Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 6, 2917-2925
https://doi.org/10.28919/jmcs/4982
ISSN: 1927-5307

SOME PROPERTIES OF GENERALIZED COMPLEMENTS OF A GRAPH

K.P. GIRIJA, SABITHA D'SOUZA*, C. DEVADAS NAYAK, PRADEEP G. BHAT

Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576104, INDIA

Abstract. Let $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of vertex set V of G. The $k-$complement of G denoted by G_k^P is defined as follows: for all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add edges between V_i and V_j which are not in G. The graph G is k-self complementary with respect to P if $G_k^P \cong G$. The k(i)-complement $G_{k(i)}^P$ of a graph G with respect to P is defined as follows: for all $V_r \in P$, remove edges inside V_r and add edges which are not in V_r. In this paper we provide sufficient conditions for G_k^P and $G_{k(i)}^P$ to be disconnected, regular, line preserving and Eulerian.

Keywords: in-degree; out-degree; Eulerian graph; k-complement; k(i)-complement.

2010 AMS Subject Classification: 05C07, 05C40, 05C45.

1. Introduction

All graphs considered in this paper will be assumed to be simple, finite and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph is connected if every pair of vertices are joined by a path and a disconnected graph is a graph consists at least two components. The complement of a graph G, denoted by $\overline{G}$ has the same vertex set as that of G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The number of edges incident to a

*Corresponding author
E-mail address: sabitha.dsouza@manipal.edu

Received August 31, 2020

vertex v in G is called degree of vertex v and is denoted by $d(v)$. The minimum degree among vertices of G is denoted by $\delta(G)$, while $\Delta(G)$ denotes maximum degree among vertices of G. If $\delta(G) = \Delta(G) = r$, then all the vertices have same degree and G is called regular of degree r. A graph is Eulerian if it has a closed trail containing all edges [3]. E. Sampathkumar et.al [6] introduced k-complement and k(i)-complement of graphs as follows.

Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of vertex set V of G. The $k-$complement of G denoted by G_k^P is defined as follows: for all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add edges between V_i and V_j which are not in G. The graph G is k-self complementary with respect to P if $G_k^P \cong G$. The k(i)-complement $G_{k(i)}^P$ of a graph G with respect to P is defined as follows: for all $V_r \in P$, remove the edges inside V_r and add edges which are not in V_r. Any graph G is k(i)-self complementary if $G_{k(i)}^P \cong G$.

In a graph G, the in-degree [5] of a vertex v denoted by $d_i(v)$, defined with respect to the partition P of $V(G)$ is the number of edges incident at v each of whose other ends are also in V_i for all $i = 1, 2, \cdots, k$. The out-degree [5] of a vertex $v \in V_i$ denoted by $d_o(v)$, defined with respect to the partition P of $V(G)$ is the number of edges incident at v each of whose other ends are not in V_i for all $i = 1, 2, \cdots, k$.

Graph partitioning problem arises in various areas of computer science, engineering and related fields. Recently, the concept of graph partition has gained importance due to its application in route planning, clustering and detection of cliques in social, pathological/biological networks and high performance computing.

For more information on generalized complements of graphs, refer to [1],[2],[4],[5],[6] and [7]. In this paper we provide some sufficient conditions for G_k^P and $G_{k(i)}^P$ to be disconnected, regular, line preserving and Eulerian.

2. Main Results

Theorem 1. *The k-complement of a disconnected graph G is disconnected if it satisfies the following conditions,*

(1) $G = H \cup rK_1$ and there exists a partite V_i such that $\langle V_i \rangle = rK_1 \cup v_j$ and $v_j \in V(H)$ covers all vertices of H.

(2) If G has at least one complete subgraph K_r and there exists a partite V_i such that $V_i = \{(V - V(K_r)) + v_j : v_j \in V(K_r)\}$.

Proof. Let $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of a disconnected graph G of order n.

(1) Let $G = H \cup rK_1$ be a disconnected graph of order n. Consider any partite V_i so that $\langle V_i \rangle = rK_1 \cup v_j$, where v_j covers all the vertices of H. Then by the definition of k-complement of graph, vertex v_j becomes isolated vertex in G_k^P. Therefore G_k^P is disconnected.

(2) Let G be a disconnected graph with at least one complete subgraph K_r as a component. Suppose V_i is a partite consists of vertices of all components of G along with v_j of K_r. Since in-degree of v_j is zero and out-degree of v_j is $r-1$, by definition of k-complement of a graph the vertex v_j becomes isolated in G_k^P. Thus G_k^P is disconnected.

□

Theorem 2. *The k-complement of a connected graph G is disconnected if it satisfies the following conditions,*

(1) There exists a partite V_i consists of independent set of vertices v_j such that every $v_j \in V_i$ covers all vertices of other partites.

(2) Graph G has a single cut vertex v_c, which belongs to the partite V_i such that $|V_i| = 1$.

(3) A partite V_i has independent set of vertices and every vertex in V_i is adjacent to each vertex of other partites.

Proof. Let $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of vertex set of a connected graph G of order n.

(1) Suppose partite V_i consists only the independent set of vertices, which covers all vertices of other $k-1$ partites. Since all vertices of V_i are independent, in-degree of each vertex of V_i is zero. By definition of k-complement of a graph every vertex of V_i becomes isolated. Hence G_k^P is disconnected.

(2) Suppose G has a single cut vertex, which belongs to singleton partite V_i. By definition of G_k^P, $n-1$ edges will be removed from v_c and hence v_c will be isolated vertex. Thus G_k^P is a disconnected graph.

(3) In-degree of all $v_j \in V_i$ is zero since each vertex inside the partite is independent. Also if every vertex from each partite is adjacent to all vertices of other partites, then by definition of G_k^P there will be no edges from vertices of one partite to other in G_k^P. Hence G_k^P is disconnected graph.

□

Theorem 3. *The $G_{k(i)}^P$ of non regular graph G is r-regular if $r = \overline{d_i(v_j)} + d_o(v_j)$, where $\overline{d_i(v_j)} = |V_i| - d_i(v_j)$.*

Proof. Let G be any non regular graph with $|V(G)| = n$ and $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of vertices of G. By the definition of $G_{k(i)}^P$, $\overline{d_i(v_j)}$ edges are added to all vertices in V_i for all $i = 1, 2, \cdots, k$, where $\overline{d_i(v_j)} = |V_i| - d_i(v_j)$ and out-degree $d_o(v_j)$ of each $v_j \in V_i$ remains same. As $\overline{d_i(v_j)} + d_o(v_j)$ gives the degree of each vertex $v_j \in V(G_{k(i)}^P)$ and if $\overline{d_i(v_j)} + d_o(v_j)$ is constant then $G_{k(i)}^P$ is a regular graph. □

Theorem 4. *The G_k^P of non regular graph G is r-regular if $r = d_i(v_j) + \overline{d_o(v_j)}$, where $\overline{d_o(v_j)} = |V_i| - d_o(v_j)$.*

Proof. It is similar to the proof of Theorem 3, noting $G_k^P \cong \overline{G_{k(i)}^P}$ the result follows. □

Theorem 5. *For any two non isomorphic graphs $G(V, E_1)$ and $H(U, E_2)$ of same order, if $\sum_{r=1}^{n} \left(\overline{d_i(v_r)} + d_o(v_r) \right) = \sum_{s=1}^{n} \left(\overline{d_i(u_s)} + d_o(u_s) \right)$ then $G_{k(i)}^P$ and $H_{k(i)}^P$ are line preserving.*

Proof. Let $G(V, E_1)$ and $H(U, E_2)$ be any two graphs of same order. $\overline{d_i(v_r)} + d_o(v_r) = d(v_r)$ in $G_{k(i)}^P$ for all $r = 1, 2, \cdots, n$ and $\overline{d_i(u_s)} + d_o(u_s) = d(u_s)$ in $H_{k(i)}^P$ for all $s = 1, 2, \cdots, n$. Now $\sum_{r=1}^{n} \left(\overline{d_i(v_r)} + d_o(v_r) \right)$ and $\sum_{s=1}^{n} \left(\overline{d_i(u_s)} + d_o(u_s) \right)$ will be the degree sum of all vertices in $G_{k(i)}^P$ and $H_{k(i)}^P$ respectively. If $\sum_{r=1}^{n} \left(\overline{d_i(v_r)} + d_o(v_r) \right) = \sum_{s=1}^{n} \left(\overline{d_i(u_s)} + d_o(u_s) \right)$, we say that $G_{k(i)}^P$ and $H_{k(i)}^P$ are line preserving.

□

Corollary 6. *For any two non isomorphic graphs $G(V, E_1)$ and $H(U, E_2)$ of same order, if $\sum_{r=1}^{n} \left(d_i(v_r) + \overline{d_o(v_r)} \right) = \sum_{s=1}^{n} \left(d_i(u_s) + \overline{d_o(u_s)} \right)$ then G_k^P and H_k^P are line preserving.*

Proof. It can be proved in the similar lines of Theorem 5. □

Theorem 7. *For any two non-isomorphic graphs $G(V,E_1)$ and $H(U,E_2)$ of same order, if $\sum_{r=1}^{n}\left(\overline{d_i(v_r)}+d_o(v_r)\right)=\sum_{s=1}^{n}\left(d_i(u_s)+\overline{d_o(u_s)}\right)$ then $G_{k(i)}^P$ and H_k^P are line preserving.*

Proof. Let $G(V,E_1)$ and $H(U,E_2)$ be any two graphs of same order. $\overline{d_i(v_r)}+d_o(v_r)=d(v_r)$ in $G_{k(i)}^P$, where $r=1,2,\cdots,n$ and $d_i(u_s)+\overline{d_o(u_s)}=d(u_s)$ in H_k^P, where $s=1,2,\cdots,n$. Now $\sum_{j=1}^{n}\left(\overline{d_i(v_j)}+d_o(v_j)\right)$ and $\sum_{k=1}^{n}\left(d_i(u_k)+\overline{d_o(u_k)}\right)$ will be the degree sum of all vertices in $G_{k(i)}^P$ and H_k^P respectively. If $\sum_{r=1}^{n}\left(\overline{d_i(v_r)}+d_o(v_r)\right)=\sum_{s=1}^{n}\left(d_i(u_s)+\overline{d_o(u_s)}\right)$, we say that $G_{k(i)}^P$ and H_k^P are line preserving.

□

Theorem 8. *The k(i)-complement of a connected graph is Eulerian if any one of the following conditions hold,*

(1) *Each partite is of odd(even) order consists of independent set of vertices with out-degree even(odd).*
(2) *Each vertex in a partite of even(odd) order has in-degree odd(even) and out-degree even (odd).*

Proof. Let G be a connected graph with partition $P=\{V_1,V_2,\cdots,V_k\}$ of $V(G)$.

(1) Suppose each odd order partite has independent set of vertices with out-degree even. Then $\langle V_i\rangle$ will be totally disconnected. Then by definition of k(i)-complement of graph, one can observe that every vertex of $G_{k(i)}^P$ will be of even degree. Therefore $G_{k(i)}^P$ is Eulerian.
(2) Suppose each vertex in a partite of even order has in-degree odd(even) and out-degree even (odd). Then in $G_{k(i)}^P$, we find that in-degree and out-degree of each vertex v in every partite V_i will be even. Hence $G_{k(i)}^P$ is Eulerian.

□

Theorem 9. *The k-complement of a disconnected graph G is Eulerian if any one of the following conditions hold,*

(1) *The order of a totally disconnected graph G and cardinality of each partite are of either even or odd.*

(2) $|V(G)|$ is odd and G has complete subgraphs as its components such that $\langle V_i \rangle$ is a complete graph.

Proof. Let G be a disconnected graph of order n with partition $P = \{V_1, V_2, \cdots, V_k\}$ of $V(G)$.

(1) Suppose order of a totally disconnected graph G and partites V_i are of either even or odd, then from definition of k-complement of a graph, every vertex v in V_i will be connected to every vertex v in V_j where $i, j = 1, 2, \cdots, k$ and $i \neq j$. Since both n and $|V_i|$ are either odd or even, degree of v in G_k^P is even. Thus G_k^P is Eulerian.

(2) Let G be a disconnected graph of complete subgraph as its components such that $|V(G)|$ be odd. Then from definition of k-complement of graphs, G_k^P is isomorphic to complete graph of odd order. Thus G_k^P is an Eulerian graph.

$\square$

Example 2.1.

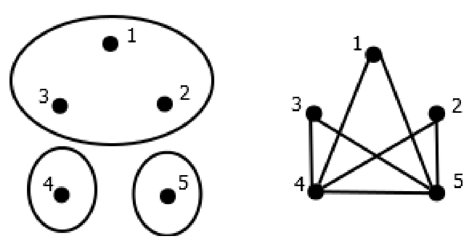

FIGURE 1. G and G_3^P

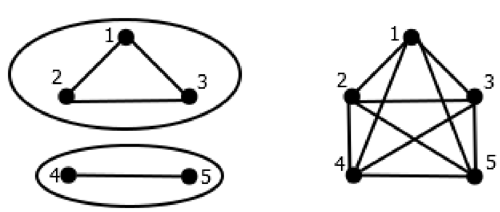

FIGURE 2. G and G_2^P

Theorem 10. *Let G be a disconnected graph such that every component of G be of even degree. Then G_k^P is Eulerian if every vertex in $\langle V_i \rangle$ is of even degree and any one of the following conditions hold good.*

(1) $|P|$ and $|V_i|$ are odd.

(2) $|P|$ and $|V_i|$ are even.

(3) $|P|$ is odd and $|V_i|$ is even.

Proof. Let $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of vertex set of G, each component of G and $\langle V_i \rangle$ be Eulerian.

(1) Suppose $|P| = k$ and $|V_i|$ are both odd, as every component in each partite of G is Eulerian, in-degree of all vertices of V_i is even. By definition of k-complement, degree of each vertex in G_k^P is even. Hence G_k^P is Eulerian.

(2) Suppose $|P| = k$ and $|V_i|$ are both even, as every component in each partite of G is Eulerian, in-degree of all vertices of V_i is even. By definition of k-complement, degree of each vertex in G_k^P is even. Hence G_k^P is Eulerian.

(3) Suppose $|P| = k$ is odd and $|V_i|$ is even, as every component in each partite of G is Eulerian, in-degree of all vertices of V_i is even. By definition of k-complement, degree of each vertex in G_k^P is even. Hence G_k^P is Eulerian.

□

Proposition 1. [6] *The k-complement and k(i) complements are related as follows*
(i) $\overline{G_k^P} \cong G_{k(i)}^P$ and (ii) $\overline{G_{k(i)}^P} \cong G_k^P$.

Theorem 11. *For any graph G*

i. $\overline{G}_{k(i)}^P \cong \overline{G}$ if and only if $\overline{G}_k^P \cong G$.
ii. $\overline{G}_k^P \cong \overline{G}$ if and only if $\overline{G}_{k(i)}^P \cong G$.

Proof. (i) Consider,

$$\overline{G}_{k(i)}^P \cong \overline{G} \tag{1}$$

$$\overline{\overline{G}_{k(i)}^P} \cong \overline{\overline{G}} \to \overline{\overline{G}_{k(i)}^P} \cong G$$

Assume that $H = \overline{G}$. Then

$$\overline{H_{k(i)}^P} \cong G$$

From Proposition 1

$$H_k^P \cong G \to \overline{G}_k^P \cong G \tag{2}$$

Conversly,

$$\overline{G}_k^P \cong G \tag{3}$$

$$\overline{\overline{G}_k^P} \cong \overline{G} \to \overline{H_k^P} \cong \overline{G}$$

From Proposition 1

$$H_{k(i)}^P \cong \overline{G} \to \overline{G}_{k(i)}^P \cong \overline{G} \tag{4}$$

Similarly we can prove (ii). □

3. Conclusion

In this paper we have obtained some sufficient conditions for G_k^P and $G_{k(i)}^P$ to be disconnected, regular, line preserving and Eulerian. To investigate the conditions for G_k^P and $G_{k(i)}^P$ to be Hamiltonian, connected and isomorphic is an open area of research.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] T. Gangopadhyay, S. P. Rao Hebbare, Paths in r-partite self complementary graphs, Discrete Math. 32 (1980), 229-244.

[2] H. J. Gowtham, Sabitha D'Souza and Pradeep G. Bhat, Laplacian energy of generalized complements of a graph, Kragujevac J. Math. 42(2) (2018), 299-315.

[3] F. Harary, Graph theory, Narosa Publishing House, New Delhi, (1989).

[4] E. Sampathkumar, L. Pushpalatha, Reconstruction of a graph of order p from its $(p-1)-$complements, Indian J. Pure Appl. Math. 27(5) (1996), 435-441.

[5] E. Sampathkumar, L. Pushpalatha, Complement of a graph: A generalization, Graphs Comb. 14 (1998), 377-392.

[6] E. Sampathkumar, L. Pushpalatha, C.V. Venkatachalam and Pradeep Bhat, Generalized complements of a graph, Indian J. Pure Appl. Math. 29(6) (1998), 625-639.

[7] S. D'Souza, H.J. Gowtham , P.G. Bhat, Energy of generalized complements of a graph, Eng. Lett. 28(1) (2020), 131-136.