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A FITTED METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS HAVING BOUNDARY LAYERS VIA DEVIATING ARGUMENT AND INTERPOLATION

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Abstract: In this paper**,** singularly perturbed differential-difference equation having boundary layers at one end (left or right) is considered. In order to obtain numerical solution to these problems, the given second order equation having boundary layer is converted into a singularly perturbed ordinary differential equation using Taylor's transformation afterwards the resultant singularly perturbed ordinary differential equation is replaced by an asymptotically equivalent to first order differential equation with a small deviating argument. Resulting first order differential equation, is solved by choosing the proper integrating factor (fitting factor) and linear interpolation formulas. The numerical results for several test examples demonstrate the applicability of the method.

Keywords: differential-difference equations; boundary layer; fitting factor; linear interpolation.

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1. INTRODUCTION

Singularly perturbed delay differential equation is a differential equation in which the highest order derivative is multiplied by a small parameter and involving a delay term. This type

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of equation arises frequently in the mathematical modelling of various practical phenomena for example: in the modelling of the human pupil-light reflex; model of HIV infection; the study of bi-stable devices in digital electronics; variational problem in control theory; first exit time problem in modelling of activation of neuronal variability; immune response; evolutionary biology; dynamics of networks of two identical amplifier; mathematical ecology; population dynamics; the modelling of biological oscillator and in a variety of models for physiological process. For a detailed theory and analytical discussion on delay differential equations having boundary layer one may refer the popular books by Bellman and Cooke [1], Driver [4], El'sgol'ts and Norkin [5], Hale [8], Nayfeh [17], O'Malley[18] and VanDyke[24]. Lange and Miura [12-13] are the first to discuss the behavior of the analytical solution of singularly perturbed differential difference equations. Chakravarthy and Reddy [3] presented an initial-value approach for solving singularly perturbed two-point boundary value problems. Fevzi Erdogan [6] described an exponentially fitted method for singularly perturbed delay differential equations. Gemechis and Reddy [7] discussed the numerical Integration of a class of Singularly Perturbed Delay Differential Equations with small shift. Kadalbajoo and Sharma [9-10] described the numerical treatment of a mathematical model arising from a model of neuronal variability. Lakshmi Sirisha and Reddy [11] presented a Fitted second order scheme for solving Singularly Perturbed Differential Difference Equations. Nageshwar Rao and Pramod Chakrravarthy [14-15] described a Fitted Numerov method for singularly perturbed parabolic partial differential equation with a small negative shift arising in control theory. Natesan and Bawa [16] presented a second order numerical scheme for singularly perturbed reaction- diffusion robin problems. Ravi Kanth and Murali [19] described a numerical approach for solving singularly perturbed convection delay problems via exponentially fitted spline method. Reddy and Awoke [20] discussed the solving singularly perturbed differential difference equations via fitted method. Reddy and Soujanya and Phaneendra [21] presented the Numerical integration method for singularly perturbed delay differential equations. Salama, A.A and Al-Amery [22] presented an

Asymptotic-numerical method for singularly perturbed differential difference equations of mixed-type. Soujanya, Reddy and K. Phaneendra [22] discussed the Numerical Solution of Singular Perturbation Problems via Deviating Argument and Exponential Fitting.

In this paper**,** singularly perturbed differential-difference equation having boundary layers at one end (left or right) is considered. In order to obtain numerical solution to these problems, the given second order equation having boundary layer is converted into a singularly perturbed ordinary differential equation using Taylor's transformation afterwards the resultant singularly perturbed ordinary differential equation is replaced by an asymptotically equivalent to first order differential equation with a small deviating argument. Resulting first order differential equation, is solved by choosing the proper integrating factor (fitting factor) and linear interpolation formulas. The numerical results for several test examples demonstrate the applicability of the method.

2. DESCRIPTION OF THE METHOD

2.1 Type-I: Delay Differential Equation having boundary layer

Consider the delay differential equation of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1, \tag{1}$$

with boundary conditions

$$y(x) = \varphi(x), \quad -\delta \le x \le 0, \tag{2}$$

and

$$y(1) = \beta, \tag{3}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ is the small delay parameter, $a(x)$, $b(x)$ and $f(x)$ are sufficiently differentiable functions in $(0, 1)$. $\varphi(x)$ is also bounded continuous function on $[0, 1]$ and β is a finite constant.

From the Taylor's series expansion

$$y'(x-\delta) \approx y'(x) - \delta y''(x). \tag{4}$$

Substituting Equation (4) into Equation (1), we get singularly perturbed ordinary differential equation:

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), \quad 0 \leq x \leq 1 \tag{5}$$

with boundary conditions

$$y(0) = \alpha \tag{6}$$

$$y(1) = \beta \tag{7}$$

where $\varepsilon' = \varepsilon - a(x)\delta$, $A(x) = a(x)$, $B(x) = b(x)$ and α is a finite constant. Further it is established that, when $a(x) \geq M > 0$ in [0, 1], boundary layer will be at $x = 0$ and when $a(x) \leq M < 0$ in [0, 1], boundary layer will be at $x = 1$, where M is some positive number. Since $0 < \delta \ll 1$, the transition from Equation (1) to Equation (5) is admitted. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [5]. Here we assume that $a(x) = a$ and $b(x) = b$ are constants.

2.2 Type-II: Differential-Difference Equation having boundary layer

Consider the differential-difference equation of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) + c(x)y(x) + d(x)y(x+\eta) = f(x), \tag{8}$$

$0 \leq x \leq 1$ with boundary conditions

$$y(x) = \varphi(x), \text{ on } -\delta \leq x \leq 0, \tag{9}$$

$$y(x) = \gamma(x), \text{ on } 1 \leq x \leq 1+\eta, \tag{10}$$

with the constant coefficients (i.e. $a(x) = a, b(x) = b, c(x) = c$ and $d(x) = d$ are constants) and $f(x)$, $\varphi(x)$ and $\gamma(x)$ are smooth functions. $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ and $0 < \eta = O(\varepsilon)$ are the delay and advanced parameters respectively.

From Taylor's series expansion

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x). \tag{11}$$

$$y(x+\eta) \approx y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x). \tag{12}$$

Substituting Equations (11)-(12) into Equation (8), we get singularly perturbed ordinary differential equation

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x),\ 0 \le x \le 1 \tag{13}$$

with boundary conditions

$$y(0) = \alpha \tag{14}$$

$$y(1) = \beta \tag{15}$$

where

$$\varepsilon' = \varepsilon + b(x)\frac{\delta^2}{2} + d(x)\frac{\eta^2}{2}, \tag{16}$$

$$A(x) = a(x) - \delta b(x) + \eta d(x), \tag{17}$$

$$B(x) = b(x) + c(x) + d(x), \tag{18}$$

Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$, the transition from Equation (8) to Equation (13) is admitted. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [5]. The behaviour of the boundary layer is given by the sign of $A(x)$ and $B(x)$. Further it is established that, if $B(x) \le 0,\ A(x) \ge M > 0$ in [0, 1] then Equation (8) has unique solution and a boundary layer at $x = 0$ and if $B(x) \le 0,\ A(x) \le M < 0$ in [0, 1] then Equation (8) has unique solution and a boundary layer at $x = 1$, where M is a positive number.

2.3. Case (i) : For Left-end boundary layer

Consider equation (5) or (13) with their boundary conditions

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x),\ 0 \le x \le 1 \tag{19}$$

$$y(0) = \alpha \tag{20}$$

$$y(1) = \beta \tag{21}$$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point x, we have

$$y\left(x - \sqrt{\varepsilon'}\right) \approx y(x) - \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x) \tag{22}$$

From equation (19) and (22), we have

$$y'(x) = p(x)y\left(x - \sqrt{\varepsilon'}\right) + q(x)y(x) + r(x) \tag{23}$$

where

$$p(x) = \frac{-2}{2\sqrt{\varepsilon'} + A(x)} \tag{24}$$

$$q(x) = \frac{2 - B(x)}{2\sqrt{\varepsilon'} + A(x)} \tag{25}$$

$$r(x) = \frac{f(x)}{2\sqrt{\varepsilon'} + A(x)} \tag{26}$$

The transition from equation (19) to (23) is valid, because of the condition that $\sqrt{\varepsilon'}$ is small. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [5].

Now, we divide the interval [0, 1] into n equal parts with constant mesh length $h = 1/n$.

Let $0 = x_0,\ x_1, \dots, x_n = 1$ be the mesh points, then we have $x_i = ih,\ i = 0, 1, 2, \dots, n$. From our earlier assumptions, $A(x)$ and $B(x)$ are constants. Therefore, $p(x)$ and $q(x)$ are constants. Equation (23) can be written as

$$y'(x) - qy(x) = py\left(x - \sqrt{\varepsilon'}\right) + r(x) \tag{27}$$

We take an integrating factor e^{-qx} to equation (27) and producing (as in B. J. McCartin [2])

$$\frac{d}{dx}\left[e^{-qx}y(x)\right] = e^{-qx}\left[py\left(x - \sqrt{\varepsilon'}\right) + r(x)\right] \tag{28}$$

On integrating equation (28) from x_i to x_{i+1}, we get

$$e^{-qx_{i+1}}y_{i+1} - e^{-qx_i}y_i = \int_{x_i}^{x_{i+1}} e^{-qx}\, py\left(x - \sqrt{\varepsilon'}\right)dx + \int_{x_i}^{x_{i+1}} e^{-qx}\, r(x)dx \tag{29}$$

Using the linear Newton's forward interpolation on $[x_i\ \ x_{i+1}]$, which we insert into the above equation, we get

$$\begin{aligned} e^{-qx_{i+1}}y_{i+1} &= e^{-qx_i}y_i \\ &\quad + p\int_{x_i}^{x_{i+1}} e^{-qx}\left[y\left(x_i - \sqrt{\varepsilon'}\right) + \frac{(x - x_i)}{h}\left\{y\left(x_{i+1} - \sqrt{\varepsilon'}\right) - y\left(x_i - \sqrt{\varepsilon'}\right)\right\}\right]dx \\ &\quad + \int_{x_i}^{x_{i+1}} e^{-qx}\left[r_i + \frac{(x - x_i)}{h}\{r_{i+1} - r_i\}\right]dx \end{aligned} \tag{30}$$

$$y_{i+1} = e^{qh} y_i + py\left(x_i - \sqrt{\varepsilon'}\right) \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, dx + \frac{py\left(x_{i+1} - \sqrt{\varepsilon'}\right)}{h} \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, (x - x_i) dx$$
$$+ \frac{py\left(x_i - \sqrt{\varepsilon'}\right)}{h} \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, (x_i - x) dx + r_i \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, dx$$
$$+ \frac{r_{i+1}}{h} \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, (x - x_i) dx$$
$$+ \frac{r_i}{h} \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, (x_i - x) dx \qquad (31)$$

After evaluating the integrals involves in equation (31), we get

$$y_{i+1} = e^{qh} y_i + J\left[py\left(x_i - \sqrt{\varepsilon'}\right) + r_i\right] + K\left[\frac{py\left(x_{i+1} - \sqrt{\varepsilon'}\right)}{h} + \frac{r_{i+1}}{h}\right]$$
$$+ L\left[\frac{py\left(x_i - \sqrt{\varepsilon'}\right)}{h} + \frac{r_i}{h}\right] \qquad (32)$$

where

$$J = \frac{e^{qh}}{q} - \frac{1}{q} \qquad (33)$$

$$K = -\frac{h}{q} - \frac{1}{q^2} + \frac{e^{qh}}{q^2} \qquad (34)$$

$$L = \frac{h}{q} + \frac{1}{q^2} - \frac{e^{qh}}{q^2} \qquad (35)$$

From finite difference approximation, we have

$$y\left(x_i - \sqrt{\varepsilon'}\right) \approx \left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) y_i + \frac{\sqrt{\varepsilon'}}{h} y_{i-1} \qquad (36)$$

$$y\left(x_{i+1} - \sqrt{\varepsilon'}\right) \approx \left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) y_{i+1} + \frac{\sqrt{\varepsilon'}}{h} y_i \qquad (37)$$

From equation (36) and equation (37), equation (32) becomes

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,2, \dots, n-1 \qquad (38)$$

where

$$E_i = -\frac{Jp\sqrt{\varepsilon'}}{h} - \frac{Lp\sqrt{\varepsilon'}}{h^2}$$

$$F_i = Jp\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + \frac{Kp\sqrt{\varepsilon'}}{h^2} + \frac{Lp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + e^{qh}$$

$$G_i = 1 - \frac{Kp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)$$

$$H_i = Jr_i + \frac{Kr_{i+1}}{h} + \frac{Lr_i}{h}$$

This is a tridiagonal system of $n-1$ equations. We solve this tridiagonal system with given two boundary conditions by Thomas algorithm.

2.4. Case (ii) : For Right-end boundary layer

Consider equation (5) or (13) with their boundary conditions

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x),\ 0 \le x \le 1 \tag{39}$$

$$y(0) = \alpha \tag{40}$$

$$y(1) = \beta \tag{41}$$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point x, we have

$$y\left(x + \sqrt{\varepsilon'}\right) \approx y(x) + \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x) \tag{42}$$

From equation (39) and (42), we have

$$y'(x) = p(x)y\left(x + \sqrt{\varepsilon'}\right) + q(x)y(x) + r(x) \tag{43}$$

where

$$p(x) = \frac{-2}{-2\sqrt{\varepsilon'} + A(x)} \tag{44}$$

$$q(x) = \frac{2 - B(x)}{-2\sqrt{\varepsilon'} + A(x)} \tag{45}$$

$$r(x) = \frac{f(x)}{-2\sqrt{\varepsilon'} + A(x)} \tag{46}$$

The transition from equation (39) to (43) is valid, because of the condition that $\sqrt{\varepsilon'}$ is small. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [5].

Now, we divide the interval [0, 1] into n equal parts with constant mesh length $h = 1/n$.

Let $0 = x_0, x_1, \dots, x_n = 1$ be the mesh points, then we have $x_i = ih$, $i = 0, 1, 2, \dots, n$. From our earlier assumptions, $A(x)$ and $B(x)$ are constants. Therefore, $p(x)$ and $q(x)$ are constants. Equation (43) can be written as

$$y'(x) - qy(x) = py\left(x + \sqrt{\varepsilon'}\right) + r(x) \tag{47}$$

We take an integrating factor e^{-qx} to equation (47) and producing (as in B. J. McCartin[2])

$$\frac{d}{dx}\left[e^{-qx}y(x)\right] = e^{-qx}\left[py\left(x + \sqrt{\varepsilon'}\right) + r(x)\right] \tag{48}$$

On integrating equation (48) from x_{i-1} to x_i, we get

$$e^{-qx_i}y_i - e^{-qx_{i-1}}y_{i-1} = \int_{x_{i-1}}^{x_i} e^{-qx}\, py\left(x + \sqrt{\varepsilon'}\right)dx + \int_{x_{i-1}}^{x_i} e^{-qx}\, r(x)dx \tag{49}$$

Using the linear Newton's backward interpolation on $[x_{i-1}\ \ x_i]$, which we insert into the above equation, we get

$$\begin{aligned} e^{-qx_i}y_i &= e^{-qx_{i-1}}y_{i-1} \\ &\quad + p\int_{x_{i-1}}^{x_i} e^{-qx}\left[y\left(x_i + \sqrt{\varepsilon'}\right) + \frac{(x - x_i)}{h}\left\{y\left(x_i + \sqrt{\varepsilon'}\right) - y\left(x_{i-1} + \sqrt{\varepsilon'}\right)\right\}\right]dx \\ &\quad + \int_{x_{i-1}}^{x_i} e^{-qx}\left[r_i + \frac{(x - x_i)}{h}\{r_i - r_{i-1}\}\right]dx \end{aligned} \tag{50}$$

$$\begin{aligned} y_i &= e^{qh}y_{i-1} + py\left(x_i + \sqrt{\varepsilon'}\right)\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\, dx + \frac{py\left(x_i + \sqrt{\varepsilon'}\right)}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\,(x - x_i)dx \\ &\quad + \frac{py\left(x_{i-1} + \sqrt{\varepsilon'}\right)}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\,(x_i - x)dx + r_i\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\, dx \\ &\quad + \frac{r_i}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\,(x - x_i)dx + \frac{r_{i-1}}{h}\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\,(x_i - x)dx \end{aligned} \tag{51}$$

After evaluating the integrals involves in equation (31), we get

$$y_i = e^{qh} y_{i-1} + J\left[py\left(x_i + \sqrt{\varepsilon'}\right) + r_i\right] + K\left[\frac{py\left(x_i + \sqrt{\varepsilon'}\right)}{h} + \frac{r_i}{h}\right] + L\left[\frac{py\left(x_{i-1} + \sqrt{\varepsilon'}\right)}{h} + \frac{r_{i-1}}{h}\right] \tag{52}$$

where

$$J = \frac{e^{qh}}{q} - \frac{1}{q} \tag{53}$$

$$K = -\frac{he^{qh}}{q} - \frac{1}{q^2} + \frac{e^{qh}}{q^2} \tag{54}$$

$$L = \frac{he^{qh}}{q} + \frac{1}{q^2} - \frac{e^{qh}}{q^2} \tag{55}$$

From finite difference approximation, we have

$$y\left(x_i + \sqrt{\varepsilon'}\right) \approx \left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) y_i + \frac{\sqrt{\varepsilon'}}{h} y_{i+1} \tag{56}$$

$$y\left(x_{i-1} + \sqrt{\varepsilon'}\right) \approx \left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) y_{i-1} + \frac{\sqrt{\varepsilon'}}{h} y_i \tag{57}$$

From equation (56) and equation (57), equation (52) becomes

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,2,\dots,n-1 \tag{58}$$

where

$$E_i = -e^{qh} - \frac{Lp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)$$

$$F_i = -1 + Jp\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) + \frac{Lp\sqrt{\varepsilon'}}{h^2} + \frac{Kp}{h}\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)$$

$$G_i = -\frac{Jp\sqrt{\varepsilon'}}{h} - \frac{Kp\sqrt{\varepsilon'}}{h^2}$$

$$H_i = Jr_i + \frac{Kr_i}{h} + \frac{Lr_{i-1}}{h}$$

This is a tridiagonal system of $n-1$ equations. We solve this tridiagonal system with given two boundary conditions by Thomas algorithm.

3. NUMERICAL EXPERIMENTS

In this section, six model examples are solved and the solutions are compared with the exact/available solutions. The exact solution of equation (8) is given by (with assumptions $f(x) = f$, $\varphi(x) = \varphi$ and $\gamma(x) = \gamma$ are constant)

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + f/c' \tag{59}$$

where

$$c' = b + c + d$$

$$m_1 = \left[-(a - \delta b + \eta d) + \sqrt{(a - \delta b + \eta d)^2 - 4\varepsilon c'}\right]/2\,\varepsilon$$

$$m_2 = \left[-(a - \delta b + \eta d) - \sqrt{(a - \delta b + \eta d)^2 - 4\varepsilon c'}\right]/2\,\varepsilon$$

$$c_1 = [-f + \gamma c' + e^{m_2}(f - \varphi c')]/[(e^{m_1} - e^{m_2})c']$$

$$c_2 = [f - \gamma c' + e^{m_1}(-f + \varphi c')]/[(e^{m_1} - e^{m_2})c']$$

Example 1. Consider the delay differential equation having left boundary layer:

$$\varepsilon y''(x) + y'(x - \delta) - y(x) = 0,\ \ 0 \le x \le 1;\ \text{ with } y(0) = 1 \text{ and } y(1) = 1.$$

The exact solution is given by

$$y = ((1 - e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x})/(e^{m_1} - e^{m_2})$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)} \text{ and } m_2 = \frac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$$

The computational results are shown in table-1 & 2, the layer behaviour in fig. 1 & 2 for different values of parameters.

Example 2. Consider the differential-differential equation having left boundary layer:

$$\varepsilon y''(x) + y'(x) - 2y(x - \delta) - 5y(x) + y(x + \eta) = 0, \qquad 0 \le x \le 1;$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 1.$$

The exact solution is given by equation (59) and computational results are shown in table-3 & 4 and the layer behaviour in fig. 3 & 4 for different values of parameters.

Example 3. Consider the differential-differential equation having left boundary layer:

$$\varepsilon y''(x) + y'(x) - 3y(x) + 2y(x+\eta) = 0,\ 0 \le x \le 1; \text{with } y(0) = 1 \text{ and } y(1) = 1.$$

The exact solution is given by equation (59) and computational results are shown in table-5 & 6, the layer behaviour in fig. 5 & 6 for different values of parameters.

Example 4. Now we consider the delay differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0,\ 0 \le x \le 1;\ \text{with } y(0) = 1 \text{ and } y(1) = -1.$$

The exact solution is given by

$$y = ((1+e^{m_2})e^{m_1 x} - (e^{m_1}+1)e^{m_2 x})/(e^{m_2} - e^{m_1})$$

where

$$m_1 = \frac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)} \text{ and } m_2 = \frac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$$

The computational results are shown in table-7 & 8, the layer behaviour in fig. 7 & 8 for different values of parameters.

Example 5. Consider the differential-differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x) - 2y(x-\delta) + y(x) - 2y(x+\eta) = 0,\ 0 \le x \le 1; \text{with } y(0) = 1 \text{ and } y(1) = -1.$$

The exact solution is given by equation (59) and computational results are shown in table-9 & 10, the layer behaviour in fig. 9 & 10 for different values of parameters.

Example 6. Consider the differential-differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x) + y(x) - 2y(x+\eta) = 0,\ 0 \le x \le 1; \text{with } y(0) = 1 \text{ and } y(1) = -1.$$

The exact solution is given by equation (59) and computational results are shown in table-11 & 12, the layer behaviour in fig. 11 & 12 for different values of parameters.

4. DISCUSSION AND CONCLUSIONS

An exponentially fitted numerical method is described to solve the singularly perturbed differential-difference equations having boundary layers at one end of the domain. To develop this method deviating argument and Newton's interpolation concepts are used. Discrete approximation is taken on equidistant mesh for the differential-difference equations. Moreover, the method is very simple and easy to implement on considered problems. Efficiency of the method is proved by numerical experiment and also by comparing the results with the available exact solution of the problem. Method is implemented on six standard test examples and it is observed that the numerical solutions approximate the exact solution very well. Computational results and layer behaviour are presented in the tables and figures and for different values of the parameters. The applicability of this method is demonstrated by solving popular model problems and the numerical results are well compared with exact solution. It is observed from the tables and figures that present method agrees with exact solution very well,

Table-1: Results for Example-1 with $h = 0.01, \varepsilon = 0.01$ and $\delta = 0.001$

x	Present Solution/ Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [7] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.38039196	0.42885272	0.37530590
0.04	0.38328086	0.38987298	0.38287105
0.06	0.39098031	0.39385789	0.39060602
0.08	0.39887072	0.40144299	0.39849726
0.1	0.40692064	0.40946339	0.40654792
0.2	0.44967374	0.45216849	0.44930761
0.3	0.49691869	0.49933011	0.49656465
0.4	0.54912744	0.55141074	0.54879207
0.5	0.60682149	0.60892343	0.60651264
0.6	0.67057717	0.67243474	0.67030411
0.7	0.74103133	0.74257036	0.74080501
0.8	0.81888776	0.82002118	0.81872102
0.9	0.90492417	0.90555021	0.90483204
1.0	1.00000000	1.00000000	1.00000000

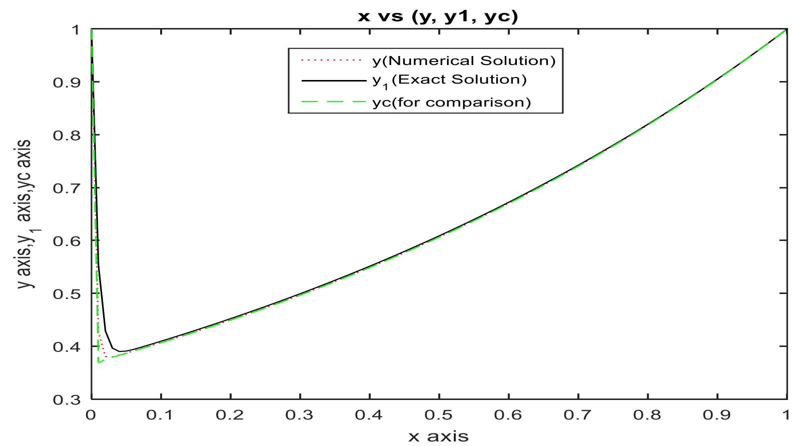

Figure-1: Example-1 with $h = 0.01, \varepsilon = 0.01$ and $\delta = 0.001$

Table-2: Results for Example-1 with $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0001$

x	Present Solution	Exact Solution	Result by [7]
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37596249	0.37560498	0.37562175
0.04	0.38300913	0.38318659	0.38296304
0.06	0.39074350	0.39092122	0.39069782
0.08	0.39863453	0.39881199	0.39858892
0.1	0.40668492	0.40686202	0.40663940
0.2	0.44944219	0.44961616	0.44939748
0.3	0.49669480	0.49686302	0.49665156
0.4	0.54891536	0.54907470	0.54887440
0.5	0.60662618	0.60677293	0.60658846
0.6	0.67040450	0.67053424	0.67037115
0.7	0.74088822	0.74099575	0.74086058
0.8	0.81878233	0.81886155	0.81876196
0.9	0.90486591	0.90490969	0.90485466
1.0	1.00000000	1.00000000	1.00000000

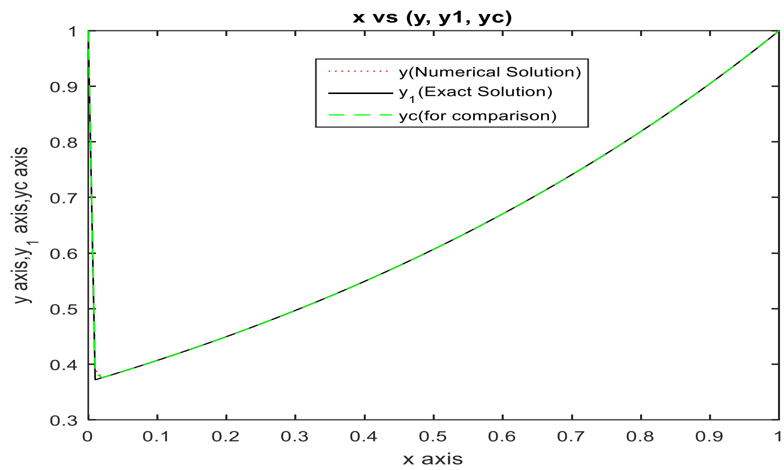

Figure-2: Example-1 with $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0001$

Table-3: Results for Example 2 with $h = 0.01, \varepsilon = 0.01,\ \delta = 0.001$ and $\eta = 0.005$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.02	0.01080579	0.12303611	0.12573638
0.04	0.00345099	0.01867136	0.01924308
0.06	0.00381696	0.00667501	0.00668484
0.08	0.00429653	0.00577240	0.00566896
0.1	0.00483695	0.00625879	0.00612688
0.2	0.00874664	0.01096036	0.01074933
0.3	0.01581654	0.01926844	0.01894343
0.4	0.02860100	0.03387416	0.03338381
0.5	0.05171911	0.05955119	0.05883196
0.6	0.09352352	0.10469173	0.10367897
0.7	0.16911831	0.18404935	0.18271239
0.8	0.30581616	0.32356102	0.32199219
0.9	0.55300647	0.56882424	0.56744355
1.0	1.00000000	1.00000000	1.00000000

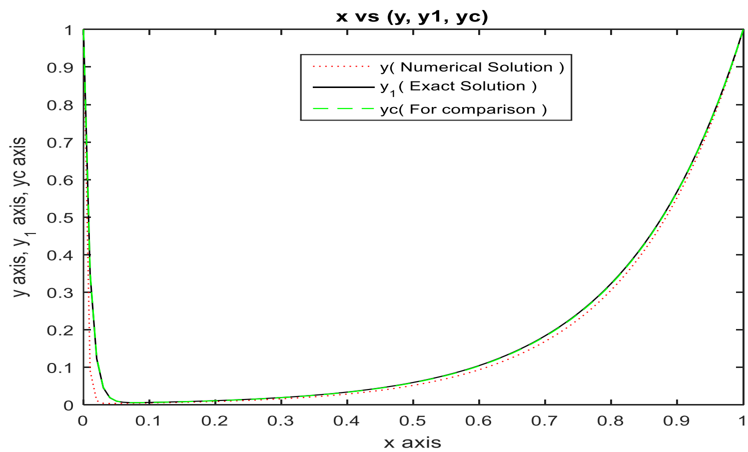

Figure-3: Example 2 with $h = 0.01, \varepsilon = 0.01,\ \delta = 0.001$ and $\eta = 0.005$

Table-4: Results for Example-2 with $h = 0.01, \varepsilon = 0.001,\ \delta = 0.0001$ and $\eta = 0.0005$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.02	0.00374013	0.00290566	0.00427314
0.04	0.00320002	0.00327352	0.00209563
0.06	0.00360595	0.00368796	0.00237638
0.08	0.00406442	0.00415486	0.00270237
0.1	0.00458117	0.00468088	0.00307310
0.2	0.00833428	0.00849532	0.00584433
0.3	0.01516212	0.01541816	0.01111458
0.4	0.02758363	0.02798241	0.02113738
0.5	0.05018141	0.05078525	0.04019845
0.6	0.09692223	0.09217010	0.07644823
0.7	0.16608320	0.16727944	0.14538701
0.8	0.30214620	0.30359530	0.27649274
0.9	0.54967827	0.55099482	0.52582577
1.0	1.00000000	1.00000000	1.00000000

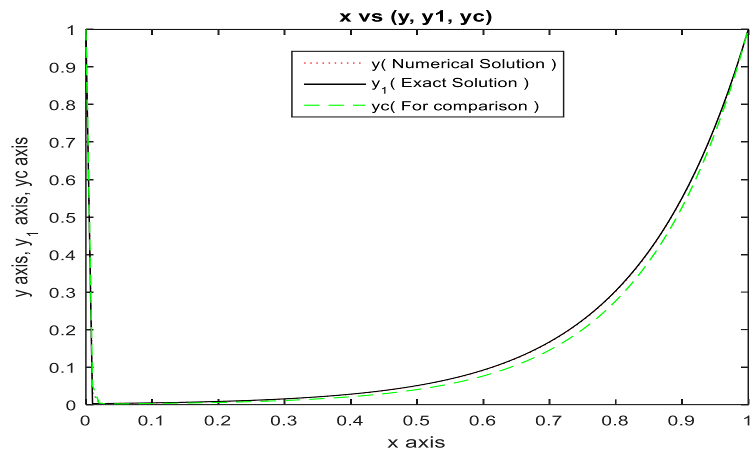

Figure-4: Example-2 with $h = 0.01, \varepsilon = 0.001,\ \delta = 0.0001$ and $\eta = 0.0005$

Table-5: Results for Example-3 with $h = 0.01, \varepsilon = 0.01,$ and $\eta = 0.005$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.02	0.38442760	0.46422815	0.46428771
0.04	0.38695520	0.40078807	0.40057622
0.06	0.39464452	0.39922966	0.39894603
0.08	0.40252891	0.40589482	0.40559750
0.1	0.41057116	0.41377113	0.41347202
0.2	0.45325780	0.45637318	0.45607943
0.3	0.50038251	0.50339060	0.50310708
0.4	0.55240673	0.55525194	0.55498387
0.5	0.60983985	0.61245625	0.61220984
0.6	0.67324423	0.67555398	0.67533653
0.7	0.74324069	0.74515229	0.74497240
0.8	0.82051461	0.82192091	0.82178862
0.9	0.90582261	0.90659853	0.90652557
1.0	1.00000000	1.00000000	1.00000000

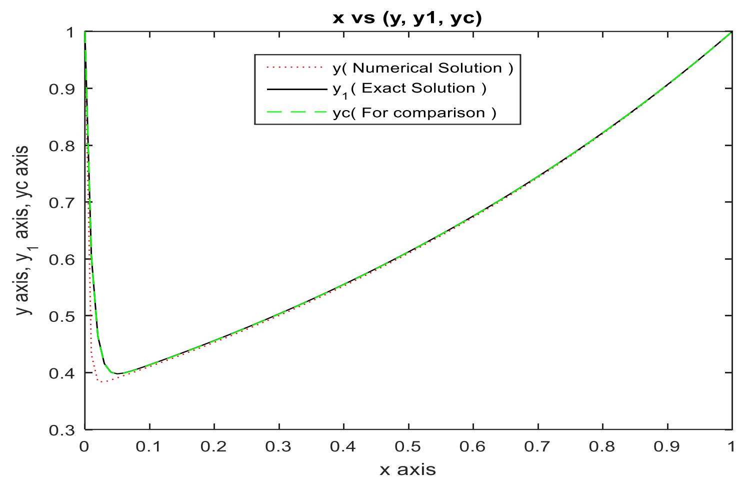

Figure-5: Example-3 with $h = 0.01, \varepsilon = 0.01$, and $\eta = 0.005$

Table-6: Results for Example-3 with $h = 0.01, \varepsilon = 0.001$, and $\eta = 0.0005$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37639205	0.37604531	0.37143446
0.04	0.38338231	0.38362663	0.37897379
0.06	0.39111619	0.39136079	0.38671244
0.08	0.39900665	0.39925088	0.39460912
0.1	0.40705630	0.40730004	0.40266706
0.2	0.44980699	0.45004640	0.44549309
0.3	0.49704754	0.49727901	0.49287394
0.4	0.54924948	0.54946871	0.54529402
0.5	0.60693387	0.60713575	0.60328928
0.6	0.67067652	0.67085497	0.66745269
0.7	0.74111367	0.74126157	0.73844026
0.8	0.81894842	0.81905737	0.81697778
0.9	0.90495769	0.90501788	0.90386823
1.0	1.00000000	1.00000000	1.00000000

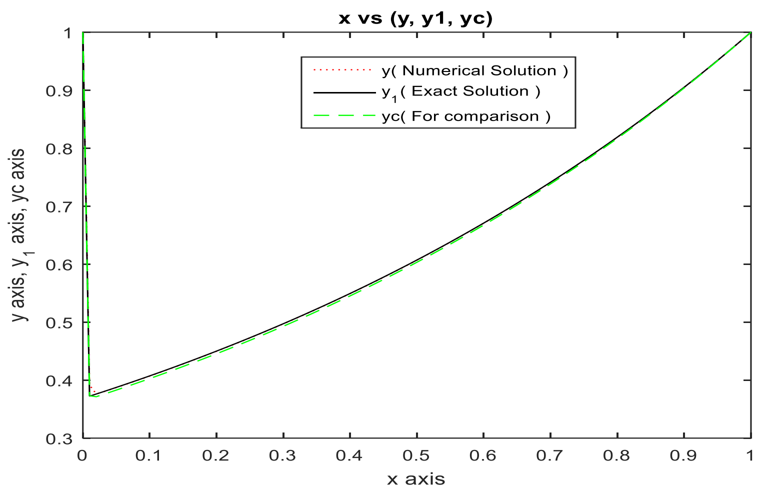

Figure-6: Example-3 with $h = 0.01, \varepsilon = 0.001,$ and $\eta = 0.0005$

Table-7: Results for Example-4 with $h = 0.01, \varepsilon = 0.01$ and $\delta = 0.001$

x	Present Solution	Exact Solution	Result by [7]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90493315	0.90589854	0.90413410
0.2	0.81890401	0.82065216	0.81745848
0.3	0.74105339	0.74342760	0.73909209
0.4	0.67060378	0.67346998	0.66823837
0.5	0.60685160	0.61009547	0.60417710
0.6	0.54916013	0.55268459	0.54625713
0.7	0.49695321	0.50067617	0.49388970
0.8	0.44970944	0.45356174	0.44654252
0.9	0.40695698	0.41058218	0.40373430
0.92	0.39890712	0.40122583	0.39567734
0.94	0.39101552	0.38624162	0.38774960
0.96	0.38317166	0.34017030	0.37889378
0.98	0.36341417	0.12554105	0.33281495
1.0	-1.00000000	-1.00000000	-1.00000000

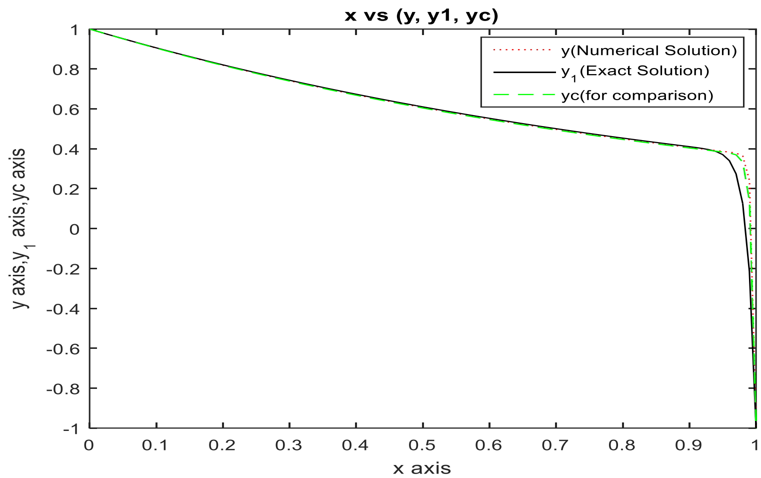

Figure-7: Example-4 with $h = 0.01, \varepsilon = 0.01$ and $\delta = 0.001$

Table-8: Results for Example-4 with $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0005$

x	Present Solution	Exact Solution	Result by [7]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90487376	0.90501768	0.90361600
0.2	0.81879653	0.81905700	0.81652188
0.3	0.74090750	0.74126107	0.73782223
0.4	0.67042776	0.67085438	0.66670798
0.5	0.60665249	0.60713507	0.60244800
0.6	0.54894393	0.54946798	0.54438165
0.7	0.49672496	0.49727823	0.49191197
0.8	0.44947338	0.45004559	0.44449953
0.9	0.40671667	0.40729922	0.40165688
0.92	0.39866635	0.39925006	0.39359720
0.94	0.39077536	0.39135997	0.38569882
0.96	0.38303793	0.38362581	0.37789776
0.98	0.37355981	0.37598358	0.36118295
1.0	-1.00000000	-1.00000000	-1.00000000

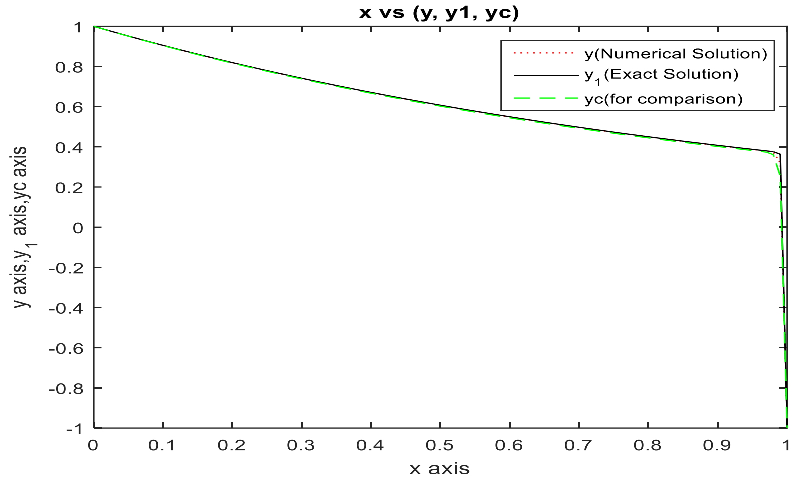

Figure-8: Example-4 with $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0005$

Table-9: Results for Example-5 with $h = 0.01, \varepsilon = 0.01,\ \delta = 0.001$ and $\eta = 0.005$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.74323362	0.74876089	0.74825389
0.2	0.55239621	0.56064288	0.55988388
0.3	0.41055943	0.41978746	0.41893529
0.4	0.30514157	0.31432044	0.31346996
0.5	0.22679147	0.23535085	0.23455512
0.6	0.16855905	0.17622151	0.17550678
0.7	0.12527875	0.13194778	0.13132363
0.8	0.09311138	0.09879733	0.09826341
0.9	0.06920350	0.07394332	0.07349177
0.92	0.06521585	0.06955834	0.06911225
0.94	0.06145744	0.06382765	0.06333588
0.96	0.05784989	0.04568524	0.04489425
0.98	0.04620247	-0.07327602	-0.07518285
1.0	-1.00000000	-1.00000000	-1.00000000

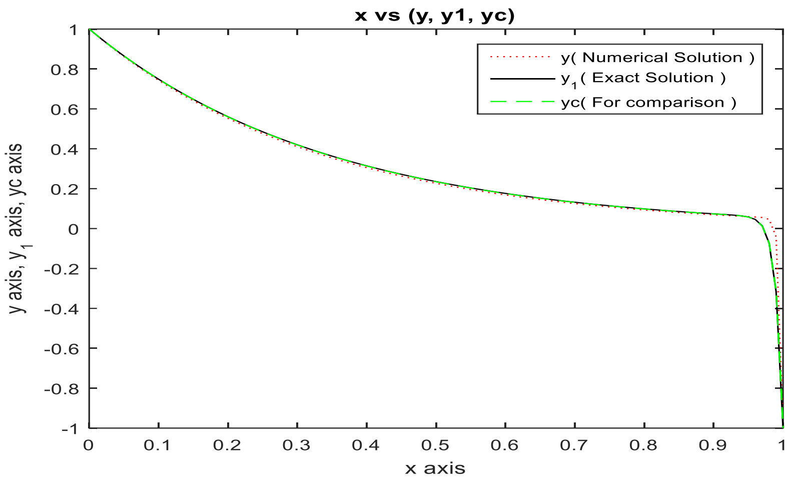

Figure-9: Example-5 with $h = 0.01, \varepsilon = 0.01,\ \delta = 0.001$ and $\eta = 0.005$

Table-10: Results for Example-5 with $h = 0.01, \varepsilon = 0.002,\ \delta = 0.0002$ and $\eta = 0.0006$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.74129930	0.74231169	0.73886435
0.2	0.54952466	0.55102664	0.54592053
0.3	0.40736225	0.40903352	0.40336121
0.4	0.30197735	0.30363036	0.29802922
0.5	0.22385560	0.22538837	0.22020317
0.6	0.16594400	0.16730842	0.16270027
0.7	0.12301417	0.12419499	0.12021343
0.8	0.09119032	0.09219140	0.08882141
0.9	0.06759932	0.06843475	0.06562698
0.92	0.06367091	0.06447538	0.06177252
0.94	0.05997078	0.06074509	0.05814445
0.96	0.05648230	0.05723062	0.05472934
0.98	0.05131385	0.05387498	0.05114502
1.0	-1.00000000	-1.00000000	-1.00000000

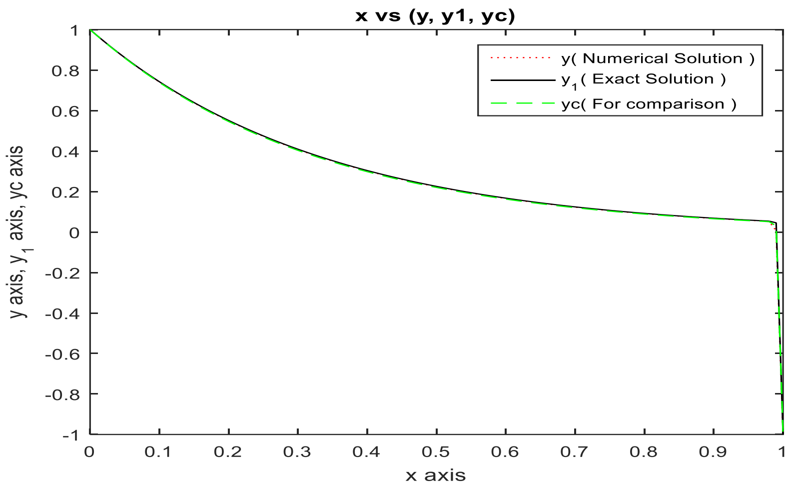

Figure-10: Example-5 with $h = 0.01, \varepsilon = 0.002,\ \delta = 0.0002$ and $\eta = 0.0006$

Table-11: Results for Example-6 with $h = 0.01, \varepsilon = 0.01,$ and $\eta = 0.005$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90582240	0.90659430	0.90652095
0.2	0.82051422	0.82191323	0.82178023
0.3	0.74324016	0.74514185	0.74496100
0.4	0.67324358	0.67554136	0.67532275
0.5	0.60983912	0.61244195	0.61219422
0.6	0.55240594	0.55523639	0.55496689
0.7	0.50038167	0.50337415	0.50308911
0.8	0.45325693	0.45635613	0.45606082
0.9	0.41057028	0.41367994	0.41337777
0.92	0.40252801	0.40530903	0.40500104
0.94	0.39464257	0.39483488	0.39449733
0.96	0.38682364	0.36705702	0.36657544
0.98	0.36826127	0.20454728	0.20357484
1.0	-1.00000000	-1.00000000	-1.00000000

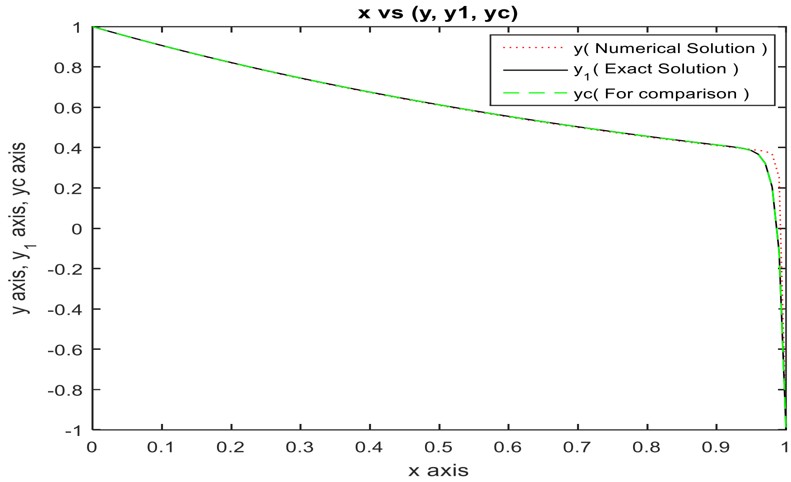

Figure-11: Example-6 with $h = 0.01, \varepsilon = 0.01, \text{and } \eta = 0.005$

Table-12: Results for Example-6 with $h = 0.01, \varepsilon = 0.002, \text{and } \eta = 0.0007$

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90500548	0.90514342	0.90467003
0.2	0.81903493	0.81928461	0.81842787
0.3	0.74123110	0.74157007	0.74040717
0.4	0.67081822	0.67122727	0.66982417
0.5	0.60709417	0.60755695	0.60596986
0.6	0.54942355	0.54992617	0.54820277
0.7	0.49723133	0.49776206	0.49594262
0.8	0.44999708	0.45054605	0.44866443
0.9	0.40724983	0.40780879	0.40589326
0.92	0.39920057	0.39976067	0.39784132
0.94	0.39131040	0.39187138	0.38994910
0.96	0.38357163	0.38413778	0.38221343
0.98	0.37349850	0.37649687	0.37447299
1.0	-1.00000000	-1.00000000	-1.00000000

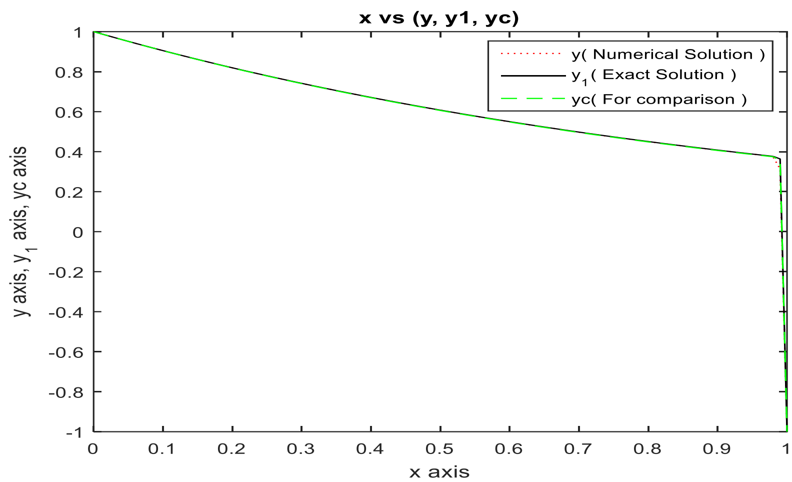

Figure-12: Example-6 with $h = 0.01, \varepsilon = 0.002,$ and $\eta = 0.0007$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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