Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 6, 2658-2673
https://doi.org/10.28919/jmcs/4979
ISSN: 1927-5307

APPLICATION OF PARTIAL MEASURE OF NONCOMPACTNESS TO TRIPLED FIXED POINTS

SACHIN V. BEDRE*

Department of Mathematics, Dr. S. D. Devsey Arts, Commerce and Science College, Wada-421303, Palghar, India
(Affilated to University of Mumbai)

Abstract. This paper aims to show some tripled fixed point theorems associated with measure of noncompactness via partially condensing mapping having mixed monotone properties in partially ordered metric spaces.

Keywords: triple fixed point; partially ordered metric space; measure of noncompactness; partially condensing mapping.

2010 AMS Subject Classification: 34A12, 34A45, 47H07, 47H10.

1. Introduction and Preliminaries

Fixed point theorems are very important for proving the existence of solutions for some nonlinear differential and integral equations. The mixed arguments from various branches of mathematics utilized for the examination of fixed point theory. The fixed point problem of contractive mappings in partially ordered metric spaces has been considered recently by Ran and Reurings [13], Bhaskar and Lakshmikantham [6], Nieto and Rodriguez-Lopez [11], Dhage [7], Shrivastava et.al. [14], and Bedre et.al. [2, 3].

*Corresponding author
E-mail address: sachin.bedre@yahoo.com
Received August 27, 2020

In [5, 4], Berinde and Borcut began the analysis of a triple hybrid fixed-point theorem for nonlinear mapping in partially ordered metric spaces and obtained its existence results which Afshari et.al.[1] further generalized with a slightly different method. The tripled fixed point theorems are well known to have nice applications to dynamic systems based on nonlinear tripled functional differential, integral and integro-differential equations to prove the existence of tripled solutions.

Throughout this study, we present a new method focused on the combination of the noncompactness measure with a new tripled fixed point theorem of partially condensing mapping $\mathscr{F}$ in X^3.

First we are reminding ourselves of some history and gathering some valuable results that are important for our further research.

Definition 1.1. X is regular if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in X and $x_n \to x^*$ as $n \to \infty$, then $x_n \leq x^*$ (resp. $x_n \geq x^*$.) for all $n \in N$.

The regularity of X can be found in and the references in Guo and Lakshmikantham [10].

Definition 1.2. A mapping $\mathscr{T} : X \to X$ is called monotone non-decreasing if $x \leq y$ implies $\mathscr{T}x \leq \mathscr{T}y$ for all $x, y \in X$.

Definition 1.3. A mapping $\mathscr{T} : X \to X$ is called monotone non-increasing if $x \leq y$ implies $\mathscr{T}x \geq \mathscr{T}y$ for all $x, y \in X$.

Definition 1.4. A mapping $\mathscr{T} : X \to X$ is called monotone if it is either monotone non-increasing or monotone non-decreasing.

Defnition 1.5.[14] A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a monotone dominating function or, in short, an M-function if it is an upper or lower semi-continuous and monotonic non-decreasing or non-increasing function satisfying the condition: $\varphi(0) = 0$.

Defnition 1.6.[14] Given a partially-ordered normed linear space E, a mapping $\mathscr{T}: E \to E$ is called partially M-Lipschitz or partially nonlinear M-Lipschitz if there is an M-function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|\mathscr{T}x - \mathscr{T}y\| \preceq \varphi(\|x-y\|)$$

for all comparable elements $x, y \in E$. The function is called an M-function of $\mathscr{T}$ on E. If $\varphi(r) = kr \ \ (k > 0)$, then $\mathscr{T}$ is called partially M-Lipschitz with the Lipschitz constant k. In particular, if $k < 1$, then $\mathscr{T}$ is called a partially M-contraction on X with the contraction constant k. Further, if $\varphi(r) < r$, for $r > 0$, then $\mathscr{T}$ is called a partially nonlinear M-contraction with an M-function φ of $\mathscr{T}$ on X.

Definition 1.7. A nondecreasing mapping $\mathscr{T}: E \to E$ is called nonlinear partial M-set-Lipschitz if there exists a M-function φ such that

$$\mu_p(\mathscr{T}(C)) \leq \varphi(\mu_p(C))$$

for all bounded chain C in E. $\mathscr{T}$ is called partial k-set-Lipschitz if $\varphi(r) = kr$, $k > 0$. $\mathscr{T}$ is called partial k-set-contraction if it is a partial k-set-Lipschitz with $k < 1$. Finally, $\mathscr{T}$ is called a nonlinear partial M-set-contraction in E if it is a nonlinear partial M-Lipschitz with $\varphi(r) < r$ for $r > 0$.

Defnition 1.8. An operator $\mathscr{T}$ on a normed linear space E into itself is called compact if $\mathscr{T}(E)$ is a relatively compact subset of E. $\mathscr{T}$ is called totally bounded if, for any bounded subset S of E, $\mathscr{T}(S)$ is a relatively compact subset of E. If $\mathscr{T}$ is continuous and totally bounded, then it is called completely continuous on E.

Defnition 1.9. An operator $\mathscr{T}$ on a normed linear space E into itself is called partially compact if $\mathscr{T}(C)$ is a relatively compact subset of E for all totally ordered set or chain C in E. The operator $\mathscr{T}$ is called partially totally bounded if, for any totally ordered and bounded subset C of E, $\mathscr{T}(C)$ is a relatively compact subset of E. If the operator $\mathscr{T}$ is continuous and partially totally bounded, then it is called partially completely continuous on E.

In comparison, Dhage[8] introduced the compatiblity principle as follows.

Definition 1.10.[8] The order relation $\leq$ and the norm $\|.\|$ in a non-empty set X are said to be compatible if $\{x_n\}$ is a monotone sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x_0 implying that the whole sequence $\{x_n\}$ converges to x_0. Similarly, given a partially-ordered normed linear space $(X, \leq, \|\cdot\|)$, the ordered relation $\leq$ and the norm $\|.\|$ are said to be compatible if $\leq$ and the metric d define through the norm are compatible.

Now, we present some preliminary findings about non-compact measurements in Banach spaces that we will use in the sequel. We emphasize that non-compact measurements are very useful tools in the theory of operator equations in Banach spaces. Quite frequently, they are used in functional equation analysis, including ordinary differential equations, partial derivative equations, etc.

Definition 1.11. A mapping $\mu_p : P_{bd,cn}(E) \to \mathbb{R}^+ = [0, \infty)$ is said to be a partial measure of noncompactness in E if it satisfies the following conditions:

(P_1) $\phi \neq (\mu_p)^{-1}(0) \subseteq P_{rcp,cn}(E)$. (kernel compactivity)

(P_2) $\mu_p(\overline{C}) = \mu_p(C)$. (closure property)

(P_3) μ_p is nondecreasing, i.e., if $C_1 \subseteq C_2 \Rightarrow \mu_p(C_1) \leq \mu_p(C_2)$. (monotonicity)

(P_4) If $\{C_n\}$ is a sequence of closed chains from $P_{bd,cn}(E)$ such that $C_{n+1} \subseteq C_n$ (n = 1, 2, ...) and if $\lim_{n\to\infty} \mu_p(C_n) = 0$, then the set $\overline{C}_\infty = \cap_{n=1}^{\infty} C_n$ is nonempty. (limit intersection property)

The family of sets described in (P_1) is said to be the kernel of the partial measure of noncompactness μ_p and is defined as

$$\ker\mu_p = \{C \in P_{bd,cn}(E) | \mu_p(C) = 0\}$$

Clearly, $\ker\mu_p \subset P_{rcp,cn}(E)$. Observe that the intersection set C_∞ from condition (P_4) is a member of the family $\ker\mu_p$. In fact, since $\mu_p(C_\infty) \subseteq \mu_p(C_n)$ for any n, we infer that $\mu_p(C_\infty) = 0$.

This yields that $C_\infty \in \ker\mu_p$. This simple observation will be essential in our further investigations. The partial measure μ_p of noncompactness is called full if it satisfies

(P$_5$) $\ker\mu_p = P_{rcp,cn}(E)$. Finally, μ_p is said to satisfy maximum property if

(P$_6$) $\mu_p(C_1 \cup C_2) = \max\{\mu_p(C_1), \mu_p(C_2)\}$.

By $P_{cl}(E)$, $P_{bd}(E)$, $P_{rcp}(E)$, $P_{cn}(E)$, $P_{bd,cn}(E)$, $P_{rcp,cn}(E)$ respectively, we denote the family of all non-empty and closed, bounded, relatively compact chains, bounded chains and relatively compact chains of E.

The following lemma is frequently used in the analytical fixed point theory of metric spaces.

Lemma 1.1. *If φ is a M-function with $\varphi(r) < r$ for $r > 0$, then $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0,\infty)$ and vice versa.*

2. Main Results

In this section, we develop a new triple fixed point theorems in partially ordered metric spaces for mapping having mixed monotone properties. These fixed point results are very interesting and may have a number of applications. It's going to serve as a key tool for developing our future theory of existence. We need to recall the following more or less well-known results and prove some lemmas before we make a formal statement of our fixed point result.

Let $(X,\leq)$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. Consider on the product space $X \times X \times X$ the following partial order: for $(x,y,z),(u,v,w) \leq X \times X \times X$, $(u,v,w) \leq (x,y,z) \Leftrightarrow x \geq u, y \leq v, z \geq w$.

Definition 2.1.[4] An element $(x,y,z) \in X$. is called a tripled fixed point of a given mapping $\mathscr{F} : X \times X \times X \to X$ if $\mathscr{F}(x,y,z) = x$, $\mathscr{F}(y,x,y) = y$, and $\mathscr{F}(z,y,x) = z$.

Definition 2.2.[4]. Let $(X,\preceq)$ be a partially ordered set and $\mathscr{F} : X \times X \times X \to X$. The mapping $\mathscr{F}$ is said to have the mixed monotone property if for any $x,y,z \in X$,

$$\begin{aligned} &x_1, x_2 \in X,\ x_1 \le x_2 \Rightarrow \mathscr{F}(x_1, y, z) \le F(x_2, y, z), \\ &y_1, y_2 \in X,\ y_1 \le y_2 \Rightarrow \mathscr{F}(x, y_1, z) \ge F(x, y_2, z), \\ &z_1, z_2 \in X,\ z_1 \le z_2 \Rightarrow \mathscr{F}(x, y, z_1) \le F(x, y, z_2). \end{aligned} \tag{2.1}$$

Lemma 2.1. *If (X, d) be a partially ordered metric space and The mapping $d_2 : X \times X \times X \to X$ is given by $d_2[(x,y,z),(u,v,w)] = \frac{1}{3}[d(x,u)+d(y,v)+d(z,w)]$ then $(X^3, \le, d_2)$ is a partially ordered metric space. Moreover, if X is complete, then (X^3, d_2) is also a complete metric space.*

Lemma 2.2. *If $(X, \le, d)$ is a regular, then $(X^3, \le, d_2)$ is also a regular partially ordered metric space.*

Proof. Let $(X, \le, d)$ is a regular partially ordered metric space. Then, by definition, if $\{x_n\}$ is a monotone nondecreasing sequence of points in X and $\lim_{n\to\infty} x_n = x^*$, then $x_n \le x^*$ for all $n \in \mathbb{N}$. Assume that $\{w_n\} = \{(x_n, y_n, z_n)\}$ be a sequence of points in X^3 such that

$$w_1 \le w_2 \le \dots \le w_n \le \dots$$

where $\{x_n\}$ and $\{z_n\}$ are monotone nondecreasing, and $\{y_n\}$ is monotone non-increasing in X. Now, let $\lim_{n\to\infty} w_n = w^*$. Then $d_2(w_n, w^*) = 0$ as $n \to \infty$ which by definition of the metric d_2 implies that

$$\lim_{n\to\infty} \frac{1}{3}[d(x_n, x^*) + d(y_n, y^*) + d(z_n, z^*)] = 0$$

. Consequently, $x_n \to x^*$, $y_n \to y^*$ and $z_n \to z^*$. Since X is regular, one has $x_n \le x^*, y_n \ge y^*$ and $z_n \le z^*$, for each $n \in N$. By definition of the order relation $\le$ we obtain

$$w_n = (x_n, y_n, z_n) \le (x^*, y^*, z^*) = w^*$$

for all $n \in \mathbb{N}$. Hence, $(X^3, \le, d_2)$ is a regular partially ordered metric space. □

Lemma 2.3. *If the order relation $\le$ and the metric d are M-compatible in metric space $(X, \le, d)$, then the order relation $\le$ and the metric d_2 are M-compatible in metric space $(X^3, \le, d_2)$.*

Proof. Let $\{w_n\} = \{(x_n, y_n, z_n)\}$ be a monotone nondecreasing sequence of points in X^3, where $\{x_n\}, \{y_n\}$ are monotone nondecreasing and $\{z_n\}$ is monotone nonincreasing sequences of points in X, respectively. Suppose that $\{w_n\}$ has a convergent subsequence $\{w_{k_n}\}$ converging to the point $w^* = (x^*, y^*, z^*)$. Then, we have

$$\begin{aligned} d_2(w_{k_n}, w^*) &= d_2((x_{k_n}, y_{k_n}, z_{k_n}), (x^*, y^*, z^*)) \\ &= \frac{1}{3}[d(x_{k_n}, x^*) + d(y_{k_n}, y^*) + d(z_{k_n}, z^*)] \\ &\to 0 \; as \; n \to \infty \end{aligned}$$

Therefore, $\{x_{k_n}\}$, $\{y_{k_n}\}$ and $\{z_{k_n}\}$ are convergent subsequences of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converging, respectively, to the points x^*, y^* and z^* in X. Since the order relation $\leq$ and the metric d are M-compatible in $(X, \leq, d)$, As a result, the original sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^*, respectively. Therefore, we have

$$\begin{aligned} d_2(w_n, w^*) &= d_2((x_n, y_n, z_n), (x^*, y^*, z^*)) \\ &= \frac{1}{3}[d(x_n, x^*) + d(y_n, y^*) + d(z_n, z^*)] \qquad (2.2) \\ &\to 0 \; as \; n \to \infty \end{aligned}$$

This shows that $w_n \to w^*$. Consequently, the order relation $\leq$ and the metric d are M-compatible in $(X^3, \leq, d_2)$ and the proof of the lemma is complete. □

Lemma 2.4. *If μ_p is a partial measure of noncompactness in a partially ordered metric space X, then the function $\widetilde{\mu_p} : \mathscr{P}_{bd,cn}(X^3) \to \mathbb{R}^+$ defined by*

$$\widetilde{\mu_p}(B \times C \times D) = \mu_p(B) + \mu_p(C) + \mu_p(D) \qquad (2.3)$$

where $(B \times C \times D) \in \mathscr{P}_{bd,cn}(X) \times \mathscr{P}_{bd,cn}(X) \times \mathscr{P}_{bd,cn}(X)$ is a partial measure of noncompactness in X^3.

Proof. We shall prove that $\widetilde{\mu_p}$ satisfies all the conditions (P_1) through (P_4) of the partial measure of noncompactness in X^3. First we prove the kernel compactivity of $\widetilde{\mu_p}$. Let $\mathscr{C} = B \times C \times D$ a chain in X^3 for some $B, C, D \in \mathscr{P}_{bd,cn}(X)$ such that $\widetilde{\mu_{\mathscr{C}}} = 0$. Then $\mu_(B) = 0, \mu_(C) = 0$ and $\mu_(D) = 0$. As a result $\phi \neq B \in \mathscr{P}_{rcp,cn}(X), \phi \neq C \in \mathscr{P}_{rcp,cn}(X)$ and $\phi \neq D \in \mathscr{P}_{rcp,cn}(X)$. Therefore, $\phi \neq B \times C \times D \in \mathscr{P}_{rcp,cn}(X) \times \mathscr{P}_{rcp,cn}(X) \times \mathscr{P}_{rcp,cn}(X)$. Consequently, $\phi \neq \mathscr{C} = B \times C \times D \in (\widetilde{\mu_p})^{-1}(\{0\}) \subset \mathscr{P}_{rcp,cn}(X^3)$.

Now, by closure property of the partial measure of noncompactness, we get

$$\begin{aligned}\widetilde{\mu}_p(\mathscr{C}) &= \widetilde{\mu}_p(B\times C\times D)\\ &= \mu_p(B)+\mu_p(C)+\mu_p(D)\\ &= \mu_p(\overline{B})+\mu_p(\overline{C})+\mu_p(\overline{D})\\ &= \widetilde{\mu}_p(\overline{B}\times\overline{C}\times\overline{D})\\ &= \widetilde{\mu}_p\overline{(B\times C\times D)}\\ &= \widetilde{\mu}_p(\overline{\mathscr{C}})\end{aligned}$$

and so, $\widetilde{\mu}_p$ satisfies the closure property.

Let $\mathscr{C}_1 = B_1\times C_1\times D_1$ and $\mathscr{C}_2 = B_2\times C_2\times D_2$ be two chains in X^3 for some chains $B_1,\ B_2,\ C_1,\ C_2$ and $D_1,\ D_2$ in X. Assume that $\mathscr{C}_1\subset\mathscr{C}_2$. Then By monotone property of $\widetilde{\mu}_p$, we obtain

$$\begin{aligned}\widetilde{\mu}_p(\mathscr{C}_1) &= \widetilde{\mu}_p(B_1\times C_1\times D_1)\\ &= \mu_p(B_1)+\mu_p(C_1)+\mu_p(D_1)\\ &\le \mu_p(B_2)+\mu_p(C_2)+\mu_p(D_2)\\ &= \widetilde{\mu}_p(B_2\times C_2\times D_2)\\ &= \widetilde{\mu}_p(\mathscr{C}_2)\end{aligned}$$

and so, $\widetilde{\mu}_p$ satisfies the monotone property.

Let $\{C_n\}$ be a sequence of closed chains from $\mathscr{P}_{bd,cn}(X^3)$ such that $\mathscr{C}_{n+1}\subset\mathscr{C}_n$ $(n=1,2,\ldots)$ and let $\lim_{n\to\infty}\mu_p(\mathscr{C}_n)=0$. Then there exist nondecreasing sequences $\{B_n\}$, $\{C_n\}$ and $\{D_n\}$ of chains in X such that $\mathscr{C}_n = B_n\times C_n\times D_n$ for each $n=1,2,\ldots$. Moreover, $\lim_{n\to\infty}\mu_p(B_n)=0$, $\lim_{n\to\infty}\mu_p(C_n)=0$ and $\lim_{n\to\infty}\mu_p(D_n)=0$

As μ_p is a partial measure, by property (P_4), we obtain

$$\overline{B}=\cap_{n=1}^{\infty}B_n\neq\phi\neq\overline{C}=\cap_{n=1}^{\infty}C_n\neq\phi\neq\cap_{n=1}^{\infty}D_n=\overline{D}$$

Hence the chain

$$\overline{\mathscr{C}}=\overline{B}\times\overline{C}\times\overline{D}=\cap_{n=1}^{\infty}(B_n\times C_n\times D_n)=\cap_{n=1}^{\infty}\mathscr{C}_n$$

is a nonempty chain in X^3.

Thus, $\widetilde{\mu}_p$ satisfies all the properties of a partial measure of noncompactness and hence it is a partial measure of noncompactness on X^3. □

The following definition is crucial for our further work.

Definition 2.3. A mapping $\mathscr{T} : E^3 \to E^3$ is called nonlinear partial M-set-contraction if there exist a M-function φ such that

$$\widetilde{\mu}_p(\mathscr{T}(\mathscr{C})) \leq \varphi(\widetilde{\mu}_p(\mathscr{C})) \tag{2.4}$$

for all bounded chains $\mathscr{C}$ of E^3, where $\varphi(r) < r$ for $r > 0$. In the special case when $\varphi(r) = kr,\ 0 < k < 1$, $\mathscr{T}$ is called a partial k-set-contraction mapping on E^3.

The following theorem is an important result in fixed point topology theory. It generalizes, in some way, Nieto and Rodriguez-Lopez[11] fixed point theorem. The consequence is an analog of Dhage[7] for a strong topology in triple metric space. In this relation, we are also referring to the paper by Heikill'a and Lakshmikantham[9] for additional related information.

Theorem 2.1. *Let $(X, \leq, d)$ be a regular partially ordered complete metric space such that the metric d and the order relation $\leq$ are compatible in every compact chain $\mathscr{C}$ of X. Suppose that $\mathscr{F} : X^3 \to X$ is a partially continuous and partially bounded mixed monotone tripled mapping satisfying*

$$\begin{aligned} &\mu_p(\mathscr{F}(B\times C\times D)) + \mu_p(\mathscr{F}(C\times B\times C)) + \mu_p(\mathscr{F}(D\times C\times B)) \\ &\qquad \leq \varphi(\mu_p(B) + \mu_p(C) + \mu_p(D)) \end{aligned} \tag{2.5}$$

for all $B, C, D \in \mathscr{P}_{bd,cn}(X)$, where φ is M-function satisfies $\varphi(r) < r,\ \ r > 0$. If there exists an element $(x_0, y_0, z_0) \in X \times X \times X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$ or $x_0 \geq F(x_0, y_0, z_0)$, $y_0 \leq F(y_0, x_0, y_0)$ and $z_0 \geq F(z_0, y_0, x_0)$, then $\mathscr{F}$ has a tripled fixed point (x^*, y^*, z^*) and the sequences $\{\mathscr{F}_n(x_0, y_0, z_0)\}$, $\{\mathscr{F}_n(y_0, x_0, y_0)\}$ and $\{\mathscr{F}_n(z_0, y_0, x_0)\}$ of successive iterations converge monotonically to x^*, y^* and z^*, respectively. Moreover, the set of all comparable tripled fixed points is compact.

Proof. Our main purpose in the immediate sequel is to prove the theorem in the event that there exists $(x_0,y_0,z_0) \in X \times X \times X$ such that $x_0 \leq F(x_0,y_0,z_0)$, $y_0 \geq F(y_0,x_0,y_0)$ and $z_0 \leq F(z_0,y_0,x_0)$. The proof for the case $x_0 \geq F(x_0,y_0,z_0)$, $y_0 \leq F(y_0,x_0,y_0)$ and $z_0 \geq F(z_0,y_0,x_0)$ is similar and can be gained by using equivalent arguments with suitable modification. We shall built a mapping $\mathscr{T} : X^3 \to X^3$ by

$$\mathscr{T}(\mathscr{Z}) = (\mathscr{F}(x,y,z), \mathscr{F}(y,x,y), \mathscr{F}(z,y,x)) \tag{2.6}$$

For all $\mathscr{Z} = (x,y,z) \in X \times X \times X = X^3$.

Obviously $\mathscr{T}$ defines a mapping $\mathscr{T} : X^3 \to X^3$. First we prove that $\mathscr{T}$ is a partially continuous on X^3. Let $\mathscr{Q} = (x,y,z)$ and $\mathscr{S} = (u,v,w)$ be two comparable elements of X^3. Without loss of generality, we may assume that $\mathscr{Q} \geq \mathscr{S}$.

Let $\varepsilon > 0$ be given. Now, by the definitions of the mapping $\mathscr{T}$ and the metric d_2, we obtain

$$\begin{aligned}
&d_2(\mathscr{T}(\mathscr{Q}), \mathscr{T}(\mathscr{S}))\\
&= d_2((\mathscr{F}(x,y,z), \mathscr{F}(y,x,y), \mathscr{F}(z,y,x)), (\mathscr{F}(u,v,w), \mathscr{F}(v,u,v), \mathscr{F}(w,v,u))) \\
&\leq \frac{1}{3}[d((\mathscr{F}(x,y,z), \mathscr{F}(u,v,w)) + d(\mathscr{F}(y,x,y), \mathscr{F}(v,u,v)) + (\mathscr{F}(z,y,x), \mathscr{F}(w,v,u)))]
\end{aligned} \tag{2.7}$$

Since $\mathscr{F}$ is partially continuous on X^3, for $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that

$$d((\mathscr{F}(x,y,z), \mathscr{F}(u,v,w)) < \frac{\varepsilon}{3}, \tag{2.8}$$

whenever

$$d_2(\mathscr{Q}, \mathscr{S}) = d_2((x,y,z),(u,v,w)) < \delta_1$$

Similarly, for $\varepsilon > 0$ there exists a $\delta_2 > 0$ such that

$$d((\mathscr{F}(y,x,y), \mathscr{F}(v,u,v)) < \frac{\varepsilon}{3}, \tag{2.9}$$

whenever

$$d_2(\mathscr{Q}', \mathscr{S}') = d_2((y,x,y),(v,u,v)) < \delta_2$$

and for $\varepsilon > 0$ there exists a $\delta_3 > 0$ such that

$$d((\mathscr{F}(z,y,x), \mathscr{F}(w,v,u)) < \frac{\varepsilon}{3}, \tag{2.10}$$

whenever

$$d_2(\mathscr{Q}'', \mathscr{S}'') = d_2((z,y,x),(w,v,u)) < \delta_3$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, from the inequalities (2.7), (2.8), (2.9) and (2.10) it follows that

$$d_2(\mathscr{Q}, \mathscr{S}) < \delta \Rightarrow d_2(\mathscr{T}(\mathscr{Q}), \mathscr{T}(\mathscr{S})) < \varepsilon. \tag{2.11}$$

Hence $\mathscr{T}$ is a partially continuous mapping on X^3 into itself.

Next we shall show that $\mathscr{T}$ is a nondecreasing map with respect to the order relation $\leq$ defined in X^3. Let $\mathscr{Q} = (x,y,z)$ and $\mathscr{S} = (u,v,w)$ be two elements in X^3 with $\mathscr{Q} \geq \mathscr{S}$. Then $x \geq u$, $y \leq v$ and $z \geq w$. From mixed monotonicity of the mapping $\mathscr{F}$ it follows that

$$\mathscr{F}(x,y,z) \geq \mathscr{F}(u,v,w), \ \ \mathscr{F}(y,x,y) \leq \mathscr{F}(v,u,v), \text{ and } \ \mathscr{F}(z,y,x) \geq \mathscr{F}(w,v,x)$$

Now, by definition of the mapping $\mathscr{T}$, we get

$$\begin{aligned}
\mathscr{T}(Q) &= (F(x,y,z), F(y,x,y), F(z,y,x)) \\
&\geq (F(u,v,w), F(v,u,v), F(w,v,u)) \\
&= \mathscr{S}(\mathscr{W})
\end{aligned}$$

which clear that $\mathscr{T}$ is a nondecreasing mapping on X^3 into itself. Next we show that $\mathscr{T}$ is a nonlinear partial M-set-contraction on X^3. Let B, C and D be three chains in X and let $\mathscr{C} = B \times C \times D$ be a chain in X^3. Then, by the definition of partial measure of noncompactness in X^3, we obtain

$$\begin{aligned}
\widetilde{\mu_p}(\mathscr{T}(\mathscr{C})) &= \widetilde{\mu_p}(\mathscr{T}(B \times C \times D)) \\
&= \widetilde{\mu_p}(\mathscr{F}(B \times C \times D) \times \mathscr{F}(C \times B \times C) \times \mathscr{F}(D \times C \times B)) \\
&= [\mu_p(\mathscr{F}(B \times C \times D)) + \mu_p(\mathscr{F}(C \times B \times C)) + \mu_p(\mathscr{F}(D \times C \times B))] \\
&= \varphi(\mu_p(B) + \mu_p(C) + \mu_p(D)) \\
&= \varphi(\widetilde{\mu_p}(B \times C \times D)) \\
&= \varphi(\widetilde{\mu_p}(\mathscr{C}))
\end{aligned}$$

for all bounded chains B, C and D in X. This shows that $\mathscr{T}$ is a nonlinear partial M-contraction on X^3 into itself.

Next, given an element $\mathscr{Q}_0 = (x_0, y_0, z_0) \in X^3$, define a sequence $\{\mathscr{Q}_n\}$ in X^3 as follows.

Set

$$\mathscr{Q}_1 = (x_1, y_1, z_1) = (\mathscr{F}(x_0, y_0, z_0), \mathscr{F}(y_0, x_0, y_0), \mathscr{F}(z_0, y_0, x_0)) = \mathscr{T}(\mathscr{Q}_0),$$

$$\mathscr{Q}_2 = (x_2, y_2, z_2) = (\mathscr{F}^2(x_0, y_0, z_0), \mathscr{F}^2(y_0, x_0, y_0), \mathscr{F}^2(z_0, y_0, x_0)) = \mathscr{T}^2(\mathscr{Q}_0),$$

$$\vdots$$

$$\mathscr{Q}_n = (x_n, y_n, z_n) = (\mathscr{F}^n(x_0, y_0, z_0), \mathscr{F}^n(y_0, x_0, y_0), \mathscr{F}^n(z_0, y_0, x_0)) = \mathscr{T}^n(\mathscr{Q}_0),$$

etc.

By hypotheses, there exists an element $(x_0, y_0, z_0) \in X^3$ such that

$$\mathscr{Q}_0 = (x_0, y_0, z_0) \leq (\mathscr{F}(x_0, y_0, z_0), \mathscr{F}(y_0, x_0, y_0), \mathscr{F}(z_0, y_0, x_0)) = \mathscr{T}(\mathscr{Q}_0) = \mathscr{Q}_1 \tag{2.12}$$

Since $\mathscr{T}$ is nondecreasing, from (2.12) it follows that

$$\mathscr{Q}_0 \leq \mathscr{Q}_1 \leq \mathscr{Q}_2 \leq \ldots \leq \mathscr{Q}_n \leq \ldots \tag{2.13}$$

Denote

$$\begin{aligned} \mathscr{C}_0 &= \{\mathscr{Q}_0, \mathscr{Q}_1, \ldots, \mathscr{Q}_n, \ldots\} \\ \mathscr{C}_1 &= \{\mathscr{Q}_1, \mathscr{Q}_2, \ldots, \mathscr{Q}_{n+1}, \ldots\} \\ &\vdots \\ \mathscr{C}_n &= \{\mathscr{Q}_n, \mathscr{Q}_{n+1}, \ldots, \mathscr{Q}_{2n}, \ldots\} \end{aligned} \tag{2.14}$$

As $\mathscr{F}$ is partially bounded, $\mathscr{T}$ is a partially bounded mapping on X^3, and so, each chain $\mathscr{C}_n$, $n = 0,\ 1,\ \ldots,$ is bounded in X^3. Moreover,

$$\mathscr{C}_0 \supset \mathscr{C}_1 \supset \mathscr{C}_2 \supset\ \ldots \mathscr{C}_n \supset\ \ldots \tag{2.15}$$

Therefore, by nondecreasing nature of $\widetilde{\mu_p}$, we obtain

$$\begin{aligned}
\widetilde{\mu_p}(\overline{\mathscr{C}_n}) &= \widetilde{\mu_p}(\mathscr{C}_n) \\
&= \widetilde{\mu_p}(\mathscr{T}(\mathscr{C}_{n-1})) \\
&\leq \varphi(\widetilde{\mu_p}(\mathscr{C}_{n-1})) \\
&\leq \varphi^2(\widetilde{\mu_p}(\mathscr{C}_{n-2})) \\
&. \\
&. \\
&. \\
&\leq \varphi^n(\widetilde{\mu_p}(\mathscr{C}_0))
\end{aligned} \tag{2.16}$$

Taking the limit superior as $n \to \infty$ in the above equality (2.16), we obtain

$$\lim_{n\to\infty} \widetilde{\mu_p}(\overline{\mathscr{C}_n}) = \lim_{n\to\infty} \widetilde{\mu_p}(\mathscr{C}_n) \leq \limsup_{n\to\infty} \varphi^n(\widetilde{\mu_p}(\mathscr{C}_0)) = \lim_{n\to\infty} \varphi^n(\widetilde{\mu_p}(\mathscr{C}_0)) = 0 \tag{2.17}$$

Hence, by condition (P_4) of μ_p,

$$\overline{\mathscr{C}}_\infty = \cap_{n=1}^{\infty} \mathscr{C}_n \neq \phi \text{ and } \mathscr{C}_\infty \subset \mathscr{P}_{rcp,cn}(X)$$

From (2.17) it follows that for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\widetilde{\mu_p}(\mathscr{C}_n) < \varepsilon \quad \forall n \geq n_0$$

This shows that $\overline{\mathscr{C}}_{n_0}$ and consequently $\overline{\mathscr{C}}_0$ is a compact chain in X. Hence, $\{\mathscr{Q}_n\}$ has a convergent subsequence. Furthermore, since the order relation $\leq$ and d are compatible in the compact chain $\mathscr{C}_0$ of X, the original sequence $\{\mathscr{Q}_n\} = \{\mathscr{T}^n\mathscr{Q}_0\}$ is convergent and converges monotonically to a point, say $\mathscr{Q}^* \in \overline{\mathscr{C}}_0$. Since the ordered metric space X is regular, we have that $\mathscr{Q}_n \leq \mathscr{Q}^*$. Finally, from the partial continuity of $\mathscr{T}$, we get

$$\mathscr{T}(\mathscr{Q}^*) = \mathscr{T}(\lim_{n\to\infty} \mathscr{Q}_n) = \lim_{n\to\infty} \mathscr{T}(\mathscr{Q}_n) = \lim_{n\to\infty} \mathscr{Q}_{n+1} = \mathscr{Q}^*$$

This further in view of the definition of mapping $\mathscr{T}$ implies that

$$(\mathscr{F}(x^*,y^*,z^*),\ \ \mathscr{F}(y^*,x^*,y^*), \mathscr{F}(z^*,y^*,x^*)) = (x^*,y^*,z^*)$$

$$x^* = \mathscr{F}(x^*,y^*,z^*), y^* = \mathscr{F}(y^*,x^*,y^*) \text{ and } z^* = \mathscr{F}(z^*,y^*,x^*)$$

As a result the tripled equations $\mathscr{F}(x^*,y^*,z^*)$, $\mathscr{F}(y^*,x^*,y^*)$ and $\mathscr{F}(z^*,y^*,x^*)$ have a tripled solution (x^*,y^*,z^*) and the sequences of successive iterations $\{\mathscr{F}^n(x_0,y_0,z_0)\}$, $\{\mathscr{F}^n(y_0,x_0,y_0)\}$ and $\{\mathscr{F}^n(z_0,y_0,x_0)\}$ converge monotonically to x^*, y^* and z^*. This completes the proof. □

Theorem 2.2. *Let $(X,\leq,d)$ be a regular partially ordered complete metric space such that the metric d and the order relation $\leq$ are compatible in every compact chain $\mathscr{C}$ of X. Suppose that $\mathscr{F}: X^3 \to X$ is a partially continuous and partially bounded mixed monotone tripled mapping satisfying*

$$\mu_p(\mathscr{F}(B\times C\times D))+\mu_p(\mathscr{F}(C\times B\times C))+\mu_p(\mathscr{F}(D\times C\times B))$$
$$\leq \mu_p(B)+\mu_p(C)+\mu_p(D)$$

for all $B,C,D \in \mathscr{P}_{bd,cn}(X)$, where μ_p is a full partial measure of noncompactness with maximum property satisfying $\mu_p(B)+\mu_p(C)+\mu_p(D) > 0$. Further, if there exists an element $(x_0,y_0,z_0) \in X\times X\times X$ such that $x_0 \leq F(x_0,y_0,z_0)$, $y_0 \geq F(y_0,x_0,y_0)$ and $z_0 \leq F(z_0,y_0,x_0)$ or $x_0 \geq F(x_0,y_0,z_0)$, $y_0 \leq F(y_0,x_0,y_0)$ and $z_0 \geq F(z_0,y_0,x_0)$, then $\mathscr{F}$ has a tripled fixed point (x^*,y^*,z^*) and the sequences $\{\mathscr{F}_n(x_0,y_0,z_0)\}$, $\{\mathscr{F}_n(y_0,x_0,y_0)\}$ and $\{\mathscr{F}_n(z_0,y_0,x_0)\}$ of successive iterations converge monotonically to x^*, y^* and z^*, respectively. Moreover, the set of all comparable tripled fixed points is compact.

Proof. As in previous theorem, Our main purpose in the immediate sequel is to prove the theorem in the event that there exists an element $(x_0,y_0,z_0) \in X\times X\times X$ such that $x_0 \leq F(x_0,y_0,z_0)$, $y_0 \geq F(y_0,x_0,y_0)$ and $z_0 \leq F(z_0,y_0,x_0)$. The proof for the case $x_0 \geq F(x_0,y_0,z_0)$, $y_0 \leq F(y_0,x_0,y_0)$ and $z_0 \geq F(z_0,y_0,x_0)$ is similar and can be gained by using equivalent arguments with suitable modification. Define a mapping $\mathscr{T}: X^3 \to X^3$ by

$$\mathscr{T}(\mathscr{X}) = (\mathscr{F}(x,y,z),\mathscr{F}(y,x,y),\mathscr{F}(z,y,x))$$

For all $\mathscr{X}=(x,y,z) \in X\times X\times X = X^3$. From Lemmas 2.2, 2.3 and 2.4, it follows that $\mathscr{T}$ is a partially continuous, partially bounded mapping on X^3 into itself. Also it is easily verified that $\mathscr{T}$ is nondecreasing on X^3. We now show that $\mathscr{T}$ is a partial condensing mapping on X^3. Let $\mathscr{C} = B\times C\times D$ be a chain in X^3, where B, C and D are bounded chains in X. Then, by definition of the mapping $\mathscr{T}$ and the partial measure of noncompactness μ_p, we obtain

$$\begin{aligned}
\widetilde{\mu_p}(\mathscr{T}(\mathscr{C})) &= \widetilde{\mu_p}(\mathscr{T}(B\times C\times D)) \\
&= \widetilde{\mu_p}(\mathscr{F}(B\times C\times D)\times\mathscr{F}(C\times B\times C)\times\mathscr{F}(D\times C\times B)) \\
&= \mu_p(\mathscr{F}(B\times C\times D))+\mu_p(\mathscr{F}(C\times B\times C))+\mu_p(\mathscr{F}(D\times C\times B)) \\
&= \mu_p(B)+\mu_p(C)+\mu_p(D) \\
&= \widetilde{\mu_p}(B\times C\times D) \\
&= \widetilde{\mu_p}(\mathscr{C})
\end{aligned}$$

Provided $\widetilde{\mu_p}(\mathscr{C}) = \mu_p(B)+\mu_p(C)+\mu_p(D) > 0$. Therefore, $\mathscr{T}$ is a condensing mapping on X^3 into itself. Given $\mathscr{Q}_0 = (x_0,y_0,z_0)\in X^3$, define a sequence $\{\mathscr{Q}_n\}$ of points of X^3 of successive iterations of $\mathscr{T}$ by

$$\mathscr{Q}_{n+1} = \mathscr{T}(\mathscr{Q}_n),\ n\ =\ 0,\ 1,\ \ldots \tag{2.18}$$

Since $\mathscr{T}$ is nondecreasing, in view of (2.12) we obtain

$$\mathscr{Q}_0 \le \mathscr{Q}_1 \le \ldots \le \mathscr{Q}_n \le \ldots$$

Let

$$\begin{aligned}
\mathscr{C} &= \{\mathscr{Q}_0,\mathscr{Q}_1,\ ..\ \mathscr{Q}_n\ \ldots\} \\
&= \{\mathscr{Q}_0\}\cap\{\mathscr{Q}_1,\ ..\ \mathscr{Q}_n\ \ldots\} \\
&= \{\mathscr{Q}_0\}\cap\{\mathscr{T}(\mathscr{C})\}
\end{aligned}$$

Clearly, $\mathscr{C}$ is a bounded chain in E^3 in view of the fact that $\mathscr{T}$ is a partially bounded mapping on E^3. Now, if $\widetilde{\mu}_p(\mathscr{C}) > 0$, then

$$\begin{aligned}
\widetilde{\mu}_p(\mathscr{C}) &= \widetilde{\mu_p}(\mathscr{Q}_0\cup\mathscr{T}(\mathscr{C})) \\
&= \max\{\mu_p(\mathscr{Q}_0),\mu_p(\mathscr{T}(\mathscr{C}))\} \\
&= \max\{0,\mu_p(\mathscr{T}(\mathscr{C}))\} \\
&= \widetilde{\mu}_p(\mathscr{T}(\mathscr{C})) \\
&< \widetilde{\mu}_p(\mathscr{C})
\end{aligned}$$

which is a contradiction. Hence $\mu_p(\mathscr{C}) = 0$ and that $\mathscr{C}$ is a compact chain in X^3. The rest of the proof is similar to Theorem 2.1 and hence we omit the details. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] H. Afshari, A. Kheiryan, H.M. Srivastava, Tripled fixed point theorems and applications to a fractional differential equation boundary value problem, Asian-Eur. J. Math. 10 (3) (2017), Article ID 1750056.

[2] S.V. Bedre, S.M. Khairnar, B.S. Desale, Tripled coincidence point theorems for nonlinear contractions in partially ordered quasi-metric spaces with a Q-function, Int. J. Math. Anal. 9 (17) (2014), 823-836.

[3] S.V. Bedre, S.M. Khairnar, B.S. Desale, Hybrid fixed point theorems for M-contraction type maps and applications to functional differential equation, in Proceedings of the International Conference on Information and Mathematical Sciences (IMS '13), pp. 390–397, Elsevier Science and Technology, October 2013.

[4] V. Berinde, M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput. 218 (10) (2010), 5929-5936.

[5] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Fuel Energy Abstr. 74 (15) (2011), 4889-4897.

[6] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal., Theory Method Appl. 65 (2006), 1379-1393.

[7] B.C. Dhage, Coupled hybrid fixed point theory in a partially ordered metric space and attractivity of nonlinear hybrid fractional integral equations, J. Fixed Point Theory Appl. 19 (4) (2017), 2541-2575.

[8] B.C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, Differ. Equ. Appl. 5 (2013), 155-184.

[9] S. Heikkil'a, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations,Marcel Dekker inc., New York, 1994.

[10] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones. Academic Press, New York, 1988.

[11] J. J. Nieto and R. Rodriguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation, Acta Math. Sin. (Engl. Ser.) 23 (12) (2007), 2205-2212.

[12] I. Podlubny, Fractional differential equations, Academic Press, New York, 1999.

[13] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.

[14] H. M. Shrivastava, S. V. Bedre, S. M. Khairnar and B. S. Desale, Krasnosel'skii type hybrid fixed point theorems and their applications to fractional integral equations, Abstr. Appl. Anal. 2014 (2014), Article ID 710746.