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GF-STRUCTURE ON THE SEMI COTANGENT BUNDLE

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Abstract. We define GF-structure on the semi-cotangent bundle and establish its existence. We find basic results for complete lifts of tensor field of type (1,1) and of type (1,2) on the semi-cotangent bundle. We also investigate the integrability conditions for GF-structure on the semi-cotangent bundle.

Keywords: complete lift; semi-cotangent bundle; Nijenhuis tensor; integrability.

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1. Introduction

Let M_n be an n-dimensional differentiable manifold and $T_p^*(M_n)$ be the cotangent space at a point $p \in M_n$, that is, the dual space to tangent space $T_p(M_n)$ at p. Any element of $T_p^* M_n$ is called a covector at $p \in M_n$. Then the set $T_p^*(M_n) = \cup_{p \in M_n} T_p^*$ is, the by definition, the cotangent bundle over the manifold M_n [8]. The semi-cotangent bundle is a pull-back bundle of the cotangent bundle. Yildirim [6] studied the semi-cotangent bundles by considering the complete lift of the vector and tensor field of type (1,1). Yildirim and Solimov [7] studied the semi-cotangent bundles and some of their lift problems. Integrability conditions of an almost complex structure on semi-cotangent bundle are established by Cayer [1]. In the present paper

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we consider methods by which the GF-structure in tangent bundle TM_n can be extended to semi-cotangent bundle $t^*(M_n)$. These methods enable us to examine GF-structure of $t^*(M_n)$ in relation to those of TM_n.

Let M_n be an n-dimensional differentiable manifold and TM_n its tangent bundle. The projection bundle $\pi_1 : TM_n \to M_n$ which denotes the natural bundle structure of TM_n over M_n [4]. Let $(x^i) = (x^{\bar{\alpha}}, x^{\alpha})$ be a system of local coordinates where x^{α} are coordinates in M_n and $x^{\bar{\alpha}}$ are fibre coordinates of tangent bundle TM_n. The indices $\alpha, \beta, ... = 1, ..., n,\ \bar{\alpha}, \bar{\beta}, ... = n = 1, ...2n$ and $i, j, ... = 1, ..., 2n$. If $(x^{i'}) = (x^{\bar{\alpha}'}, x^{\alpha'})$ is another system of local adapted coordinates in the tangent bundle TM_n, where

$$
\begin{aligned}
x^{\bar{\alpha}'} &= \frac{\partial x^{\alpha'}}{\partial x^{\beta}} y^{\beta} \\
x^{\alpha'} &= x^{\alpha'}(x^{\beta})
\end{aligned}
\tag{1.1}
$$

The Jacobian of (1.1) is given by the matrix

$$
A^{i'}_{j} = \left(\frac{\partial x^{i'}}{\partial x^{j}} \right) = \begin{pmatrix} \frac{\partial x^{\alpha'}}{\partial x^{\beta}} & \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\delta}} y^{\delta} \\ & \\ 0 & \frac{\partial x^{\alpha'}}{\partial x^{\beta}} y^{\beta} \end{pmatrix}. \tag{1.2}
$$

Let $T^*_x(M_n)$ be the cotangent space at a point $x \in M_n$ and element of $T^*_x(M_n)$ is called covector at $x \in M_n$. If a covector $p^*_x(M_n)$ whose components are given by p_{α} with respect to the natural coframe $\{dx^{\alpha}\}$ that is $p = p_i dx^i$, then by definition the set $t^*(M_n)$ of all points $(x^K) = (x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\bar{\alpha}}})$, where $x^{\bar{\bar{\alpha}}} = p_{\alpha}; K, L, ... = 1, ..., 3n$ with projection $\pi_2 : t^*(M_n) \to T^*(M_n)$ that is $\pi_2 : (x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\bar{\alpha}}}) \to (x^{\bar{\alpha}}, x^{\alpha})$ is a semi-cotangent (pull back) bundle of the cotangent bundle by submersion $\pi_1 : T^*(M_n) \to M_n$ [3].

The pull back bundle $t^*(M_n)$ of the cotangent bundle $T^*(M_n)$ also has the natural bundle structure over M_n. Its projection bundle $\pi : t^*(M_n) \to M_n$ which denotes the natural bundle structure of $t^*(M_n)$ over M_n and defined by $\pi : (x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\bar{\alpha}}}) \to (x^{\alpha})$, and it is easily verified that $\pi = \pi_1 \circ \pi_2$. Hence, $(t^*(M_n), \pi_1 \circ \pi_2)$ is the composite bundle [5] or step like bundle.

2. Complete Lifts of Vector Fields

If $(x^{\bar{\alpha}'}, x^{\alpha'}, x^{\bar{\bar{\alpha}}'})$ is system of local adapted coordinates in the $t^*(M_n)$, then we have

$$\begin{aligned} x^{\bar{\alpha}'} &= \frac{\partial x^{\alpha'}}{\partial x^{\beta}} y^{\beta} \\ x^{\alpha'} &= x^{\alpha'}(x^{\beta}) \\ x^{\bar{\bar{\alpha}}'} &= \frac{\partial x^{\beta}}{\partial x^{\alpha'}} p_{\beta} \end{aligned} \tag{2.1}$$

The Jacobian of equation (2.1) is given by the matrix

$$\bar{A} = \left(A_L^{K'}\right) = \begin{pmatrix} \frac{\partial x^{\alpha'}}{\partial x^{\beta}} & \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta}\partial x^{\delta}} y^{\delta} & 0 \\ 0 & \frac{\partial x^{\alpha'}}{\partial x^{\beta}} y^{\beta} & 0 \\ 0 & p_{\alpha}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial^2 x^{\alpha}}{\partial x^{\beta'}\partial x^{\alpha'}} & \frac{\partial x^{\beta}}{\partial x^{\alpha'}} \end{pmatrix} \tag{2.2}$$

it is obvious that the condition $Det\bar{A} \neq 0$ is to the condition $Det\left(\frac{\partial x^{\alpha'}}{\partial x^{\beta}}\right) \neq 0$.

Suppose that $X \in \Im_0^1(M_n)$ and X^{α} are components of X then we have $X = X^{\alpha}\partial_{\alpha}$. The complete lift ${}^CX$ of X to tangent bundle $T(M_n)$ is defined by ${}^CX = X^{\alpha}\partial_{\alpha} + (y^{\beta}\partial_{\beta}X^{\alpha})\partial\bar{\alpha}$ [8]. On putting

$${}^{CC}X = ({}^{CC}X^{\alpha}) = \begin{pmatrix} y^{\delta}\partial_{\delta}X^{\alpha} \\ X^{\alpha} \\ -p_{\delta}(\partial_{\alpha}X^{\delta}) \end{pmatrix} \tag{2.3}$$

by the virtue of equation (2.2), we have ${}^{CC}X' = \bar{A}({}^{CC}X)$. The vector field ${}^{CC}X$ is called the complete lift of ${}^CX$ to the semi-cotangent byndle $t^*(M_n)$.

Suppose that $\omega \in \Im_1^0(M_n), F \in \Im_1^1(T(M_n))$ and $T \in \Im_2^1(M_n)$. The vertical lift ${}^{VV}\omega$ of the ${}^V\omega$, $\gamma F \in \Im_0^1(t^*(M_n))$ and $\gamma T \in \Im_1^1(t^*(M_n))$ have the components on the semi-cotangent bundle $t^*(M_n)$

$${}^{VV}\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix}, \gamma F = (\gamma F^K) = \begin{pmatrix} 0 \\ 0 \\ p_{\beta}F_{\alpha}^{\beta} \end{pmatrix} \tag{2.4}$$

$$\gamma T = \left(\gamma T_L^K\right) = \begin{pmatrix} 0 & 0 & 0 \\ \\ 0 & 0 & 0 \\ \\ 0 & p_\delta T^\delta_{\beta\alpha} & 0 \end{pmatrix} \tag{2.5}$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\bar{\alpha}}})$ where $\omega_\alpha, F^\beta_\alpha$ and $T^\delta_{\beta\alpha}$ are local components of ω, F and T respectively.

If f is a function in M_n, we have write ${}^{VV}f$ the vertical lift of the function f on $t^*(M_n)$ is

$$ {}^{VV}f = {}^{V}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. \tag{2.6}$$

Let $X, Y \in \Im^1_0(T(M_n)), f \in \Im^0_0(M_n), \omega \in \Im^0_1(M_n)$ and $F \in \Im^1_1(T(M_n))$, we have

$$(i)\ {}^{CC}(X+Y) = {}^{CC}X + {}^{CC}Y, \tag{2.7}$$

$$(ii)\ {}^{CC}X^{VV}f = {}^{CC}(Xf), \tag{2.8}$$

$$(iii)\ [{}^{CC}X, {}^{CC}Y] = {}^{CC}[X, Y]$$

$$\Rightarrow \pounds_{{}^{CC}X}({}^{CC}Y) = {}^{CC}(\pounds_X Y) \tag{2.9}$$

$$(iv)\ [{}^{CC}X, {}^{VV}\omega] = {}^{VV}(\pounds_X \omega) \tag{2.10}$$

$$(v)\ [{}^{CC}X, \gamma F] = \gamma(\pounds_X F) \tag{2.11}$$

where $\pounds_X$ the operator of Lie derivation with respect to X.

3. Complete Lift of Tensor Fields of Type (1,1) and of Type (1,2)

Let $F \in \Im^1_1(T(M_n))$ and F^α_β local components of F. Then we have $F = F^\alpha_\beta \partial_\alpha \otimes dx^\beta$ [8]. Making use (2.2), we define F^{CC} for tensor field of type (1,1) on $t^*(M_n)$ whose components are given by

$$(3.1) \qquad {}^{CC}F = \left({}^{CC}F_L^K\right) = \begin{pmatrix} F_\beta^\alpha & y^\delta \partial_\delta F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix}$$

with respect to the coordinates $((x^{\bar{\alpha}}, x^\alpha, x^{\bar{\bar{\alpha}}})$ on $t^*(M_n)$, where $F_\beta^\alpha F_\theta^\beta = a^2 \delta_\theta^\beta$. The tensor field of type (1,1) ${}^{CC}F$ is called the complete lift of ${}^{C}F$ to the semi-cotangent bundle $t^*(M_n)$.

We now have the following propositions [6]:

Proposition 3.1. *Let $X \in \Im_0^1(T(M_n)), \omega \in \Im_1^0(M_n)$ and $F \in \Im_1^1(T(M_n))$ then*

$$(3.2) \qquad {}^{CC}F^{CC}X = {}^{CC}(FX) + \gamma(\pounds_X F)$$

$$(3.3) \qquad {}^{CC}F^{VV}\omega = {}^{VV}(\omega \circ F)$$

$$(3.4) \qquad \pounds_{{}^{CC}X}{}^{CC}F = 0 \; if \; \pounds_X F = 0$$

where $\pounds_X$ the operator of Lie derivation with respect to X.

Proposition 3.2. *Let $X \in \Im_0^1(T(M_n)), \omega \in \Im_1^0(M_n), F \in \Im_1^1(T(M_n))$ and $S, T \in \Im_2^1(M_n)$, then*

$$(3.5) \qquad (\gamma S)^{CC}X = \gamma(S_X)$$

$$(3.6) \qquad (\gamma S)(\gamma F) = 0$$

$$(3.7) \qquad {}^{CC}F(\gamma G) = \gamma(G \circ F)$$

$$(3.8) \qquad (\tilde{\gamma}S)\gamma(\pounds_X F) = \begin{pmatrix} 0 & y^\delta S_{\delta\beta}^\alpha & 0 \\ 0 & 0 & 0 \\ 0 & -p_\sigma S_{\beta\alpha}^\sigma & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ p_\sigma(\pounds_X F)_\alpha^\sigma) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $\pounds_X$ the operator of Lie derivation with respect to X.

Theorem 3.1. *If $F \in \Im_1^1(T(M_n))$ and $S \in \Im_2^1(M_n)$ then*

$$^{CC}F(\tilde{\gamma}S) = \tilde{\gamma}(SF), \tag{3.9}$$

where $S \in \Im_2^1(M_n)$ is defined by $(SF)(X,Y) = S(X,FY)$ for any $X,Y \in \Im_0^1(T(M_n))$ then

Proof: If $Z \in \Im_0^1(T(M_n))$, then from equations (3.5) and (3.7), we find

$$^{CC}F(\tilde{\gamma}S)^{CC}Z =^{CC} F(\tilde{\gamma}S)^{CC}Z =^{CC} F(\tilde{\gamma}S_Z) = \tilde{\gamma}(S_Z F) \tag{3.10}$$

But we have by equation (3.5),

$$(\tilde{\gamma}SF)^{CC}Z = \tilde{\gamma}(SF)_Z$$

Since $(SF)_Z Y = (SF)(Z,Y) = S(Z,FY) = (S_Z F)Y$, for all $Y \in \Im_0^1(T(M_n))$. again from equation (3.5),

$$\tilde{\gamma}(S_Z F) = \tilde{\gamma}(SF)_Z = \tilde{\gamma}(SF)^{CC}Z, \tag{3.11}$$

From equations (3.10) and (3.11), we get

$$(^{CC}F(\tilde{\gamma}S))^{CC}Z = \tilde{\gamma}(SF)^{CC}Z \Rightarrow^{CC} F(\tilde{\gamma}S) = \tilde{\gamma}(SF).$$

Hence, the proof is completed.

Theorem 3.2. *Let $F \in \Im_1^1(T(M_n))$ and $S \in \Im_2^1(M_n)$, then $(\tilde{\gamma}S)^{CC}F = \tilde{\gamma}(SF)$ if and only if*

$$S(X,FY) = S(FX,Y)$$

for any $X,Y \in \Im_0^1(M_n)$.

Proof: Let $X \in \Im_0^1(M_n)$. Then by virtue of equation (3.5), we get

$$\tilde{\gamma}(SF)^{CC}X = \tilde{\gamma}(SF)_X$$

On other hand, by equations (3.2), (3.5) and (3.6), we obtain

$$\begin{aligned} ((\tilde{\gamma}S)^{CC}F)^{CC}X &= (\tilde{\gamma}S)(^{CC}F^{CC}X) \\ &= (\tilde{\gamma}S)(^{CC}(FX) + \gamma(\pounds_X F)) \\ &= \tilde{\gamma}S_{FX}, \text{ as } \tilde{\gamma}S\gamma(\pounds_X F) = 0. \end{aligned}$$

Now, $S_{FX} = (SF)_X$ if and only if $S_{FX}Y = (SF)_X Y$ that is if and only if

$$S(FX,Y) = S(X,FY),\ \forall X,Y \in \Im^1_0(T(M_n)).$$

Next, using equatin (3.5), we get

$$\tilde{\gamma} S_{FX} = \tilde{\gamma}(SF)_X$$

i.e.

$$(\tilde{\gamma} S)^{CC} F)^{CC} X = (\tilde{\gamma}(SF))^{CC} X$$

if and only if

$$S_{FX} = (SF)_X.$$

Hence the proof is completed.

4. GF-Structure in the Semi-Cotangent Bundle

Let M_n be n-dimensional differentiable manifold of class C^∞ and $T(M_n)$ its tangent bundle. Suppose there exists a tensor field F of type (1,1) in $T(M_n)$ satisfying

$$F^2 = a^2 I \tag{4.1}$$

where a is any real or complex number. Then manifold $T(M_n)$ is said to posses a GF-structure [2].

Let F and $G \in \Im^1_1(T(M_n))$, then the torsion $N_{F,G}$ of the tensor field F and G of type (1,1) is the tensor field $N_{F,G}$ of type (1,1) defined by [8]

$$\begin{aligned} 2N_{F,G}(X,Y) &= [FX,GY] + [GX,FY] - F[GX,Y] - G[FX,Y] \\ &- F[X,GY] - G[FX,Y] + (FG+GF)[X,Y] \end{aligned} \tag{4.2}$$

where $X,Y \in \Im^1_0(T(M_n))$.

If we put $F = G$, then we have

$$N_F = N_{F,F}(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y] \tag{4.3}$$

which is the Nijenhuis tensor of F.

Theorem 4.1. *If $F \in \mathfrak{I}_1^1(T(M_n))$ is GF-structure in $\mathfrak{I}_1^1(t^*(M_n))$ and $N = N_F$ then*

$$(\tilde{\gamma}N)^{CC}F = \tilde{\gamma}(NF)$$

Proof: By Theorem (3.2), It is sufficient to show that

$$N(FX,Y) = N(X,FY),\ \forall \in \mathfrak{I}_0^1(TM_n).$$

This can be verified as follows

$$N[FX,Y] = [F^2X,FY] - F[F^2X,Y] - F[FX,FY] + F^2[FX,Y]$$

since $F^2 = a^2I$, we get

$$N[FX,Y] = a^2[X,FY] - a^2F[X,Y] - F[FX,FY] + a^2[FX,Y]$$

and

$$N[FX,Y] = [FX,F^2Y] - F[FX,FY] - F[X,F^2Y] + F^2[X,FY]$$

$$N[FX,Y] = a^2[FX,Y] - F[FX,FY] - a^2F[X,Y] + a^2[X,FY]$$

Thus, we have

$$N[FX,Y] = N[FX,Y].$$

The proof is completed.

Theorem 4.2. *If $F \in \mathfrak{I}_1^1(T(M_n)),\ F^2 = a^2I$ then*

$$({}^{CC}F)^2 = a^2I - \gamma(N_F). \tag{4.4}$$

Proof: By the virtue of equations (2.5) and (3.1), we have

$$
\begin{aligned}
({}^{CC}F)^2 &= \begin{pmatrix} F^{\alpha}_{\beta} & y^{\delta}\partial_{\delta}F^{\alpha}_{\beta} & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta}) & F^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} F^{\beta}_{\theta} & y^{\delta}\partial_{\delta}F^{\beta}_{\theta} & 0 \\ 0 & F^{\beta}_{\theta} & 0 \\ 0 & p_{\sigma}(\partial_{\theta}F^{\sigma}_{\beta} - \partial_{\beta}F^{\sigma}_{\theta}) & F^{\theta}_{\beta} \end{pmatrix} \\
&= \begin{pmatrix} a^2\delta^{\beta}_{\theta} & 0 & 0 \\ 0 & a^2\delta^{\beta}_{\theta} & 0 \\ 0 & 0 & a^2\delta^{\theta}_{\beta} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -p_{\sigma}(N_F)^{\sigma}_{\theta\alpha} & 0 \end{pmatrix} \\
&= a^2 I - \gamma(N_F) \qquad (4.5)
\end{aligned}
$$

where $F^{\alpha}_{\beta}F^{\beta}_{\theta} = a^2\delta^{\beta}_{\theta}$.

Theorem 4.3. *Let ${}^{CC}F$ be GF-structure in $t^*(M_n)$ and $N_{{}^{CC}F}$ be the Nijenhuis tensor of ${}^{CC}F$. then*

$$N_{{}^{CC}F} = N({}^{CC}X, {}^{CC}Y) = 0$$

if and only if $N_F = 0$, where $X, Y \in \Im^1_0(T(M_n))$ and N_F be Nijenhuis tensor of $F \in \Im^1_1(T(M_n))$.

Proof: By the definition of Nijenhuis tensor, we have

$$
\begin{aligned}
N_{{}^{CC}F} = N({}^{CC}X, {}^{CC}Y) &= [{}^{CC}F\,{}^{CC}X, {}^{CC}F\,{}^{CC}Y] - {}^{CC}F[{}^{CC}F\,{}^{CC}X, {}^{CC}Y] \\
&\quad - {}^{CC}F[{}^{CC}X, {}^{CC}F\,{}^{CC}Y] + {}^{CC}F^2[{}^{CC}X, {}^{CC}Y] \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{N}({}^{CC}X, {}^{CC}Y) &= [{}^{CC}F\,{}^{CC}X, {}^{CC}F\,{}^{CC}Y] - {}^{CC}F[{}^{CC}F\,{}^{CC}X, {}^{CC}Y] \\
&\quad - {}^{CC}F[{}^{CC}X, {}^{CC}F\,{}^{CC}Y] + a^2[{}^{CC}X, {}^{CC}Y]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[{}^{CC}(FX) + \gamma \pounds_X F, {}^{CC}(FY) + \gamma \pounds_Y F\right] \\
&- {}^{CC}F\left[{}^{CC}(FX) + \gamma \pounds_X F, {}^{CC}Y\right] \\
&- {}^{CC}F\left[{}^{CC} - X, {}^{CC}(FY) + \gamma \pounds_Y F\right] \\
&+ a^{2CC}[X,Y]
\end{aligned}
$$

$$
\begin{aligned}
&= {}^{CC}\{[FX,FY] - F[FX,Y] - F[X,FY] + a^2[X,Y]\} \\
&- \gamma\{\pounds_X(\pounds_{FY}F - F\pounds_Y F) - \pounds_Y(\pounds_{FX}F - F\pounds_X F) \\
&- \pounds_{F[X,Y]}F + F\pounds_{[X,Y]}F\}
\end{aligned}
$$

where we used the relation

$$\pounds_X \pounds_Y F - \pounds_Y \pounds_X F = \pounds_{[X,Y]}F.$$

Thus, we have

$$\tilde{N}({}^{CC}X, {}^{CC}Y) = {}^{CC}(N(X,Y)) + \gamma P \tag{4.7}$$

where P is tensor field of type (1,1) in $T(M_n)$ given by

$$P = \pounds_Y \pounds_{FX}F - \pounds_X \pounds_{FY}F + (\pounds_X F)(\pounds_Y F) - (\pounds_X F)(\pounds_Y F) - (\pounds_{[X,Y]}F)F$$

Since $\tilde{N}^{CC} = 0$, then from (4.7), we have

$${}^{CC}(N(X,Y)) + \gamma P = 0$$

This shows that $N(X,Y) = 0$ for all $X,Y \in \Im_0^1(T(M_n))$. Thus, F is integrable. Hence the proof is completed.

Theorem 4.4. *Let F be a GF-structure on $T(M_n)$, then the complete lift of ${}^{CC}F$ of F on $t^*(M_n)$ is a GF-structure on $t^*(M_n)$ iff F is integrable.*

Proof: In the view of Theorem (4.2), we have

$$({}^{CC}F)^2 = F^2 - \gamma(N_F)$$

since F is a GF-structure i.e. $F^2 = a^2 I$, then

$$({}^{CC}F)^2 = a^2I - \gamma(N_F)$$

So, $({}^{CC}F)^2 = a^2I$ if and only if $N_F = 0$ Hence, $({}^{CC}F)^2$ gives GF-structure on on $t^*(M_n)$ iff F is integrable.

Theorem 4.5. *Let M_n be a differentiable manifold and its tangent bundle $T(M_n)$ admiiting with the Nijenhuis tensor N_F. Then*

$$ {}^{CC}F + \frac{1}{2a^2}\gamma(NF) \tag{4.8}$$

defines GF-structure on $t^*(M_n)$.

Proof:

$$\begin{aligned}\left({}^{CC}F + \frac{1}{2a^2}\gamma(NF)\right)^2 &= \left({}^{CC}F + \frac{1}{2a^2}\gamma(NF)\right)\left({}^{CC}F + \frac{1}{2a^2}\gamma(NF)\right)\\ &= ({}^{CC}F)^2 + \frac{1}{2a^2}{}^{CC}F\gamma(NF) + \frac{1}{2a^2}\gamma(NF){}^{CC}F\\ &= ({}^{CC}F)^2 + \frac{1}{2a^2}\gamma(NF^2) + \frac{1}{2a^2}\gamma(NF^2)\\ &= ({}^{CC}F)^2 + \frac{1}{a^2}\gamma(NF^2)\end{aligned}$$

using (4.1) and (4.4), we have

$$\begin{aligned}\left({}^{CC}F + \frac{1}{2a^2}\gamma(NF)\right)^2 &= a^2I - \gamma N + \frac{1}{a^2}\gamma(NF^2)\\ \left({}^{CC}F + \frac{1}{2a^2}\gamma(NF)\right)^2 &= a^2I\end{aligned}$$

which proves the theorem.

Theorem 4.6. *The GF-structure ${}^{CC}F + \frac{1}{2a^2}\gamma(NF)$ in $t^*(M_n)$ is integrable iff the GF-structure F in $T(M_n)$is integrable.*

Proof: Let us suppose that F is integrale, then $N = 0$. Hence

$$ {}^{CC}F + \frac{1}{2a^2}\gamma(NF) = {}^{CC}F$$

and Theorem 4.4 implies ${}^{CC}F$ is also integrable.

Conversely, we suppose that ${}^{CC}F+\frac{1}{2a^2}\gamma(NF)$ is integrable, then the Nijenhuis tensor $\tilde{N}$ of ${}^{CC}F+\frac{1}{2a^2}\gamma(NF)$ is zero in $t^*(M_n)$. Taking account of the definition of the Nijenhuis tensor and theorem 4.5, we have

$$\begin{aligned}
\tilde{N}({}^{CC}X,{}^{CC}Y) &= \left[\left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)^{CC}X,\left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)^{CC}Y\right]\\
&- \left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)\left[\left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)^{CC}X,{}^{CC}Y\right]\\
&- \left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)\left[{}^{CC}X,\left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)^{CC}Y\right]\\
&+ a^2[{}^{CC}X,{}^{CC}Y]
\end{aligned}$$

$$\begin{aligned}
&= \left[{}^{CC}(FX)+\gamma\left(£_XF+\frac{1}{2a^2}\gamma(NF)_X\right),{}^{CC}(FY)+\gamma\left(£_YF+\frac{1}{2a^2}\gamma(NF)_Y\right)\right]\\
&- \left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)\left[{}^{CC}(FX)+\gamma\left(£_XF+\frac{1}{2a^2}\gamma(NF)_X\right),{}^{CC}Y\right]\\
&- \left({}^{CC}F+\frac{1}{2a^2}\gamma(NF)\right)\left[{}^{CC}X,{}^{CC}(FY)+\gamma\left(£_YF+\frac{1}{2a^2}\gamma(NF)_Y\right)\right]\\
&+ a^{2CC}[X,Y]
\end{aligned}$$

$$\begin{aligned}
&= {}^{CC}\{[FX,FY]-F[FX,Y]-F[X,FY]+a^2[X,Y]\}\\
&+ \gamma\{£_Y£_{FX}F+\frac{1}{2a^2}£_{FX}(NF)_Y-£_X£_{FY}F-\frac{1}{2a^2}£_{FX}(NF)_X\\
&+ \left(£_XF+\frac{1}{2a^2}(NF)_X\right)\left(£_YF+\frac{1}{2a^2}(NF)_Y\right)\\
&- \left(£_YF+\frac{1}{2a^2}(NF)_Y\right)\left(£_XF+\frac{1}{2a^2}(NF)_X\right)\}\\
&- (£_{[X,Y]}F)F+\frac{1}{2a^2}(£_Y(NF)_X)F-\frac{1}{2a^2}(£_X(NF)_Y)F\\
&- \frac{1}{2a^2}(NF)_{[FX,Y]}-\frac{1}{2a^2}(NF)_{[X,FY]}
\end{aligned}$$

where

$$£_X£_YF-£_Y£_XF=£_{[X,Y]}F. \tag{4.9}$$

Thus, we have

$$\tilde{N}(^{CC}X,^{CC}Y) =^{CC} (N(X,Y)) + \gamma P \tag{4.10}$$

where P is tensor field of type (1,1) in $T(M_n))$ given by

$$\begin{aligned} P &= £_Y £_{FX} F + \frac{1}{2a^2} £_{FX} (NF)_Y - £_X £_{FY} F - \frac{1}{2a^2} £_{FX} (NF)_X \\ &+ (£_X F)(£_Y F) + \frac{1}{2a^2}(£_Y F)(NF)_X) + \frac{1}{2a^2}(£_X F)(NF)_Y \\ &+ \frac{1}{4a^4}(NF)_X (NF)_Y - (£_X F)(£_Y F) - \frac{1}{2a^2}(£_Y F)(NF)_X \\ &- \frac{1}{2a^2}(£_X F)(NF)_Y - \frac{1}{4a^4}(NF)_X (NF)_Y - (£_{[X,Y]} F)F \\ &+ \frac{1}{2a^2}(£_Y (NF)_X)F - \frac{1}{2a^2}(£_X (NF)_Y)F \\ &- \frac{1}{2a^2}(NF)_{[FX,Y]} - \frac{1}{2a^2}(NF)_{[X,FY]} \end{aligned}$$

Since $^{CC}\tilde{N} = 0$, then from (4.10), we have

$$^{CC}(N(X,Y)) + \gamma P = 0$$

This shows that $N(X,Y) = 0$ for all $X, Y \in \Im_0^1(T(M_n))$. Thus, F is integrable. Hence the proof is completed.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] H. Cayir, Integrability conditions and Tachibana operators according to $^{CC}F - \frac{1}{2}\gamma(NF)$ on semi-cotangent bundle $t^*(M_n)$, Karaelmas Fen ve Müh. Derg. 7 (2017), 165-170.

[2] K.L. Duggal, On differentiable structures defined by algebraic equation I, Nijenhuis tensor. Tensor N.S., 22 (2) (1971), 238-242.

[3] D. Husemoller, Fibre Bundles. New York, NY, USA, Springer, 1994.

[4] M.N.I. Khan, Tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold, Facta Univ. Ser. Math. Inform. 35 (1) (2020), 167-178.

[5] W.A. Poor, Differential Geometric Structures, McGraw-Hill, New York, 1981.

[6] F. Yildirim, On a special class of semi-cotangent bundle, Proceedings of the Institute of Mathematics and Mechanics, Nat. Acad. Sci. Azerbaijan, 41 (1) (2015), 25–38.

[7] F. Yıldırım, A. Salimov, Semi-cotangent bundle and problems of lifts, Turk. J. Math. 38 (2014), 325-339.

[8] K. Yano, S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker, Inc., New York, 1973.