).$$

This shows that λ is r-fuzzy ℓ-open.

Remark 2.18. r-fuzzy pre-ℓ-open and r-fuzzy $*$-dense-in-itself are independent notions as shown by Example 2.19.

Example 2.19. Let $X = \{x, y, z\}$ be a set and $\mu_1, \mu_2 \in I^X$ defined as follows: $\mu_1 = \{\frac{x}{0.5}, \frac{y}{0.4}, \frac{z}{0.6}\}$ and $\mu_2 = \{\frac{x}{0.5}, \frac{y}{0.3}, \frac{z}{0.4}\}$. Define $\tau, \ell^1, \ell^2 : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \mu_1, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell^1(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise}. \end{cases}$$

$$\ell^2(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.6}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, (1) In (X, τ, ℓ^1), μ_2 is $\frac{1}{2}$-fuzzy $*$-dense-in-itself set but it is not $\frac{1}{2}$-fuzzy pre-ℓ-open.

(2) In (X, τ, ℓ^2), μ_1 is $\frac{1}{2}$-fuzzy pre-ℓ-open set but it is neither $\frac{1}{2}$-fuzzy $*$-dense-in-itself nor $\frac{1}{2}$-fuzzy ℓ-open.

Theorem 2.20. Let (X, τ, ℓ) be a fuzzy ideal topological space and $r \in I_\circ$.

(1) If λ is r-fuzzy ℓ-open and r-fuzzy $*$-perfect (resp. r-fuzzy semi-closed), then $\lambda = I_\tau(\lambda_r^*, r)$.

(2) r-fuzzy ℓ-open and r-fuzzy open are equivalent if λ is r-fuzzy $*$-perfect.

Proof. (1) Let λ be r-fuzzy ℓ-open and r-fuzzy $*$-perfect, $\lambda \le I_\tau(\lambda_r^*, r)$ and $\lambda = \lambda_r^*$. Thus $\lambda = I_\tau(\lambda_r^*, r)$. Other case, let λ be r-fuzzy ℓ-open and r-fuzzy semi-closed, $I_\tau(\lambda_r^*, r) \le I_\tau(C_\tau(\lambda, r), r) \le \lambda$. Thus $\lambda = I_\tau(\lambda_r^*, r)$.

(2) Obvious.

Theorem 2.21. Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda, \mu \in I^X$ and $r \in I_\circ$. Then,

(1) λ is r-fuzzy semi-ℓ-open iff $C^*_\tau(\lambda, r) = C^*_\tau(I_\tau(\lambda, r), r)$.

(2) λ is r-fuzzy semi-ℓ-open iff there exists $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $\mu \leq \lambda \leq C^*_\tau(\mu, r)$.

(3) If λ is r-fuzzy semi-ℓ-open such that $\lambda \leq \mu \leq C^*_\tau(\lambda, r)$, then μ is also r-fuzzy semi-ℓ-open.

Proof. (1) Let λ be r-fuzzy semi-ℓ-open, $\lambda \leq C^*_\tau(I_\tau(\lambda, r), r)$ and

$$C^*_\tau(\lambda, r) \leq C^*_\tau(C^*_\tau(I_\tau(\lambda, r), r), r) = C^*_\tau(I_\tau(\lambda, r), r).$$

Thus $C^*_\tau(\lambda, r) = C^*_\tau(I_\tau(\lambda, r), r)$. The converse is obvious.

(2) Let λ be r-fuzzy semi-ℓ-open, $\lambda \leq C^*_\tau(I_\tau(\lambda, r), r)$. Take $I_\tau(\lambda, r) = \mu$, $\mu \leq \lambda \leq C^*_\tau(\mu, r)$. Conversely, let $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $\mu \leq \lambda \leq C^*_\tau(\mu, r)$. Then $\mu \leq I_\tau(\lambda, r)$ and hence $\lambda \leq C^*_\tau(\mu, r) \leq C^*_\tau(I_\tau(\lambda, r), r)$. This shows that μ is r-fuzzy semi-ℓ-open.

(3) Let λ be r-fuzzy semi-ℓ-open and $\lambda \leq \mu \leq C^*_\tau(\lambda, r)$. Then,

$$\lambda \leq C^*_\tau(I_\tau(\lambda, r), r) \leq C^*_\tau(I_\tau(\mu, r), r).$$

Since $\mu \leq C^*_\tau(\lambda, r)$, $\mu \leq C^*_\tau(\lambda, r) \leq C^*_\tau(C^*_\tau(I_\tau(\mu, r), r), r) = C^*_\tau(I_\tau(\mu, r), r)$. This shows that μ is r-fuzzy semi-ℓ-open.

Theorem 2.22. Let (X, τ, ℓ) be a fuzzy ideal topological space. Then for each $\lambda \in I^X$ and $r \in I_\circ$, we define an operator $C^\ell_\tau : I^X \times I_\circ \to I^X$ as follows:

$$C^\ell_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \ : \ \lambda \leq \mu, \ \mu \text{ is } r\text{-fuzzy } \ell\text{-closed}\}.$$

For each $\lambda, \ \nu \in I^X$, the operator C^ℓ_τ satisfies the following properties:

(1) $C^\ell_\tau(\underline{1}, r) = \underline{1}$.

(2) $\lambda \leq C^\ell_\tau(\lambda, r)$.

(3) If $\lambda \leq \nu$, then $C^\ell_\tau(\lambda, r) \leq C^\ell_\tau(\nu, r)$.

(4) If $\ell(\lambda) \geq r$, then $C^\ell_\tau(\lambda, r) = \underline{1}$.

(5) $C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r) = C^{\ell}_{\tau}(\lambda,r)$.

(6) $C^{\ell}_{\tau}(\lambda,r) \vee C^{\ell}_{\tau}(\nu,r) \leq C^{\ell}_{\tau}(\lambda \vee \nu,r)$.

(7) $\lambda = C^{\ell}_{\tau}(\lambda,r)$ iff λ is r-fuzzy ℓ-closed.

Proof. (1) Since $\underline{0} \leq I_{\tau}(\underline{0}^{*}_{r},r)$ this implies $\underline{1}$ is r-fuzzy ℓ-closed. From the definition of C^{ℓ}_{τ}, $C^{\ell}_{\tau}(\underline{1},r) = \underline{1}$.

(2), (3), (4) and (7) are easily proved from the definition of C^{ℓ}_{τ}, Remark 2.12 and Lemma 2.14.

(5) From (2) and (3), we have $C^{\ell}_{\tau}(\lambda,r) \leq C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r)$. Now we show that $C^{\ell}_{\tau}(\lambda,r) \geq C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r)$. Suppose that $C^{\ell}_{\tau}(\lambda,r) \not\geq C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r)$. There exist $x \in X$ and $t \in (0,1)$ such that $C^{\ell}_{\tau}(\lambda,r)(x) < t < C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r)(x)$. (A)

Since $C^{\ell}_{\tau}(\lambda,r)(x) < t$, by the definition of C^{ℓ}_{τ}, there exists r-fuzzy ℓ-closed μ_1 with $\lambda \leq \mu_1$ such that $C^{\ell}_{\tau}(\lambda,r)(x) \leq \mu_1(x) < t$. Since $\lambda \leq \mu_1$, we have $C^{\ell}_{\tau}(\lambda,r) \leq \mu_1$. Again, by the definition of C^{ℓ}_{τ}, we have $C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r) \leq \mu_1$. Hence $C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r)(x) \leq \mu_1(x) < t$, it is a contradiction for (A). Thus, $C^{\ell}_{\tau}(\lambda,r) \geq C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r)$. Then $C^{\ell}_{\tau}(C^{\ell}_{\tau}(\lambda,r),r) = C^{\ell}_{\tau}(\lambda,r)$

(6) Since λ and $\nu \leq \lambda \vee \nu$ implies $C^{\ell}_{\tau}(\lambda,r) \leq C^{\ell}_{\tau}(\lambda \vee \nu,r)$ and $C^{\ell}_{\tau}(\nu,r) \leq C^{\ell}_{\tau}(\lambda \vee \nu,r)$. Thus, $C^{\ell}_{\tau}(\lambda,r) \vee C^{\ell}_{\tau}(\nu,r) \leq C^{\ell}_{\tau}(\lambda \vee \nu,r)$.

Theorem 2.23. Let (X,τ,ℓ) be a fuzzy ideal topological space. Then for each $\lambda \in I^X$ and $r \in I_{\circ}$, we define an operator $I^{\ell}_{\tau} : I^X \times I_{\circ} \to I^X$ as follows:

$$I^{\ell}_{\tau}(\lambda,r) = \bigvee\{\mu \in I^X \ : \ \mu \leq \lambda, \ \mu \text{ is } r\text{-fuzzy } \ell\text{-open}\}.$$

For each $\lambda, \ \nu \in I^X$, the operator I^{ℓ}_{τ} satisfies the following properties:

(1) $I^{\ell}_{\tau}(\underline{0},r) = \underline{0}$.

(2) $I^{\ell}_{\tau}(\lambda,r) \leq \lambda$.

(3) $I^{\ell}_{\tau}(\lambda^c,r) = (C^{\ell}_{\tau}(\lambda,r))^c$.

(4) If $\ell = \ell_1$, then $I^{\ell}_{\tau}(\lambda,r) = \underline{0}$.

(5) $I^{\ell}_{\tau}(I^{\ell}_{\tau}(\lambda,r),r) = I^{\ell}_{\tau}(\lambda,r)$.

(6) $I^{\ell}_{\tau}(\lambda,r) \wedge I^{\ell}_{\tau}(\nu,r) \geq I^{\ell}_{\tau}(\lambda \wedge \nu,r)$.

(7) $\lambda = I^{\ell}_{\tau}(\lambda,r)$ iff λ is r-fuzzy ℓ-open.

Proof. It is similarly proved as in Theorem 2.22.

3. Continuity of Fuzzy Multifunctions via Fuzzy Ideals

Definition 3.1. A fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$ is called:

(1) Fuzzy upper ℓ-continuous (resp. almost ℓ-continuous and weakly ℓ-continuous) iff $F^u(\mu)$ is r-fuzzy ℓ-open (resp. $F^u(\mu) \leq I_\tau([F^u(I_\eta(C_\eta(\mu,r),r))]^*_r,r)$ and $F^u(\mu) \leq I_\tau([F^u(C_\eta(\mu,r))]^*_r,r)$) for each $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $r \in I_o$.

(2) Fuzzy lower ℓ-continuous (resp. almost ℓ-continuous and weakly ℓ-continuous) iff $F^l(\mu)$ is r-fuzzy ℓ-open (resp. $F^l(\mu) \leq I_\tau([F^l(I_\eta(C_\eta(\mu,r),r))]^*_r,r)$ and $F^l(\mu) \leq I_\tau([F^l(C_\eta(\mu,r))]^*_r,r)$) for each $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $r \in I_o$.

The following implications hold:

$$\ell\text{-continuity} \quad \Rightarrow \quad \text{almost } \ell\text{-continuity} \quad \Rightarrow \quad \text{weakly } \ell\text{-continuity.}$$

In general the converses are not true.

Remark 3.2. (1) Fuzzy upper (resp. lower) ℓ-continuity and fuzzy upper (resp. lower) semi-continuity [11] are independent notions as shown by Example 3.5.

(2) Fuzzy upper (resp. lower) almost ℓ-continuity and fuzzy upper (resp. lower) almost continuity [11] are independent notions as shown by Example 3.3.

(3) Fuzzy upper (resp. lower) weakly ℓ-continuity and fuzzy upper (resp. lower) weakly continuity [11] are independent notions as shown by Example 3.4.

Example 3.3. Let $X=\{x_1,x_2\}, Y=\{y_1,y_2,y_3\}$ and $F:X\multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1)=0.1,\ G_F(x_1,y_2)=1.0,\ G_F(x_1,y_3)=0.3,\ G_F(x_2,y_1)=0.5,\ G_F(x_2,y_2)=0.1$ and $G_F(x_2,y_3)=1.0$. Define $\tau_1,\tau_2,\ell^1,\ell^2:I^X\longrightarrow I$ and $\eta_1,\eta_2:I^Y\to I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1, & \text{if} \quad \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if} \quad \lambda=\underline{0.5},\\ 0, & \text{otherwise},\end{cases}\qquad \tau_2(\lambda)=\begin{cases}1, & \text{if} \quad \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if} \quad \lambda=\underline{0.4},\\ 0, & \text{otherwise},\end{cases}$$

$$\ell^1(\nu)=\begin{cases}1, & \text{if} \quad \nu=\underline{0},\\ \frac{1}{2}, & \text{if} \quad \underline{0}<\nu<\underline{0.4},\\ 0, & \text{otherwise},\end{cases}\qquad \ell^2(\nu)=\begin{cases}1, & \text{if} \quad \nu=\underline{0},\\ \frac{1}{2}, & \text{if} \quad \underline{0}<\nu<\underline{0.3},\\ 0, & \text{otherwise},\end{cases}$$

$$\eta_1(\mu)=\begin{cases}1, & \text{if} \quad \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{3}, & \text{if} \quad \mu=\underline{0.4},\\ \frac{1}{2}, & \text{if} \quad \mu=\underline{0.5},\\ 0, & \text{otherwise},\end{cases}\qquad \eta_2(\mu)=\begin{cases}1, & \text{if} \quad \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{3}, & \text{if} \quad \mu=\underline{0.4},\\ 0, & \text{otherwise}.\end{cases}$$

Then, (1) $F:(X,\tau_1,\ell^1)\multimap(Y,\eta_1)$ is fuzzy upper (resp. lower) almost ℓ-continuous but it is not fuzzy upper (resp. lower) ℓ-continuous.

(2) $F:(X,\tau_1,\ell^2)\multimap(Y,\eta_2)$ is fuzzy upper (resp. lower) almost ℓ-continuo-us but it is not fuzzy upper (resp. lower) almost continuous.

(3) $F:(X,\tau_2,\ell^1)\multimap(Y,\eta_2)$ is fuzzy upper (resp. lower) almost continuous but it is not fuzzy upper (resp. lower) almost ℓ-continuous.

Example 3.4. Let $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$ and $F:X\multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1)=0.8,\ G_F(x_1,y_2)=0.3, G_F(x_1,y_3)=0.3,\ G_F(x_2,y_1)=0.1,\ G_F(x_2,y_2)=1.0,\ G_F(x_2,y_3)=0.1,\ G_F(x_3,y_1)=0.1,\ G_F(x_3,y_2)=0.2,\ G_F(x_3,y_3)=1.0$. Define $\mu_1\in I^X$ and $\mu_2\in I^Y$ as follows: $\mu_1=\{\frac{x_1}{0.4},\frac{x_2}{0.1},\frac{x_3}{0.2}\}$ and $\mu_2=\{\frac{y_1}{0.4},\frac{y_2}{0.1},\frac{y_3}{0.2}\}$. Define $\tau_1,\tau_2,\ell^1,\ell^2,\ell^3:$

$I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell^1(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.4}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell^2(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{3}{4}, & \text{if} \quad \mu = \mu_2, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell^3(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{2}{3}, & \text{if} \quad \underline{0} < \nu < \underline{0.9}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, (1) $F : (X, \tau_1, \ell^1) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) weakly ℓ-continuous but it is not fuzzy upper (resp. lower) almost ℓ-continuous.

(2) $F : (X, \tau_2, \ell^2) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) weakly ℓ-continuous but it is not fuzzy upper (resp. lower) weakly continuous.

(3) $F : (X, \tau_1, \ell^3) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) weakly continuous but it is not fuzzy upper (resp. lower) weakly ℓ-continuous.

Example 3.5. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = 1.0$, $G_F(x_1, y_3) = 0.3$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = 0.1$ and $G_F(x_2, y_3) = 1.0$. Define $\tau_1, \tau_2, \ell^1, \ell^2 : I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.2}, \\ \frac{3}{4}, & \text{if} \quad \lambda = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\ell^1(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{2}{3}, & \text{if} \quad \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell^2(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \mu = \underline{0.2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, (1) $F : (X, \tau_1, \ell^1) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) semi-continuous but it is not fuzzy upper (resp. lower) ℓ-continuous.

(2) $F : (X, \tau_2, \ell^2) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) ℓ-continuous but it is not fuzzy upper (resp. lower) semi-continuous.

Corollary 3.6 Let $F : (X, \tau, \ell) \multimap (Y, \eta)$ be a fuzzy multifunction (resp. normalized fuzzy multifunction). Then every fuzzy lower (resp. upper) ℓ-continuous multifunction is fuzzy lower (resp. upper) precontinuous.

Corollary 3.7. Let $F : (X, \tau, \ell) \multimap (Y, \eta)$ be a fuzzy multifunction (resp. normalized fuzzy multifunction). If we take $\ell = \ell_0$, we have F is fuzzy lower (resp. upper) ℓ-continuous iff it is fuzzy lower (resp. upper) precontinuous.

Theorem 3.8. For a fuzzy multifunction $F : (X, \tau, \ell) \multimap (Y, \eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy lower ℓ-continuous.

(2) $F^u(\mu)$ is r-fuzzy ℓ-closed, if $\eta(\mu^c) \geq r$.

(3) $C^{\ell}_{\tau}(F^u(\mu), r) \leq F^u(C_\eta(\mu, r))$.

(4) $F^l(I_\eta(\mu, r)) \leq I^{\ell}_{\tau}(F^l(\mu), r)$.

Proof. (1) $\Rightarrow$ (2) Let $\mu \in I^Y$ with $\eta(\mu^c) \geq r$. Then by Definition 3.1(2), $F^l(\mu^c) = (F^u(\mu))^c$ is r-fuzzy ℓ-open. Thus, $F^u(\mu)$ is r-fuzzy ℓ-closed.

(2) $\Rightarrow$ (3) Let $\mu \in I^Y$. Then by (2), $F^u(C_\eta(\mu, r))$ is r-fuzzy ℓ-closed. Hence, we obtain $C^{\ell}_{\tau}(F^u(\mu), r) \leq F^u(C_\eta(\mu, r))$.

(3) $\Rightarrow$ (4) Since $(F^u(C_\eta(\mu,r)))^c = F^l(I_\eta(\mu^c,r))$ and $(C^\ell_\tau(F^u(\mu),r))^c = I^\ell_\tau(F^l(\mu^c),r)$. Hence, we obtain $I^\ell_\tau(F^l(\mu),r) \geq F^l(I_\eta(\mu,r))$ for each $\mu \in I^Y$.

(4) $\Rightarrow$ (1) Let $\mu \in I^Y$ with $\eta(\mu) \geq r$. Then by (4) and $\mu = I_\eta(\mu,r)$, $F^l(\mu) \leq I^\ell_\tau(F^l(\mu),r)$. Thus, $F^l(\mu)$ is r-fuzzy ℓ-open.

The following theorem is similarly proved as in Theorem 3.8.

Theorem 3.9. For a fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy upper ℓ-continuous.

(2) $F^l(\mu)$ is r-fuzzy ℓ-closed, if $\eta(\mu^c) \geq r$.

(3) $C^\ell_\tau(F^l(\mu),r) \leq F^l(C_\eta(\mu,r))$.

(4) $F^u(I_\eta(\mu,r)) \leq I^\ell_\tau(F^u(\mu),r)$.

Theorem 3.10. For a fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy lower almost ℓ-continuous.

(2) $F^l(\mu)$ is r-fuzzy ℓ-open, if μ is r-fuzzy regular open.

(3) $F^l(I_\eta(C_\eta(\mu,r),r))$ is r-fuzzy ℓ-open, $\forall\ \mu \in I^Y$.

(4) $F^u(\mu)$ is r-fuzzy ℓ-closed, if μ is r-fuzzy regular closed.

(5) $F^u(C_\eta(I_\eta(\mu,r),r))$ is r-fuzzy ℓ-closed, $\forall\ \mu \in I^Y$.

(6) $C^\ell_\tau(F^u(\mu),r) \leq F^u(C_\eta(I_\eta(\mu,r),r))$, if μ is r-fuzzy semi-open.

(7) $F^l(I_\eta(C_\eta(\mu,r),r)) \leq I^\ell_\tau(F^l(\mu),r)$, if μ is r-fuzzy semi-closed.

Proof. (1) $\Rightarrow$ (2) Let μ be r-fuzzy regular open set. Then by (1),

$$F^l(\mu) \leq I_\tau([F^l(I_\eta(C_\eta(\mu,r),r))]^*_r,r) = I_\tau([F^l(\mu)]^*_r,r).$$

Thus, $F^l(\mu)$ is r-fuzzy ℓ-open.

(2) $\Rightarrow$ (3) Since $I_\eta(C_\eta(\mu,r),r)$ is r-fuzzy regular open set for each $\mu \in I^Y$. Then by (2), $F^l(I_\eta(C_\eta(\mu,r),r))$ is r-fuzzy ℓ-open.

(3) $\Rightarrow$ (1) Let $\mu \in I^Y$ with $\eta(\mu) \geq r$. Then by (3),

$$F^l(\mu) \leq F^l(I_\eta(C_\eta(\mu,r),r)) \leq I_\tau([F^l(I_\eta(C_\eta(\mu,r),r))]^*_r,r).$$

Thus, F is fuzzy lower almost ℓ-continuous.

(2) $\Leftrightarrow$ (4) Let μ be r-fuzzy regular closed set. Then by (2), $F^l(\mu^c) = (F^u(\mu))^c$ is r-fuzzy ℓ-open. Thus, $F^u(\mu)$ is r-fuzzy ℓ-closed.

(4) $\Leftrightarrow$ (5) Since $C_\eta(I_\eta(\mu,r),r)$ is r-fuzzy regular closed set for each $\mu \in I^Y$. Then by (4), $F^u(C_\eta(I_\eta(\mu,r),r))$ is r-fuzzy ℓ-closed.

(5) $\Rightarrow$ (6) Let μ be r-fuzzy semi-open set. Then by (5), $F^u(C_\eta(I_\eta(\mu,r),r))$ is r-fuzzy ℓ-closed. Hence, we obtain $C^\ell_\tau(F^u(\mu),r) \leq F^u(C_\eta(I_\eta(\mu,r),r))$.

(6) $\Rightarrow$ (7) Let μ be r-fuzzy semi-closed set. Then by (6), $C^\ell_\tau(F^u(\mu^c),r) \leq F^u(C_\eta(I_\eta(\mu^c,r),r))$. Thus, $F^l(I_\eta(C_\eta(\mu,r),r)) \leq I^\ell_\tau(F^l(\mu),r)$.

(7) $\Rightarrow$ (2) Let μ be r-fuzzy regular open set. Then by (7) and $\mu = I_\eta(C_\eta(\mu,r),r)$, $F^l(\mu) \leq I^\ell_\tau(F^l(\mu),r)$. Thus, $F^l(\mu)$ is r-fuzzy ℓ-open.

The following theorem is similarly proved as in Theorem 3.10.

Theorem 3.11. For a fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy upper almost ℓ-continuous.

(2) $F^u(\mu)$ is r-fuzzy ℓ-open, if μ is r-fuzzy regular open.

(3) $F^u(I_\eta(C_\eta(\mu,r),r))$ is r-fuzzy ℓ-open, $\forall\ \mu \in I^Y$.

(4) $F^l(\mu)$ is r-fuzzy ℓ-closed, if μ is r-fuzzy regular closed.

(5) $F^l(C_\eta(I_\eta(\mu,r),r))$ is r-fuzzy ℓ-closed, $\forall\ \mu \in I^Y$.

(6) $C^\ell_\tau(F^l(\mu),r) \leq F^l(C_\eta(I_\eta(\mu,r),r))$, if μ is r-fuzzy semi-open.

(7) $F^u(I_\eta(C_\eta(\mu,r),r)) \leq I^\ell_\tau(F^u(\mu),r)$, if μ is r-fuzzy semi-closed.

Corollary 3.12. A fuzzy multifunction $F:(X,\tau,\ell) \multimap (Y,\eta,\ell)$ is fuzzy upper (resp. lower) weakly ℓ-continuous if $F^u(\mu) \leq I_\tau([F^u(C_\eta(\mu,r))]^*_r,r)$ (resp. $F^l(\mu) \leq I_\tau([F^l(C_\eta(\mu,r))]^*_r,r)$) for each r-fuzzy regular open $\mu \in I^Y$.

Definition 3.13. A fuzzy multifunction $F:(X,\tau,\ell) \multimap (Y,\eta,\ell)$ is called fuzzy upper (resp. lower) ℓ-irresolute iff $F^u(\mu)$ (resp. $F^l(\mu)$) is r-fuzzy ℓ-open for each r-fuzzy ℓ-open $\mu \in I^Y$.

Remark 3.14. Fuzzy upper (resp. lower) ℓ-irresolute and fuzzy upper (resp. lower) ℓ-continuous are independent notions because r-fuzzy ℓ-open and r-fuzzy open are independent notions.

The following theorems are similarly proved as in Theorem 3.8.

Theorem 3.15. For a fuzzy multifunction $F:(X,\tau,\ell) \multimap (Y,\eta,\ell)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy lower ℓ-irresolute.

(2) $F^u(\mu)$ is r-fuzzy ℓ-closed, if μ is r-fuzzy ℓ-closed.

(3) $C^\ell_\tau(F^u(\mu),r) \leq F^u(C^\ell_\eta(\mu,r))$.

(4) $F^l(I^\ell_\eta(\mu,r)) \leq I^\ell_\tau(F^l(\mu),r)$.

Theorem 3.16. For a fuzzy multifunction $F:(X,\tau,\ell) \multimap (Y,\eta,\ell)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy upper ℓ-irresolute.

(2) $F^l(\mu)$ is r-fuzzy ℓ-closed, if μ is r-fuzzy ℓ-closed.

(3) $C^\ell_\tau(F^l(\mu),r) \leq F^l(C^\ell_\eta(\mu,r))$.

(4) $F^u(I^\ell_\eta(\mu,r)) \leq I^\ell_\tau(F^u(\mu),r)$.

Theorem 3.17. Let $F:(X,\tau,\ell) \multimap (Y,\eta,\ell)$ and $H:(Y,\eta,\ell) \multimap (Z,\gamma,\ell)$ be two fuzzy multifunctions. Then we have the following:

(1) $H\circ F$ is fuzzy upper (resp. lower) ℓ-continuous if F is fuzzy upper (resp. lower) ℓ-irresolute and H is fuzzy upper (resp. lower) ℓ-continuous.

(2) $H\circ F$ is fuzzy upper (resp. lower) almost ℓ-continuous if F is fuzzy upper (resp. lower) ℓ-irresolute and H is fuzzy upper (resp. lower) almost ℓ-continuous.

(3) $H\circ F$ is fuzzy upper (resp. lower) weakly ℓ-continuous if F is fuzzy upper (resp. lower) ℓ-irresolute and H is fuzzy upper (resp. lower) ℓ-continuous.

(4) $H\circ F$ is fuzzy upper (resp. lower) ℓ-irresolute if F and H are fuzzy upper (resp. lower) ℓ-irresolute.

(5) $H\circ F$ is fuzzy lower ℓ-continuous if F is fuzzy lower ℓ-continuous and H is fuzzy lower semi-continuous

Proof. Obvious.

Definition 3.18. Let (X,τ,ℓ) be a fuzzy ideal topological space and $r\in I_\circ$. Then $\lambda\in I^X$ is called r-fuzzy ℓ-compact (resp., r-fuzzy almost ℓ-compact and r-fuzzy nearly ℓ-compact) iff for every family $\{\mu_i\in I^X \mid \mu_i \text{ is } r\text{-fuzzy } \ell\text{-open }\}_{i\in\Gamma}$ such that $\lambda\leq\bigvee_{i\in\Gamma}\mu_i$, there exists a finite subset $\Gamma_\circ$ of Γ such that $\lambda\leq\bigvee_{i\in\Gamma_\circ}\mu_i$ (resp., $\lambda\leq\bigvee_{i\in\Gamma_\circ}C^\ell_\tau(\mu_i,r)$ and $\lambda\leq\bigvee_{i\in\Gamma_\circ}I^\ell_\tau(C^\ell_\tau(\mu_i,r),r)$).

Remark 3.19. r-fuzzy ℓ-compact (resp., r-fuzzy almost ℓ-compact and r-fuzzy nearly ℓ-compact) and r-fuzzy compact (resp., r-fuzzy almost compact and r-fuzzy nearly compact) are independent notions because r-fuzzy ℓ-open and r-fuzzy open are independent notions.

The following implications hold:

r-fuzzy ℓ-compact $\Rightarrow$ r-fuzzy nearly ℓ-compact $\Rightarrow$ r-fuzzy almost ℓ-compact.

Theorem 3.20. Let $F : (X,\tau,\ell) \multimap (Y,\eta)$ be a crisp fuzzy upper ℓ-continuous and compact-valued. Then if $\lambda \in I^X$ is r-fuzzy ℓ-compact, $F(\lambda)$ is r-fuzzy compact .

Proof. Let $\lambda \in I^X$ be r-fuzzy ℓ-compact and $\{\mu_i \in I^Y \mid \eta(\mu_i) \geq r\}_{i\in\Gamma}$ with $F(\lambda) \leq \bigvee_{i\in\Gamma}\mu_i$. Since $\lambda = \bigvee_{x_t\in\lambda} x_t$, $F(\lambda) = F(\bigvee_{x_t\in\lambda} x_t) = \bigvee_{x_t\in\lambda} F(x_t) \leq \bigvee_{i\in\Gamma}\mu_i$. It follows that for each $x_t \in \lambda$, $F(x_t) \leq \bigvee_{i\in\Gamma}\mu_i$. Since F is compact-valued, then there exists finite subset Γ_{x_t} of Γ such that $F(x_t) \leq \bigvee_{n\in\Gamma_{x_t}}\mu_n = \mu_{x_t}$. Since $x_t \leq F^u(F(x_t)) \leq F^u(\mu_{x_t})$, we have $\lambda = \bigvee_{x_t\in\lambda} x_t \leq \bigvee_{x_t\in\lambda} F^u(\mu_{x_t})$. Since $\eta(\mu_{x_t}) \geq r$, then from Definition 3.1(1) we have $F^u(\mu_{x_t})$ is r-fuzzy ℓ-open. Hence $\{F^u(\mu_{x_t}) : F^u(\mu_{x_t})$ is r-fuzzy ℓ-open, $x_t \in \lambda\}$ is a family covering the set λ. Since λ is r-fuzzy ℓ-compact, then there exists finite index set N of Γ_{x_t} such that $\lambda \leq \bigvee_{n\in N} F^u(\mu_{x_{(t_n)}})$. Then, $F(\lambda) \leq F(\bigvee_{n\in N} F^u(\mu_{x_{(t_n)}})) = \bigvee_{n\in N} F(F^u(\mu_{x_{(t_n)}})) \leq \bigvee_{n\in N}\mu_{x_{(t_n)}}$. Thus, $F(\lambda)$ is r-fuzzy compact.

4. CONCLUSIONS

In the present work, a new forms of sets called r-fuzzy ℓ-open, r-fuzzy semi-ℓ-open, r-fuzzy pre-ℓ-open, r-fuzzy α-ℓ-open and r-fuzzy β-ℓ-open sets are introduced on a fuzzy ideal topological space (X,τ,ℓ) in Šostak sense. Also, the relations of these sets with each other are investigated with the help of examples. Moreover, the concepts of fuzzy upper (resp. lower) ℓ-continuous, almost ℓ-continuous and weakly ℓ-continuous multifunctions are introduced and some properties of these multifunctions along with their mutual relationships are specified.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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