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A MULTI-REGION DISCRETE-TIME MATHEMATICAL MODELING AND OPTIMAL CONTROL OF AN ELECTORAL BEHAVIOR

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Abstract. We study in this work, a discrete mathematical model that describes the dynamics of citizens who have the right to vote and their electoral behavior with regard to a political party during an awareness program in many regions whose individuals have a mutual influence. Also, we propose an optimal strategy for an awareness program or an election campaign that helps politicians to distinguish between different regions and categories of voters in order to increase the registered rate in the electoral process in many regions and obtain the greatest possible number of votes with a minimal effort. Pontryagin's maximum principle, in discrete time, is used to characterize the optimal controls and the optimality system is solved by an iterative method. The numerical simulation is carried out using Matlab. Consequently, the obtained results confirm the performance of the optimization strategy.

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1. INTRODUCTION

Political participation is one of the most important elements of modern democracy in all countries around the world. This political participation can be official through parties that occupy official positions or can be non-official through voting by individuals and those two forms of participation both achieve the common good. The electoral process is a key element in political participation. It allows the citizen to choose the best political party and thus contribute to the political decision-making in the country. The benefit of such political participation is to ensure the peaceful transfer of power and to ensure that the ruling elites obtain a legitimate and democratic mandate from the people on a regular and periodic basis through a free and fair electoral process that results in political, economic and social stability.

The high percentage of citizens registered and voting in the electoral process indicates that the majority of people are ready to participate in the elections, whether parliamentary or local. It also indicates that a large group of people is convinced of the possibility of change and reform through political participation. But when these rates are low, it often means that there are some administrative or cultural barriers that make registration in electoral lists and voting very difficult. These low rates can also indicate the presence of behavior from the political boycott and non-registration in the electoral rolls due to electoral and political corruption or lack of citizens' confidence in the political class. Consequently, countries, especially the developing ones, that seek to improve their image and political position, are in urgent need of improving many democratic indicators, including increasing the rate of registration in the electoral lists and voting during the electoral process. There are several mathematical modeling studies that have emerged which aim at understanding the electoral process and describing the dynamics of political systems. Some studies have used a statistical approach, while others have followed an epidemiological approach. In epidemiology, in general, the compartment model is used to describe the spread of infectious diseases. In this model, the population is divided into different classes according to the status of individuals with respect to the disease (people at risk of infection, infected persons or people who have recovered from these diseases). Infection transmits by contact between individuals. Similarly, during the electoral process, the population is divided into several classes (voters, non-voters, potential voters, majority party voters, and opposition party voters). Moreover, interpersonal interaction is also a key factor in the voting process which can result in conviction in party programs and conviction in voting for a particular party or a party opposing it. This influence occurs because people who register on the electoral lists or support a particular party often affect their social network (family, friends, tribes, group membership, etc.) to vote for a party. This contact is also made by political parties in a direct or indirect way through media, social media or direct election campaigns with the public so as to persuade them to vote for a particular party in itself or its candidate. The aforementioned influence is very similar to the phenomenon of infection. So, this epidemiological approach is more appropriate for modeling the electoral process. In this work, we propose a discrete mathematical model that describes the dynamics of citizens who have the right to vote and their electoral behavior with regard to a political party during an awareness programs in electoral campaigns. Also, we propose an optimal strategy for an awareness program in election campaigns that helps politicians to distinguish between different categories of voters in order to increase the participation rate in the electoral process and obtain the greatest possible number of votes with a minimal effort. In order to achieve these objectives, we applied the theory of optimal control of dynamic systems ,especially, the discrete-time models described by difference equations. The latter have many advantages, for example, the statistic data is collected at discrete times, the use of discrete-time models may avoid some mathematical complexities such as the choice of a function space and regularity of the solution, the numerical exploration of discrete-time models is rather straightforward and therefore can be easily implemented by non-mathematicians. (see,[18],[19],[20],[22],[23]and the references cited therein).

The electoral phenomenon differs from one region to another and from one area to another.This is mainly due to the specificity and characteristics of each region. This difference is clearly apparent between villages and cities and happens basically due to the popularity of political parties according to each region and area. The aim of this study is to highlight the impact of awareness campaigns and election campaigns on persuasion and influence on individuals who are unregistered in the electoral lists. Also, to support a political party in each region separately, then measure the mutual effect between two neighboring regions, especially, in the underdeveloped and developing countries. To this end, we will develop the previous model taking into account the evolution of the variables (The potential electors; The registered; The unregistered; The temporary abstainers, The voters for the political party; The voters against the political party.) by regions and areas. We note that this subject of multi region or multi groupes models has beeig widley studied by many researchers for exemple [1],[3],[4],[27].

The paper is organized as follows. In Section 2, we present our multi-regions discrete mathematical model that describes the dynamics of citizens who have the right to vote and their electoral behavior. In Section 3, we present the optimal control problem for the proposed model and we characterize these optimal controls using Pontryagin's maximum principle in discrete time. Numerical simulations and discussion are given in Section 4. Finally, we conclude the paper in Section 5.

2. MULTI-REGIONS DISCRETE MATHEMATICAL MODEL

We consider a mathematical model $PRUA^tV^fV^a$ which is based on a multi-regions discrete-time that describes the spatial-temporal dynamics of citizens who have the right to vote and their electoral behavior with regard to a political party. We propose the global area of interest Ω represented by the geographic union of the regions Ω_j which are the fields of this study. These regions are supposed to be connected by the movements of their populations and represent sub-domains of Ω_j or regions which represent a country, a city, a town, or a small domain. Let $\Omega = \cup_{j=1}^{s}\Omega_j$, and let $N_i^{\Omega_j}$ be the citizens of domain Ω_j at time i, i.e., the number of residents are physically present in Ω_j.

The $P-R-U-A^t-V^f-V^a$ be the number of citizens associated to the domain Ω_j are noted by the states P^{Ω_j}, R^{Ω_j},U^{Ω_j},$A^{t\,\Omega_j}$, $V^{f\,\Omega_j}$ and $V^{a\,\Omega_j}$ and we note that the transmission of the idea of voting takes place through the contact between individuals or they are convinced by this idea through media and government declarations. The unit of i can correspond to years or periods of the electoral campaigns. The following system describes a multi-regions discrete-time of the population dynamics who have the right to vote and their electoral behavior with regard to a political party:

$$
(1)\quad \begin{cases}
P_{i+1}^{\Omega_j} = \Lambda^{\Omega_j} + (1-\mu^{\Omega_j} - \alpha_1^{\Omega_j} - \alpha_2^{\Omega_j})P_i^{\Omega_j} \\
R_{i+1}^{\Omega_j} = (1-\mu^{\Omega_j} - \beta^{\Omega_j})R_i^{\Omega_j} + \alpha_1^{\Omega_j}P_i^{\Omega_j} + \theta^{\Omega_j}U_i^{\Omega_j} - \sum\limits_{q=1}^{s} \gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{f\,\Omega_q}}{N_i^{\Omega_j}} - \sum\limits_{q=1}^{s} \gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} \\
U_{i+1}^{\Omega_j} = (1-\mu^{\Omega_j} - \theta^{\Omega_j})U_i^{\Omega_j} + \alpha_2^{\Omega_j}P_i^{\Omega_j} \\
A_{i+1}^{t\,\Omega_j} = (1-\mu^{\Omega_j} - a^{\Omega_j} - c^{\Omega_j})A_i^{t\,\Omega_j} + \beta^{\Omega_j}R_i^{\Omega_j} + b^{\Omega_j}V_i^{f\,\Omega_j} + d^{\Omega_j}V_i^{a\,\Omega_j} \\
V_{i+1}^{f\,\Omega_j} = (1-\mu^{\Omega_j} - b^{\Omega_j})V_i^{f\,\Omega_j} + a^{\Omega_j}A_i^{t\,\Omega_j} + \sum\limits_{q=1}^{s}(e_{jq}^{\Omega_j} - f_{jq}^{\Omega_j})\frac{V_i^{f\,\Omega_j}V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} + \sum\limits_{q=1}^{s} \gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{f\,\Omega_q}}{N_i^{\Omega_j}} \\
V_{i+1}^{a\,\Omega_j} = (1-\mu^{\Omega_j} - d^{\Omega_j})V_i^{a\,\Omega_j} + c^{\Omega_j}A_i^{t\,\Omega_j} + \sum\limits_{q=1}^{s} \gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} - \sum\limits_{q=1}^{s}(e_{jq}^{\Omega_j} - f_{jq}^{\Omega_j})\frac{V_i^{f\,\Omega_j}V_i^{a\,\Omega_q}}{N_i^{\Omega_j}}
\end{cases}
$$

where $P_0^{\Omega_j} \geq 0$, $R_0^{\Omega_j} \geq 0$, $U_0^{\Omega_j} \geq 0$, $A_0^{t\,\Omega_j} \geq 0$, $V_0^{f\,\Omega_j} \geq 0$ and $V_0^{a\,\Omega_j} \geq 0$ are the given initial states.

We divide the population denoted by N^{Ω_j} into six compartments:

The potential electors (P^{Ω_j}) are the individuals entitled to participate in the elections but are not yet registered on the electoral lists in region Ω_j. The class of potential electors is increased by the recruitment of individuals into the compartment P^{Ω_j}at a rate Λ^{Ω_j} in region Ω_j. It is assumed that potential electors can acquire abstainer behavior (and become the boycotters of the elections) because they are not convinced of the electoral process at a rate $\alpha_2^{\Omega_j}$. In other words, it is assumed that the acquisition of an abstainer behavior is analogous to acquiring disease infection. Although not considered in this study, other factors such as economic and social conditions can lead to the abstention of the elections. Also, the potential electors can register on the electoral lists due the government declaration on the elections which facilitates the process of registration on the electoral lists at a rate $\alpha_1^{\Omega_j}$ the matter that makes them convinced of the importance of participation in the elections. Therefore, potential electors become real electors. Finally, potential electors suffer natural death (at a rate μ^{Ω_j}).

The registered (R^{Ω_j}) are the individuals who are registered on the electoral lists and wish to vote in the elections in region Ω_j. This compartment is increased at the rates $\alpha_1^{\Omega_j}$and θ^{Ω_j} when the potential electors and the unregistered individuals register on the electoral lists because due to their conviction of the importance of participation in the elections. Also, they register because the electoral system facilitates the registration on the electoral lists in a simple and innovative way for all the citizens and in all the areas of their residence. The number of some registred individuals decreases when the electors acquire abstainer behavior due to the non conviction of the electoral programs at a rate β^{Ω_j}. It is decreased by the contact with the voters for a political party and the voters against a political party at the rates $\gamma_{1jq}^{\Omega_j}$ and $\gamma_{2jq}^{\Omega_j}$. It is decreased by natural death (at a rate μ^{Ω_j}).

The unregistered (U^{Ω_j}) are the individuals who have a position of not registering on the electoral lists for several reasons, such as the lack of awareness of the importance of political participation in the development and political reforms, the absence of an electoral system that facilitates the registration on the electoral lists in a simple and innovative way for all citizens and in all areas of their residence, the loss of confidence in the political class that does not keep its promises and the spread of electoral and political corruption. The unregistered population is increased by the potential electors at a rate $\alpha_2^{\Omega_j}$ and decreased by natural death (at the rate μ^{Ω_j}). The unregistered individuals register on the electoral lists due to conviction (at the rate θ^{Ω_j}).

The temporary abstainers $(A^{t\,\Omega_j})$ are the individuals who have a hesitant position of abstaining the elections or , sometimes, their preoccupations or working conditions do not allow them to go voting in region Ω_j and, thus, their participation in the elections is not permanent. The population of temporary abstainers is increased by registered individuals (at a rate β^{Ω_j}) as well as the voters for a political party (at a rate b^{Ω_j}) and the voters against the political party (at a rate d^{Ω_j}) respectively. Also, when the electors, via effective contact with temporary abstainers, they change their attitude towards participation in the elections or sometimes their preoccupations or working conditions do not allow them to go voting . It is decreased by natural death (at the rate μ^{Ω_j}) and reversion to voting (at the rate a^{Ω_j} and c^{Ω_j}).

The voters for the political party $(V^{f\,\Omega_j})$ are the population of voters for the political party which is increased when the registered individuals move to vote for it due to contact with the voters for the political party in region Ω_j at a rate γ_{1jk}. It is also increased when the temporary abstainers become voters due to effective conviction of the importance of participating in the elections influenced by the electoral programs at a rate a^{Ω_j} and also when voters against the political party turn to vote for it due to contact with voters for the political party at a rate e^{Ω_j}. The number of some individuals of this population decreases by natural death (at the rate μ^{Ω_j})

and by reversion to voting against the political party due to contact with voters for the political party (at the rate f^{Ω_j}) or abstaining the elections temporarily (at the rate c^{Ω_j}).

The voters against the political party $(V^{a\,\Omega_j})$ are the population of voters against the political party which is increased when the electors move to vote against it in region Ω_j due to contact with the registered individuals (at a rate $\gamma_{2jk}^{\Omega_j}$) and is increased when the temporary abstainers become voters (at a rate c^{Ω_j}) and also when voters for the political party turn to vote against it due to the contact occuring between them (at a rate f^{Ω_j}). This population is decreased by natural death (at the rate μ^{Ω_j}) and by reversion to vote for the political party (at the rate e^{Ω_j}) or abstaining the elections temporarily (at the rate d^{Ω_j}).

The variables $P_i^{\Omega_j}$,$R_i^{\Omega_j}$; $U_i^{\Omega_j}$, $A_i^{t\,\Omega_j}$, $V_i^{f\,\Omega_j}$ and $V_i^{a\,\Omega_j}$ are the numbers of the individuals in the six classes at time i respectively. The unit i can correspond to periods, phases or years**.** It depends on the frequency of the survey studies as needed. The graphical representation of the proposed model is shown in Figure 1.

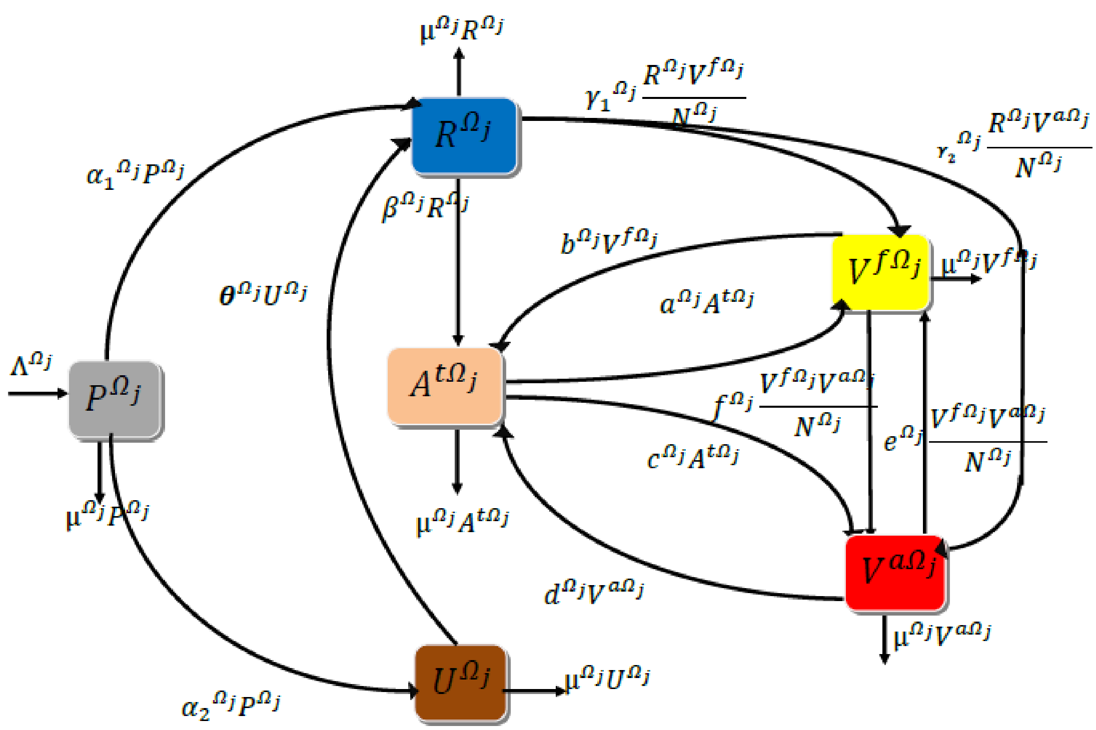

FIGURE 1

The total population size at time i in Ω_j is denoted by $N_i^{\Omega_j}$ with $N_i^{\Omega_j} = P_i^{\Omega_j} + R_i^{\Omega_j} + U_i^{t\,\Omega_j} + A_i^{t\,\Omega_j} + V_i^{f\,\Omega_j} + V_i^{a\,\Omega_j}$. The dynamics of this model are governed by the following nonlinear system of difference equations:

where $P_0^{\Omega_j} \geq 0$, $R_0^{\Omega_j} \geq 0$, $U_0^{\Omega_j} \geq 0$, $A_0^{t\,\Omega_j} \geq 0$, $V_0^{f\,\Omega_j} \geq 0$ and $V_0^{a\,\Omega_j} \geq 0$ are the given initial states.

3. THE OPTIMAL CONTROL PROBLEM

Our objective in this proposed strategy of control is to minimize the number of temporary abstainers $A_i^{t\,\Omega_j}$, maximize the number of the registered individuals on the electoral lists $R_i^{\Omega_j}$ and the voters for the political party $V_i^{f\,\Omega_j}$ during the time step $i = 0$ to T in domain Ω_j and also minimize the cost spent in an awereness program and an election campaign.
In the model (1), there are three controls $u^{\Omega_j} = (u_0^{\Omega_j}, u_{1,}^{\Omega_j} .....u_{T-1}^{\Omega_j})$, $v^{\Omega_j} = (v_{0,}^{\Omega_j} v_{1,}^{\Omega_j} ....v_{T-1}^{\Omega_j})$, $w^{\Omega_j} = (w_{0,}^{\Omega_j} w_{1,}^{\Omega_j} .....w_{T-1}^{\Omega_j})$. The first control u^{Ω_j} can be interpreted as the proportion to be adopted for the awareness campaign effort (time, money and human resources) to motivate the registered individuals to vote for a political party in the electoral process. The control v^{Ω_j} represents the electoral campaign effort to influence the temporary abstainers to participate in the electoral process and support a political party by voting for it. Finally, w^{Ω_j} measures the effort to persuade the unregistered individuals to register on the electoral lists. So, the controlled mathematical system is given by the following system of difference equations.

$$(2)\qquad \begin{cases} P_{i+1}^{\Omega_j} = \Lambda^{\Omega_j} + (1 - \mu^{\Omega_j} - \alpha_1^{\Omega_j} - \alpha_2^{\Omega_j})P_i^{\Omega_j} \\ R_{i+1}^{\Omega_j} = (1 - \mu^{\Omega_j} - \beta^{\Omega_j})R_i^{\Omega_j} + \alpha_1^{\Omega_j} P_i^{\Omega_j} + \theta^{\Omega_j} U_i^{\Omega_j} - \sum\limits_{q=1}^{s} \gamma_{1jq}^{\Omega_j} \frac{R_i^{\Omega_j} V_i^{f\,\Omega_q}}{N_i^{\Omega_j}} \\ \qquad - \sum\limits_{q=1}^{s} \gamma_{2jq}^{\Omega_j} \frac{R_i^{\Omega_j} V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} - u_i^{\Omega_j} R_i^{\Omega_j} + w_i^{\Omega_j} U_i^{\Omega_j} \\ U_{i+1}^{\Omega_j} = (1 - \mu^{\Omega_j} - \theta^{\Omega_j})U_i^{\Omega_j} + \alpha_2^{\Omega_j} P_i^{\Omega_j} - w_i^{\Omega_j} U_i^{\Omega_j} \\ A_{i+1}^{t\,\Omega_j} = (1 - \mu^{\Omega_j} - a^{\Omega_j} - c^{\Omega_j})A_i^{t\,\Omega_j} + \beta^{\Omega_j} R_i^{\Omega_j} + b^{\Omega_j} V_i^{f\,\Omega_j} + d^{\Omega_j} V_i^{a\,\Omega_j} - v_i^{\Omega_j} A_i^{t\,\Omega_j} \\ V_{i+1}^{f\,\Omega_j} = (1 - \mu^{\Omega_j} - b^{\Omega_j})V_i^{f\,\Omega_j} + a^{\Omega_j} A_i^{t\,\Omega_j} + \sum\limits_{q=1}^{s} (e_{jq}^{\Omega_j} - f_{jq}^{\Omega_j}) \frac{V_i^{f\,\Omega_j} V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} \\ \qquad + \sum\limits_{q=1}^{s} \gamma_{1jq}^{\Omega_j} \frac{R_i^{\Omega_j} V_i^{f\,\Omega_q}}{N_i^{\Omega_j}} + u_i^{\Omega_j} R_i^{\Omega_j} + \phi^{\Omega_j} v_i^{\Omega_j} A_i^{t\,\Omega_j} \\ V_{i+1}^{a\,\Omega_j} = (1 - \mu^{\Omega_j} - d^{\Omega_j})V_i^{a\,\Omega_j} + c^{\Omega_j} A_i^{t\,\Omega_j} + \sum\limits_{q=1}^{s} \gamma_{2jq}^{\Omega_j} \frac{R_i^{\Omega_j} V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} \\ \qquad - \sum\limits_{q=1}^{s} (e_{jq}^{\Omega_j} - f_{jq}^{\Omega_j}) \frac{V_i^{f\,\Omega_j} V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} + (1 - \phi^{\Omega_j}) v_i^{\Omega_j} A_i^{t\,\Omega_j} \end{cases}$$

where $P_0^{\Omega_j} \geq 0$, $R_0^{\Omega_j} \geq 0, U_0^{\Omega_j} \geq 0$, $A_0^{t\,\Omega_j} \geq 0$, $V_0^{f\,\Omega_j} \geq 0$ and $V_0^{a\,\Omega_j} \geq 0$ are the given initial states.

From the system of difference equations (2), we can extract two models. The first controlled model achieves the objective of increasing the participation rate in the electoral process and the second controlled model leads to the increase of the number of voters for a political party. Then, the problem is to minimize the objective functional

$$\begin{aligned} J(u^{\Omega_j}, v^{\Omega_j}, w^{\Omega_j})) &= h_T^{\Omega_j} A_T^{t\,\Omega_j} - k_T^{\Omega_j} R_T^{\Omega_j} - l_T^{\Omega_j} V_T^{f\,\Omega_j} + \sum_{i=0}^{T-1} \left(h_i^{\Omega_j} A_i^{t\,\Omega_j} - k_i^{\Omega_j} R_i^{\Omega_j} - l_i^{\Omega_j} V_i^{f\,\Omega_j} \right) \\ &\quad + \sum_{i=0}^{T-1} \left(\frac{m_i^{\Omega_j}}{2} (u_i^{\Omega_j})^2 + \frac{n_i^{\Omega_j}}{2} (v_i^{\Omega_j})^2 + \frac{o_i^{\Omega_j}}{2} (w_i^{\Omega_j})^2 \right) \end{aligned} \tag{3}$$

Where the parameters $h_i^{\Omega_j} > 0$, $k_i^{\Omega_j} > 0$, $l_i^{\Omega_j} > 0$**,** $m_i^{\Omega_j} > 0$, $n_i^{\Omega_j} > 0$ and $o_i^{\Omega_j} > 0$ and 0 for $i \in \{0, ..., T\}$ are the cost coefficients. They are selected to weigh the relative importance of $A_i^{t\,\Omega_j}$, $R_i^{\Omega_j}$, $V_i^{f\,\Omega_j}$, $u_i^{\Omega_j}$, $v_i^{\Omega_j}$ and $w_i^{\Omega_j}$ at time i in region Ω_j. T is the final time.
In other words, we seek the optimal controls u^{Ω_j}, v^{Ω_j} and w^{Ω_j} such that

$$J(u_i^{\Omega_j *}, v_i^{\Omega_j *}, w_i^{\Omega_j *}) = \min_{(u^{\Omega_j}, v^{\Omega_j}, w^{\Omega_j}) \in U_{ad}} J(u^{\Omega_j}, v^{\Omega_j}, w^{\Omega_j})$$

where U_{ad} is the set of admissible controls defined by

$$U_{ad} = \{(u_i^{\Omega_j}, v_i^{\Omega_j}, w_i^{\Omega_j}) : a^{\Omega_j} \leq u_i^{\Omega_j} \leq b^{\Omega_j}, c^{\Omega_j} \leq v_i^{\Omega_j} \leq d^{\Omega_j}, e^{\Omega_j} \leq w_i^{\Omega_j} \leq f^{\Omega_j} \text{ , for } i = 1, 2, ..., T-1\} \tag{4}$$

The sufficient condition for the existence of an optimal control $(u_i^{\Omega_j *}, v_i^{\Omega_j *}, w_i^{\Omega_j *})$ for problem (2-3) comes from the following theorem.

Theorem 1: There exists the optimal control $u_i^{\Omega_j *}, v_i^{\Omega_j *}$ and $w_i^{\Omega_j *}$ such that $J(u_i^{\Omega_j *}, v_i^{\Omega_j *}, w_i^{\Omega_j *}) = \min_{(u^{\Omega_j}, v^{\Omega_j}, w^{\Omega_j}) \in U_{ad}} J(u_i^{\Omega_j}, v_i^{\Omega_j}, w_i^{\Omega_j})$ subjet to the control system (2) with initial conditions.

Proof. See Dabbs ([7], Theorem 1)."Dabbs. K (2010) Optimal control in discrete pest control models. University of Tennessee Honors Thesis Projects" □

We apply the discrete version of Pontryagin's Maximum Principle [9], [11], [18], [19]. The key idea is introducing the adjoint function to attach the system of difference equations to the objective functional resulting in the formation of a function called the Hamiltonian. This principle converts the problem of finding the control to optimize the objective functional subject to the state difference equation with initial condition to find the control to optimize Hamiltonian pointwise (with respect to the control).

Now, we have the Hamiltonian H_i at time step i, defined by

$$H_i = h_i^{\Omega_j} A_i^{t\,\Omega_j} - k_i^{\Omega_j} R_i^{\Omega_j} - l_i^{\Omega_j} V_i^{f\,\Omega_j} + \frac{m_i^{\Omega_j}}{2}(u_i^{\Omega_j})^2 + \frac{n_i^{\Omega_j}}{2}(v_i^{\Omega_j})^2 + \frac{o_i^{\Omega_j}}{2}(w_i^{\Omega_j})^2 + \sum_{q=1}^{6} \zeta_{q,i+1}^{j} f_{q,i+1} \tag{5}$$

where $f_{j,i+1}$ is the right side of the system of difference equations (2) of the j^{th} state variable at time step $i+1$.

Theorem 2: Given an optimal control $(u_i^{\Omega_j *}, v_i^{\Omega_j *}, w_i^{\Omega_j *}) \in U_{ad}$, and solutions $P_i^{\Omega_j}$,$R_i^{\Omega_j}$; $U_i^{\Omega_j}$, $A_i^{t\,\Omega_j}$, $V_i^{f\,\Omega_j}$ and $V_i^{a\,\Omega_j}$of corresponding state system (2), there exists adjoint functions, $\zeta_{1,i}^{j}$, $\zeta_{2,i}^{j}$, $\zeta_{3,i}^{j}$, $\zeta_{4,i}^{j}$, $\zeta_{5,i}^{j}$,and $\zeta_{6,i}^{j}$ satisfying

$$\left\{\begin{aligned}
\zeta_{1,i}^{j} =& \zeta_{1,i+1}^{j}\left\{\Lambda^{\Omega_j} + (1-\mu^{\Omega_j} - \alpha_1^{\Omega_j} - \alpha_2^{\Omega_j})\right\} + \zeta_{2,i+1}^{j}\alpha_1^{\Omega_j} + \zeta_{3,i+1}^{j}\alpha_2^{\Omega_j}\\
\zeta_{2,i}^{j} =& -k_i^{\Omega_j} + \zeta_{2,i+1}^{j}\left\{(1-\mu^{\Omega_j} - \beta^{\Omega_j}) - \sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}} - \sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{V_i^{a\,\Omega_j}}{N_i^{\Omega_j}} - u_i^{\Omega_j}\right\}\\
&+\beta^{\Omega_j}\zeta_{4,i+1}^{j} + \zeta_{5,i+1}^{j}\left\{\sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}} + u_i^{\Omega_j}\right\} + \zeta_{6,i+1}^{j}\sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{V_i^{a\,\Omega_j}}{N_i^{\Omega_j}}\\
\zeta_{3,i}^{j} =& \zeta_{2,i+1}^{j}\left\{\theta^{\Omega_j} + w_i^{\Omega_j}\right\} + \zeta_{3,i+1}^{j}\left\{1-\mu^{\Omega_j} - \theta^{\Omega_j} - w_i^{\Omega_j}\right\}\\
\zeta_{4,i}^{j} =& h_i^{\Omega_j} + \zeta_{4,i+1}^{j}\left\{1-\mu^{\Omega_j} - a^{\Omega_j} - c^{\Omega_j} - v_i^{\Omega_j}\right\} + \zeta_{5,i+1}^{j}\left\{a^{\Omega_j} + \phi v_i^{\Omega_j}\right\}\\
&+\zeta_{6,i+1}^{j}\left\{c^{\Omega_j} + (1-\phi)v_i^{\Omega_j}\right\}\\
\zeta_{5,i}^{j} =& -l_i^{\Omega_j} - \zeta_{2,i+1}^{j}\sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}} + b^{\Omega_j}\zeta_{4,i+1}^{j} + \zeta_{5,i+1}^{j}\left\{1-\mu^{\Omega_j} - b^{\Omega_j} + \sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}}\right\}\\
&-\zeta_{6,i+1}^{j}\sum_{q=1}^{s}(e_{jq}^{\Omega_j} - f_{jq}^{\Omega_j})\frac{V_i^{a\,\Omega_j}}{N_i^{\Omega_j}}\\
\zeta_{6,i}^{j} =& -\zeta_{2,i+1}^{j}\sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}} + d^{\Omega_j}\zeta_{4,i+1}^{j} + \zeta_{5,i+1}^{j}\sum_{q=1}^{s}(e_{jq}^{\Omega_j} - f_{jq}^{\Omega_j})\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}}\\
&+\zeta_{6,i+1}^{j}\left\{1-\mu^{\Omega_j} - d^{\Omega_j} + \sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}} - \sum_{q=1}^{s}(e_{jq}^{\Omega_j} - f_{jq}^{\Omega_j})\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}}\right\}
\end{aligned}\right.$$

with the transversality conditions at time T

$$\begin{aligned}
&\zeta_{1,i}^{j} = \frac{\partial H_i}{\partial P_i^{\Omega_j}},\ \zeta_{1,T}^{j} = 0,\ \zeta_{2,i}^{j} = \frac{\partial H_i}{\partial R_i^{\Omega_j}},\ \zeta_{2,T}^{j} = -k_T^{\Omega_j},\ \zeta_{3,i}^{j} = \frac{\partial H_i}{\partial U_i^{\Omega_j}},\ \zeta_{3,T}^{j} = 0\\
&\zeta_{4,i}^{j} = \frac{\partial H_i}{\partial A_i^{t\,\Omega_j}},\ \zeta_{4,T}^{j} = h_T^{\Omega_j},\ \zeta_{5,i}^{j} = \frac{\partial H_i}{\partial V_i^{f\,\Omega_j}},\ \zeta_{5,T}^{j} = -l_T^{\Omega_j} \text{ and } \zeta_{6,i}^{j} = \frac{\partial H_i}{\partial V_i^{a\,\Omega_j}},\ \zeta_{6,T}^{j} = 0
\end{aligned} \tag{6}$$

Futhermore, for $i = 0, 1, ..., T-1$, the optimal controls $u_i^{*\Omega_j}$, $v_i^{*\Omega_j}$ and $w_i^{*\Omega_j}$ are given by

$$(7) \qquad u_i^{*\Omega_j} = \min\left(b^{\Omega_j}, \max\left(a^{\Omega_j}, \frac{1}{m_i^{\Omega_j}} R_i^{\Omega_j}\left(\zeta_{2,i+1}^j - \zeta_{5,i+1}^j\right)\right)\right).$$

$$(8) \qquad v_i^{*\Omega_j} = \min\left(d^{\Omega_j}, \max\left(c^{\Omega_j}, \frac{1}{n_i^{\Omega_j}} A_i^{t\,\Omega_j}\left(\zeta_{4,i+1}^j - \zeta_{5,i+1}^j\phi - \zeta_{6,i+1}^j(1-\phi)\right)\right)\right).$$

$$(9) \qquad w_i^{*\Omega_j} = \min\left(f^{\Omega_j}, \max\left(e^{\Omega_j}, \frac{1}{o_i^{\Omega_j}} U_i^{\Omega_j}\left(\zeta_{3,i+1}^j - \zeta_{2,i+1}^j\right)\right)\right).$$

Proof. The Hamiltonian at time step i is given by

$$\begin{aligned}
H_i \;=\;& h_i^{\Omega_j} A_i^{t\,\Omega_j} - k_i^{\Omega_j} R_i^{\Omega_j} - l_i^{\Omega_j} V_i^{f\,\Omega_j} + \frac{m_i^{\Omega_j}}{2}(u_i^{\Omega_j})^2 + \frac{n_i^{\Omega_j}}{2}(v_i^{\Omega_j})^2 + \frac{o_i^{\Omega_j}}{2}(w_i^{\Omega_j})^2 \\
&+\zeta_{1,i+1}^j\left\{\Lambda^{\Omega_j} + (1-\mu^{\Omega_j}-\alpha_1^{\Omega_j}-\alpha_2^{\Omega_j})P_i^{\Omega_j}\right\} + \zeta_{2,i+1}^j\left\{(1-\mu^{\Omega_j}-\beta^{\Omega_j})R_i^{\Omega_j} + \alpha_1^{\Omega_j}P_i^{\Omega_j}\right\} \\
&+\zeta_{2,i+1}^j\left\{\theta^{\Omega_j}U_i^{\Omega_j} - \sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{f\,\Omega_q}}{N_i^{\Omega_j}} - \sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} - u_i^{\Omega_j}R_i^{\Omega_j} + w_i^{\Omega_j}U_i^{\Omega_j}\right\} \\
&+\zeta_{3,i+1}^j\left\{(1-\mu^{\Omega_j}-\theta^{\Omega_j})U_i^{\Omega_j} + \alpha_2^{\Omega_j}P_i^{\Omega_j} - w_i^{\Omega_j}U_i^{\Omega_j}\right\} \\
&+\zeta_{4,i+1}^j\left\{(1-\mu^{\Omega_j}-a^{\Omega_j}-c^{\Omega_j})A_i^{t\,\Omega_j} + \beta^{\Omega_j}R_i^{\Omega_j} + b^{\Omega_j}V_i^{f\,\Omega_j} + d^{\Omega_j}V_i^{a\,\Omega_j} - v_i^{\Omega_j}A_i^{t\,\Omega_j}\right\} \\
&+\zeta_{5,i+1}^j\left\{(1-\mu^{\Omega_j}-b^{\Omega_j})V_i^{f\,\Omega_j} + a^{\Omega_j}A_i^{t\,\Omega_j} + \sum_{q=1}^{s}(e_{jq}^{\Omega_j}-f_{jq}^{\Omega_j})\frac{V_i^{f\,\Omega_j}V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} + \sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{f\,\Omega_q}}{N_i^{\Omega_j}}\right\} \\
&+\zeta_{5,i+1}^j\left\{u_i^{\Omega_j}R_i^{\Omega_j} + \phi v_i^{\Omega_j}A_i^{t\,\Omega_j}\right\} \\
&+\zeta_{6,i+1}^j\left\{(1-\mu^{\Omega_j}-d^{\Omega_j})V_i^{a\,\Omega_j} + c^{\Omega_j}A_i^{t\,\Omega_j} + \sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_j}V_i^{a\,\Omega_q}}{N_i^{\Omega_j}} - \sum_{q=1}^{s}(e_{jq}^{\Omega_j}-f_{jq}^{\Omega_j})\right\} \\
&+\zeta_{6,i+1}^j(1-\phi)v_i^{\Omega_j}A_i^{t\,\Omega_j}
\end{aligned}$$

$$\begin{aligned}
\zeta_{1,i}^j =& \frac{\partial H_i}{\partial P_i^{\Omega_j}} \\
=& \zeta_{1,i+1}^j\left\{\Lambda^{\Omega_j} + (1-\mu^{\Omega_j}-\alpha_1^{\Omega_j}-\alpha_2^{\Omega_j})\right\} + \zeta_{2,i+1}^j\alpha_1^{\Omega_j} + \zeta_{3,i+1}^j\alpha_2^{\Omega_j} \\
\zeta_{2,i}^j =& \frac{\partial H_i}{\partial R_i^{\Omega_j}} \\
=& -k_i^{\Omega_j} + \zeta_{2,i+1}^j\left\{(1-\mu^{\Omega_j}-\beta^{\Omega_j}) - \sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}} - \sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{V_i^{a\,\Omega_j}}{N_i^{\Omega_j}} - u_i^{\Omega_j}\right\} \\
&+\beta^{\Omega_j}\zeta_{4,i+1}^j + \zeta_{5,i+1}^j\left\{\sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}} + u_i^{\Omega_j}\right\} + \zeta_{6,i+1}^j\sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{V_i^{a\,\Omega_j}}{N_i^{\Omega_j}} \\
\zeta_{3,i}^j =& \frac{\partial H_i}{\partial U_i^{\Omega_j}} \\
=& \zeta_{2,i+1}^j\left\{\theta^{\Omega_j} + w_i^{\Omega_j}\right\} + \zeta_{3,i+1}^j\left\{1-\mu^{\Omega_j}-\theta^{\Omega_j}-w_i^{\Omega_j}\right\}
\end{aligned}$$

$$
\begin{aligned}
\zeta_{4,i}^{j} &= \frac{\partial H_i}{\partial A_i^{t\,\Omega_j}} \\
&= h_i^{\Omega_j} + \zeta_{4,i+1}^{j}\left\{1-\mu^{\Omega_j}-a^{\Omega_j}-c^{\Omega_j}-v_i^{\Omega_j}\right\} + \zeta_{5,i+1}^{j}\left\{a^{\Omega_j}+\phi v_i^{\Omega_j}\right\} \\
&\quad + \zeta_{6,i+1}^{j}\left\{c^{\Omega_j}+(1-\phi)v_i^{\Omega_j}\right\} \\
\zeta_{5,i}^{j} &= \frac{\partial H_i}{\partial V_i^{f\,\Omega_j}} \\
&= -l_i^{\Omega_j} - \zeta_{2,i+1}^{j}\sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}} + b^{\Omega_j}\zeta_{4,i+1}^{j} + \zeta_{5,i+1}^{j}\left\{1-\mu^{\Omega_j}-b^{\Omega_j}+\sum_{q=1}^{s}\gamma_{1jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}}\right\} \\
&\quad - \zeta_{6,i+1}^{j}\sum_{q=1}^{s}(e_{jq}^{\Omega_j}-f_{jq}^{\Omega_j})\frac{V_i^{a\,\Omega_j}}{N_i^{\Omega_j}} \\
\zeta_{6,i}^{j} &= \frac{\partial H_i}{\partial V_i^{a\,\Omega_j}} \\
&= -\zeta_{2,i+1}^{j}\sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}} + d^{\Omega_j}\zeta_{4,i+1}^{j} + \zeta_{5,i+1}^{j}\sum_{q=1}^{s}(e_{jq}^{\Omega_j}-f_{jq}^{\Omega_j})\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}} \\
&\quad + \zeta_{6,i+1}^{j}\left\{1-\mu^{\Omega_j}-d^{\Omega_j}+\sum_{q=1}^{s}\gamma_{2jq}^{\Omega_j}\frac{R_i^{\Omega_q}}{N_i^{\Omega_j}}-\sum_{q=1}^{s}(e_{jq}^{\Omega_j}-f_{jq}^{\Omega_j})\frac{V_i^{f\,\Omega_j}}{N_i^{\Omega_j}}\right\}
\end{aligned}
$$

For $i = 0,1,...,T-1$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's Maximum Principle, in discrete time, given in [9, 11, 18, 19] such that

$$
(10)\qquad \begin{cases}
\zeta_{1,i}^{j} = \frac{\partial H_i}{\partial P_i^{\Omega_j}}, \ \zeta_{1,T}^{j} = 0 \\
\zeta_{2,i}^{j} = \frac{\partial H_i}{\partial R_i^{\Omega_j}}, \ \zeta_{2,T}^{j} = -k_T^{\Omega_j} \\
\zeta_{3,i}^{j} = \frac{\partial H_i}{\partial U_i^{\Omega_j}}, \ \zeta_{3,T}^{j} = 0 \\
\zeta_{4,i}^{j} = \frac{\partial H_i}{\partial A_i^{t\,\Omega_j}}, \ \zeta_{4,T}^{j} = h_T^{\Omega_j} \\
\zeta_{5,i}^{j} = \frac{\partial H_i}{\partial V_i^{f\,\Omega_j}}, \ \zeta_{5,T}^{j} = -l_T^{\Omega_j} \\
\zeta_{6,i}^{j} = \frac{\partial H_i}{\partial V_i^{a\,\Omega_j}}, \ \zeta_{6,T}^{j} = 0
\end{cases}
$$

For $i = 0,1,...,T-1$, the optimal controls u_i^*, v_i^* and w_i^* can be solved from the optimality condition,

$$
(11)\qquad \frac{\partial H_i}{\partial u_i^{\Omega_j}} = 0,\ \frac{\partial H_i}{\partial v_i^{\Omega_j}} = 0 \text{ and } \frac{\partial H_i}{\partial w_i^{\Omega_j}} = 0.
$$

that is

$$\begin{aligned}
\frac{\partial H_i}{\partial u_i^{\Omega_j}} &= m_i^{\Omega_j} u_i^{\Omega_j} - \zeta_{2,i+1}^{j} R_i^{\Omega_j} + \zeta_{5,i+1}^{j} R_i^{\Omega_j} = 0. \\
\frac{\partial H_i}{\partial v_i^{\Omega_j}} &= n_i^{\Omega_j} v_i^{\Omega_j} - \zeta_{4,i+1}^{j} A_i^{t\,\Omega_j} + \zeta_{5,i+1}^{j} \phi A_i^{t\,\Omega_j} + \zeta_{6,i+1}^{j} (1-\phi) A_i^{t\,\Omega_j} = 0. \\
\frac{\partial H_i}{\partial w_i^{\Omega_j}} &= o_i^{\Omega_j} w_i^{\Omega_j} + \zeta_{2,i+1}^{j} U_i^{\Omega_j} - \zeta_{3,i+1}^{j} U_i^{\Omega_j} = 0.
\end{aligned}$$

We have

$$\begin{aligned}
u_i^{\Omega_j} &= \frac{1}{m_i^{\Omega_j}} R_i^{\Omega_j} \left(\zeta_{2,i+1}^{j} - \zeta_{5,i+1}^{j} \right). \\
v_i^{\Omega_j} &= \frac{1}{n_i^{\Omega_j}} A_i^{t\,\Omega_j} \left(\zeta_{4,i+1}^{j} - \zeta_{5,i+1}^{j} \phi - \zeta_{6,i+1}^{j} (1-\phi) \right). \\
w_i^{\Omega_j} &= \frac{1}{o_i^{\Omega_j}} U_i^{\Omega_j} \left(\zeta_{3,i+1}^{j} - \zeta_{2,i+1}^{j} \right).
\end{aligned} \tag{12}$$

However, the control attached to this case will be eliminated and removed.

By the bounds in U_{ad} of the controls, it is easy to obtain $u_i^{*\Omega_j}, v_i^{*\Omega_j}$ and $w_i^{*\Omega_j}$ in the form of (7-8-9). □

4. NUMERICAL SIMULATION AND DISCUSSION

In this formulation, there were initial conditions for the state variables and terminal conditions for the adjoints i.e. the optimality system is a two-point boundary value problem with separated boundary conditions at times step $i = 0$ and $i = T$. We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the controls at the first iteration and then before the next iteration we update the controls by using the characterization. We continue until convergence of successive iterates is achieved.

This model is valid for all regions of countries that give their citizens the freedom to register in electoral lists and to vote during the electoral process, but the simulation will be limited to the study of a model that is composed of two regions Ω_1 and Ω_2. The Ω_1 represents the rural regions and the Ω_2 represents the urban regions. The objective is to highlight the specificities of each region when formulating control strategies that seek to achieve different objectives such as:

-Increasing the percentage of registrants on the electoral lists by targeting the unregistered individuals based on the optimal control w_i.

-Increasing the percentage of voters participatig in the electoral process based on the optimal control u_i.

-Increasing the number of voters and supporters of a particular political party based on the optimal control u_i and a percentage from v_i.

Table1: The description of the parameters used for the definition of discrete time systems of rural regions (1)

P_0	R_0	U_0	A_0^t	V_0^f	V_0^a	N	Λ
6.10^3	4.10^3	$2,5.10^3$	8.10^2	10^3	1.10^3	$14,8.10^3$	$1,5.10^3$
μ	α_1	α_2	β	θ	${}^1\gamma_{11}$	${}^1\gamma_{12}$	${}^2\gamma_{11}$
$0,054$	$0,0644$	$0,02936$	$0,05771$	$0,01$	$0,002147$	$0,001047$	$0,001082$
${}^2\gamma_{12}$	a	b	c	d	e_{11}	e_{12}	f_{11}
$0,8000$	$0,010$	$0,088$	0.1	0.0020	0.0427	0.0427	0.0836
f_{12}	ϕ						
0.0336	0.8717						

Table 1: we used just an arbitrary academic data

Table2: The description of parameters used for the definition of discrete time systems of urban regions (2)

P_0	R_0	U_0	A_0^t	V_0^f	V_0^a	N	Λ
8.10^4	3.10^4	6.10^4	2.10^4	2.10^4	4.10^4	25.10^4	1.10^4
μ	α_1	α_2	β	θ	${}^1\gamma_{11}$	${}^1\gamma_{12}$	${}^2\gamma_{11}$
$0,00381$	$0,0644$	$0,32346$	$0,04$	$0,054225$	0.1	0.00254	$0,1$
${}^2\gamma_{12}$	a	b	c	d	e_{11}	e_{12}	f_{11}
$0,01$	$0,010$	0.002264	0.1	0.009591	0.003769	0.016531	0.012
f_{12}	ϕ						
0.02	0.8717						

Table 2: we used just an arbitrary academic data

4.1. The rural region. In this section, we present the results obtained by applying the proposed control strategies to the rural region.(See the figures below).

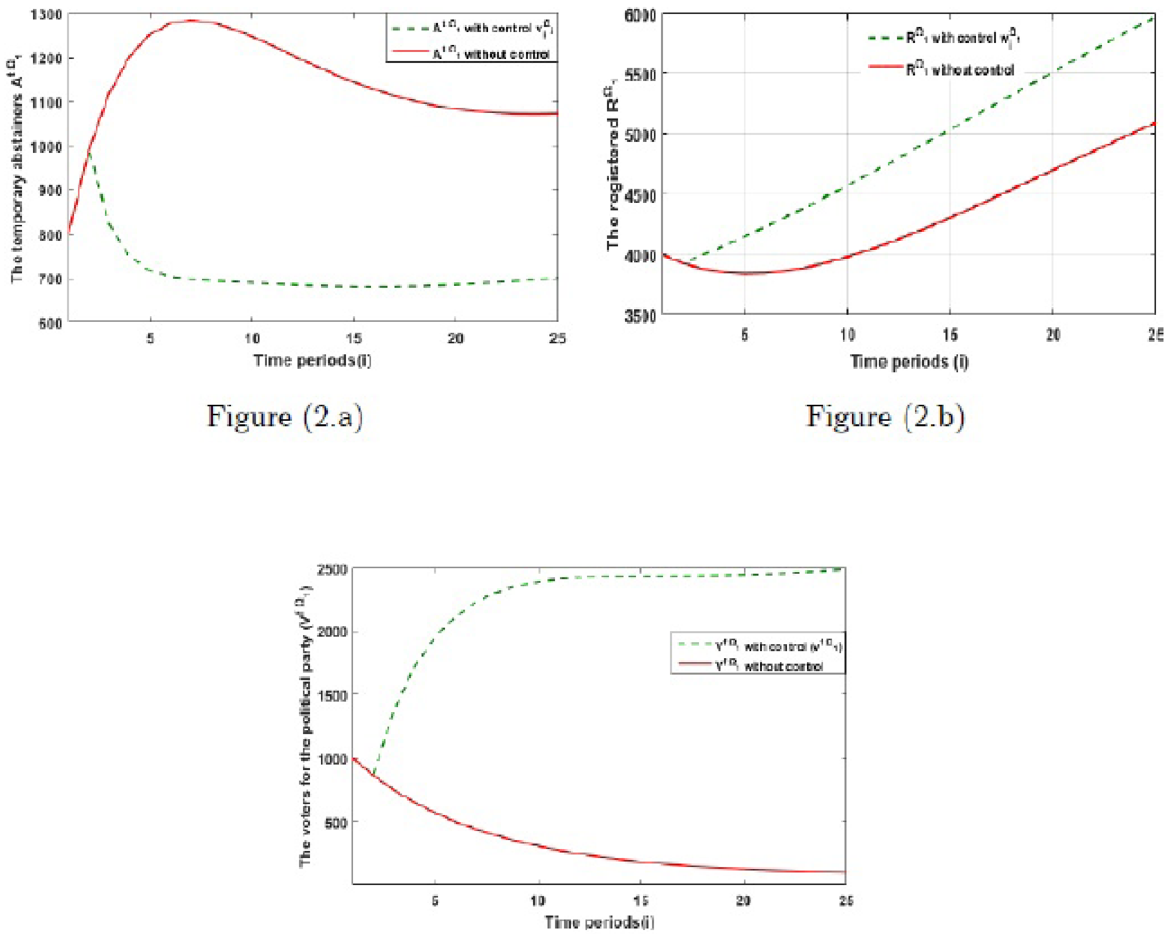

Figure (2.a) Figure (2.b)

Figure (2.c)

FIGURE 2

From Figure (2), we observe that the registration rate on the electoral lists increased from 27.30% (R without controls) to 33.60% (R with controls) (Figure 2.b). This increase occured due to the adoption of the proposed strategy of control ,therefore, we remark that the registered individuals are more persuaded to register on the electoral lists through awareness raising.

Also, we easily observe that the number of temporary abstainers decreases significantly by 65.32% (Figure 2.a). The decrease of the number of temporary abstainers and the increase of registered individuals have positively influenced the particapation rate in voting by an increase of 23.37% to 38.38% and the growth of the individuals who vote for a political party by 98.56% (Figure 2.c) in the end of implementing those strategies.

4.2. Urban regions. In this section, we present the results obtained by applying the proposed control strategies on an urban region. (See the figure (3)). With regards to Figure (3), we notice

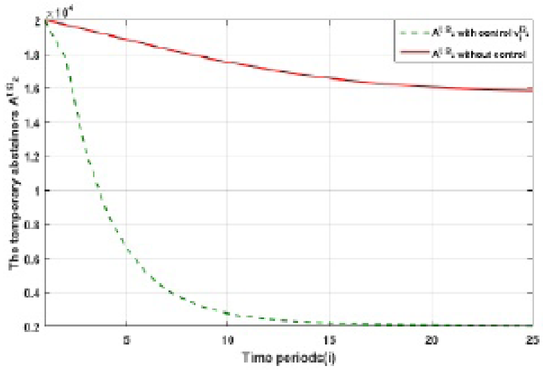

Figure (3.a)

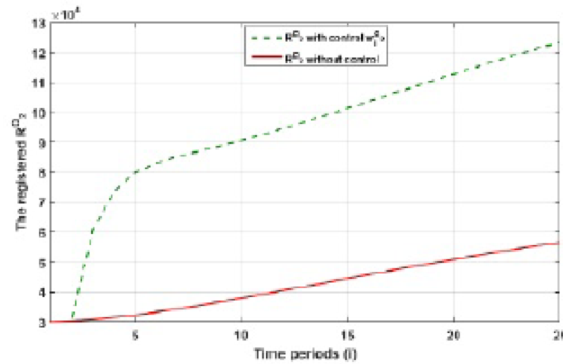

Figure (3.b)

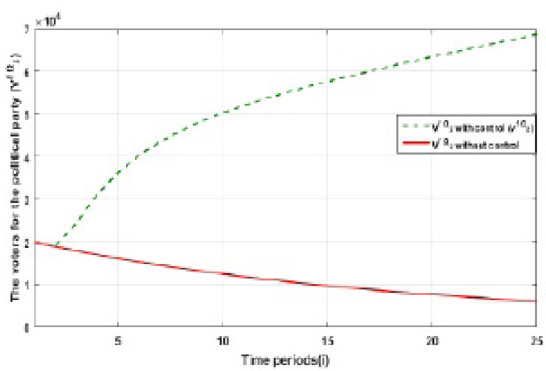

Figure (3.c)

FIGURE 3

that the registration rate on the electoral lists increases significantly from 21.71% to 37.09% (R without and with controls) (figure (3.b). This growth is achieved when we use the proposed strategy of control and we observe that the registered individuals are convinced to register on the electoral lists through awareness raising activities. As a result, this number is considerably increased.

Also, we observe in figure (3.a) that the number of temporary abstainers decreases clearly by 87.21% (A^t with controls). The decrease of the number of temporary abstainers and the increase of the registered individuals has positively influenced the particapation rate in voting by an increase from 18.09 % to 48.45% and the growth of the individuals who vote for a political party by 91.09% (V^f with controls) (figure 3.c).

4.3. Reciprocal influence between rural and urban regions. In this section, we present the results obtained in a region composed of a rural and an urban area characterized by the mutual effect between the individuals of these two milieus on the level of electoral behavior between them. (See Figure (4)). This positive decrease in the number of the temporary abstainers with

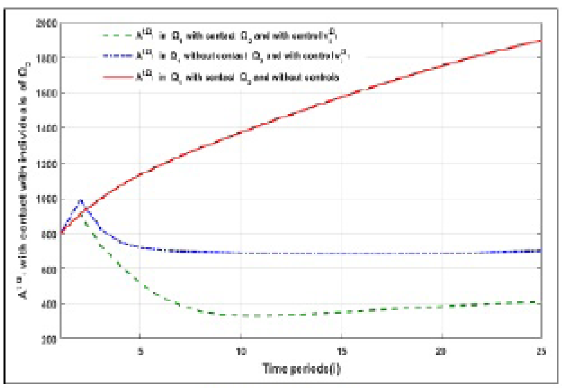

Figure (4.a)

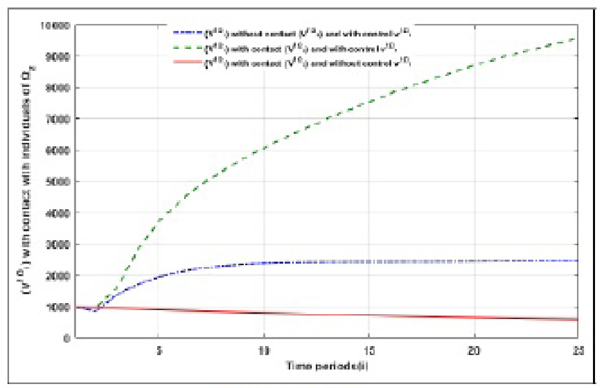

Figure (4.b)

FIGURE 4

contact and with controls is much better than the result obtained in a rural region in which we did not take into account the reciprocal influence between the individuals of this region and the individuals of the urban region (this decrease represents 78.45% with contact and with controls compared to 63.02% without contact and with controls) (see Figure (4.a)). The same improvement in the results appears at the level of the increase of the voters for a political party with contact and with controls (this increase represents 93.72% with contact and with controls compared to 75.84% without contact and with controls) (see Figure (4.b)).Thus, the reciprocal influence between these two regions is significantly important.

This effect occurs in various forms:

-Family relations and friendship between members of rural and urban areas.

-The imigration of a number of people from rural areas to work in the neighbouring urban areas, thus, the contact and influence between the two regions is achieved.

- The strong mutual influence that occurs within the means of social media (Facebook, Twitter, Phone ....), which brings together all members of the community, whether they are villagers or urban citizens. These forms of mutual influence between individuals are general and not limited to rural and urban areas.

Finally, these results prove the effectiveness of the proposed control strategies and the validity of the mathematical model.

5. CONCLUSION

In this work, we formulated a multi-region discrete mathematical model that describes the dynamics of citizens who have the right to vote and their electoral behavior during an awareness program or an electoral campaign. Also, we proposed an optimal strategy for an awareness program or an election campaign that helps politicians to distinguish between different categories of voters in many regions in order to increase the participation rate in the electoral process and obtain the greatest possible number of votes with a minimal effort. In this strategy, we introduced three controls; The first control represented the awereness campaign effort (time, money and human resources) to motivate the potential electors to register on the electoral process.The second control characterized the electoral campaign effort to make an infuence on the temporary abstainers to participate in the electoral process and support a political party by voting for it. Finally, the third control measured the required persuasion effort to change the position of voters in favor of a political party. Pontryagin's maximum principle, in discrete time, was used to characterize the optimal controls and the optimality system was solved by an iterative method.

DATA AVAILABILITY

No data were used to support this study.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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